MATHEMATICS OF COMPUTATION, VOLUME 33, NUMBER 148
OCTOBER 1979, PAGES 1333–1336

New Primes of the Form $k \cdot 2^n + 1$

By Robert Baillie

Abstract. All primes of the form $k \cdot 2^n + 1$ for k odd, $1 \leqslant k < 150$, $1 \leqslant n \leqslant 1500$, have now been computed. Those not previously published are given here. Numbers with $151 \leqslant k \leqslant 999$ and $n \leqslant 600$ were also tested. Three new factors of Fermat numbers and a large pair of twin primes were found.

This report extends those of Matthew and Williams [5], Robinson [8], and Shippee [9], who searched for primes of the form $k \cdot 2^n + 1$.

The method used to test $k \cdot 2^n + 1$ for primality was stated originally by Proth [6], and is proven in [7]. Given $N = k \cdot 2^n + 1$ with $k < 2^n$, we look for a number D which makes the Jacobi symbol $(D/N) = -1$. If $3 \nmid k$, we may take $D = 3$; if $3 \mid k$, a (usually short) search is conducted for a suitable D. Then N is prime if and only if $D^{(N-1)/2} \equiv -1 \pmod{N}$.

Each prime found was checked to see if it was a factor of a Fermat number. Three factors were found: $629 \cdot 2^{257} + 1 \mid F_{255}$, $247 \cdot 2^{302} + 1 \mid F_{298}$, and $225 \cdot 2^{547} + 1 \mid F_{544}$. For a list of all other Fermat numbers known to be composite, see [4], [5], and [9].

Also, for each prime $N = k \cdot 2^n + 1$ found here, as well as those in [5] and [8] with $n \geqslant 200$, $N - 2$ was checked for primality using Theorem 13 of [1]. The largest pair of twin primes thus found was $297 \cdot 2^{546} \pm 1$. The largest pair previously published is $156 \cdot 5^{202} \pm 1$ [11]. (Since this note was originally submitted, an even larger pair of 303-digit twin primes was published. See [2].)

The table lists the primes of the form $k \cdot 2^n + 1$ for $1 \leqslant k < 150$, $1 \leqslant n \leqslant 1500$ which were not given in [5] and [8]. π_k denotes the number of primes for a given k for $1 \leqslant n \leqslant 1500$. Note that some values of k, such as $k = 81$, yield many primes, while some, like $k = 47$ give very few. Usually, $47 \cdot 2^n + 1$ is divisible by a small prime. In fact, $47 \cdot 2^n + 1$ is divisible by 3, 5, 7, or 13 unless $n \equiv 7 \pmod{12}$. By contrast, $81 \cdot 2^n + 1$ is never divisible by 3 or 7; if $5 \mid 81 \cdot 2^n + 1$, then $n \equiv 2 \pmod{4}$; if $13 \mid 81 \cdot 2^n + 1$, then $n \equiv 2 \pmod{12}$, so in this case, $81 \cdot 2^n + 1$ is also divisible by 5. Thus, $81 \cdot 2^n + 1$ has more of a chance to produce primes than does $47 \cdot 2^n + 1$.

If n is fixed, the sequence $\{a_k\} = \{k \cdot 2^n + 1\}$ is prime for infinitely many k, by Dirichlet's theorem. If k is fixed, does $\{b_n\} = \{k \cdot 2^n + 1\}$ likewise have infinitely many primes? The answer is, "Not necessarily." Stark [10] gives a value of

Received January 12, 1979.
AMS (MOS) subject classifications (1970). Primary 10A25, 10A40, 10–04.
Key words and phrases. Fermat numbers, factoring, twin primes.

k (k = 2935 36333 15419 25531) for which b_n is always composite. See [3, p. 420] for a brief discussion of this problem.

These calculations took several hundred hours of background time on a CDC 6500. The author wishes to thank the computer operators and system staff of the Computer-Based Education Research Laboratory for their cooperation and patience in providing the computer time for this computation.

TABLE

Primes of the form $k \cdot 2^n + 1$

k	π_k	n	k	π_k	n
1	5		55	13	
3	19		57	21	1312, 1399
5	12		59	6	1085
7	19		61	5	
9	25	1305, 1411, 1494	63	33	1150, 1290, 1441
11	12		65	22	1317
13	10		67	22	1148, 1366
15	21		69	12	1450
17	10		71	13	
19	5	1246	73	12	
21	18		75	27	1018, 1054
23	12		77	10	
25	20	1280, 1328	79	7	1330
27	24	1076, 1090	81	38	1384
29	18	1053, 1175	83	6	
31	6		85	14	1300
33	18	1420	87	15	1268, 1302
35	18	1423, 1443	89	8	
37	20	1240	91	4	
39	27	1057	93	22	1020, 1110, 1478
41	6		95	22	1039
43	14	1076	97	9	
45	16	1374	99	31	
47	2	1483	101	21	1103
49	11	1202	103	10	
51	23	1485	105	27	
53	11		107	9	

TABLE (*continued*)

k	π_k	n
109	5	
111	11	
113	13	1045
115	11	1298
117	15	1416
119	10	1115
121	10	
123	17	1173
125	15	1205, 1279, 1411
127	14	1098, 1420
129	12	1071
131	21	1, 3, 9, 13, 19, 21, 25, 51, 55, 97, 153, 165, 199, 261, 285, 361, 373, 465, 475, 529, 765
133	13	4, 6, 10, 16, 30, 124, 174, 192, 336, 600, 720, 1092, 1138
135	33	1, 2, 4, 6, 10, 15, 18, 20, 30, 31, 35, 38, 39, 51, 85, 90, 106, 108, 202, 238, 253, 282, 330, 361, 452, 459, 646, 895, 922, 1201, 1402, 1441, 1462
137	8	3, 27, 39, 83, 203, 395, 467, 875
139	3	2, 14, 914
141	33	1, 3, 5, 7, 8, 12, 15, 20, 31, 33, 37, 41, 61, 65, 91, 93, 103, 117, 133, 137, 141, 160, 291, 303, 343, 488, 535, 555, 556, 640, 756, 897, 917
143	7	53, 77, 293, 333, 393, 809, 825
145	11	6, 16, 28, 70, 76, 250, 276, 312, 562, 636, 1366
147	20	8, 11, 15, 18, 19, 26, 44, 60, 84, 90, 91, 134, 155, 179, 258, 275, 475, 620, 824, 888
149	20	3, 7, 9, 15, 17, 27, 33, 35, 57, 125, 127, 137, 191, 513, 819, 827, 921, 931, 1047, 1147

Note Added in Proof. I subsequently tested the range $20 \leqslant n \leqslant 100$, $30001 \leqslant k < 130000$ and found another Fermat factor:

$$92341 \cdot 2^{96} + 1 \mid F_{93}.$$

Computer-Based Education Research Laboratory
University of Illinois at Urbana-Champaign
Urbana, Illinois 61801

1. JOHN BRILLHART, D. H. LEHMER & JOHN L. SELFRIDGE, "New primality criteria and factorizations of $2^m \pm 1$," *Math. Comp.*, v. 29, 1975, pp. 620–647. MR 52 #5546.

2. R. E. CRANDALL & M. A. PENK, "A search for large twin prime pairs," *Math. Comp.*, v. 33, 1979, pp. 383–388.

3. RICHARD K. GUY, "Some unsolved problems," in *Computers in Number Theory* (A. O. L. Atkin and B. J. Birch, Eds.), Academic Press, New York, 1971, pp. 415–422.

4. JOHN C. HALLYBURTON, JR. & JOHN BRILLHART, "Two new factors of Fermat numbers," *Math. Comp.*, v. 29, 1975, pp. 109–112. MR 51 #5460. For a correction, see *Math. Comp.*, v. 30, 1976, p. 198. MR 52 #13599.

5. G. MATTHEW & H. C. WILLIAMS, "Some new primes of the form $k \cdot 2^n + 1$," *Math. Comp.*, v. 31, 1977, pp. 797–798. MR 55 #12605.

6. F. PROTH, "Théorèmes sur les nombres premiers," *C. R. Acad. Sci. Paris*, v. 87, 1878, p. 926.

7. RAPHAEL M. ROBINSON, "The converse of Fermat's theorem," *Amer. Math. Monthly*, v. 64, 1957, pp. 703–710. MR 20 #4520.

8. RAPHAEL M. ROBINSON, "A report on primes of the form $k \cdot 2^n + 1$ and on factors of Fermat numbers," *Proc. Amer. Math. Soc.*, v. 9, 1958, pp. 673–681. MR 20 #3097.

9. D. E. SHIPPEE, "Four new factors of Fermat numbers," *Math. Comp.*, v. 32, 1978, p. 941.

10. H. M. STARK, *An Introduction to Number Theory*, Markham, Chicago, 1970, p. 110. MR 40 #7186.

11. H. C. WILLIAMS, "Primality testing on a computer," *Ars Combinatoria*, v. 5, 1978, pp. 127–185.