MATHEMATICS OF COMPUTATION, VOLUME 34, NUMBER 149
JANUARY 1980, PAGES 227–233

Solution of Linear Equations With Rational Toeplitz Matrices*

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Abstract. We associate a sequence of Toeplitz matrices with the rational formal power series $T(z)$. An algorithm for solving linear equations with a Toeplitz matrix from this sequence is given. The algorithm requires $O(n)$ operations to solve a set of n equations, for n sufficiently large.

1. Introduction. In this paper, we present an algorithm for solving a system of $N + 1$ linear equations with special Toeplitz structure:

$$T_N x = y. \tag{1}$$

For every $n \geqslant 0$, let $T_n = \{t_{i-j}, 0 \leqslant i, j \leqslant n\}$ be a Toeplitz matrix. We assume that T_n^{-1} exists for $0 \leqslant n \leqslant N$ and that the formal Laurent series

$$T(z) = \sum_{k=-\infty}^{\infty} t_k z^k \tag{2}$$

is a rational function of z. Then, for sufficiently large N, our algorithm requires $O(N)$ operations to compute the solution to (1). The algorithms of Levinson [5], Bareiss [1], and Zohar [13], which exploit only the Toeplitz structure of T_N, require $O(N^2)$ operations to solve (1).

This problem is motivated by an important special case of (1) arising in linear least squares estimation theory. When $T(z)$ is rational and matrices T_n are symmetric and positive definite for all n, T_N is the covariance matrix of $N + 1$ samples from a wide-sense stationary autoregressive moving-average stochastic process. Trench [10], in a somewhat overlooked paper, outlined an algorithm for solving the linear equations associated with certain estimation problems; his algorithm requires $O(N)$ operations but, as noted in [10], the details are "tedious to write out" except in the *banded case* when $T(z)$ in (2) is a finite series. For the banded case, alternative algorithms can be developed using the result that the (banded) Cholesky factors of T_N can be obtained in $O(N)$ operations; see Morf [6] and Rissanen [7].

Coupling Trench's work on inversion of nonsymmetric banded Toeplitz matrices [11] with Zohar's results [13] leads to an efficient algorithm for solution of general banded Toeplitz systems in $O((p + q)N)$ operations, where $t_i = 0$ for $i > p$ and $i < -q$

Received August 4, 1978; revised January 30, 1979.
AMS (MOS) subject classifications (1970). Primary 65F30; Secondary 15A57.
Key words and phrases. Toeplitz matrix, linear equations.
*Supported in part by the National Science Foundation under grant ENG77-28523.

[3]. In this paper we will generalize the solution algorithms of [3] and [13] to a class of "almost-Toeplitz" systems, leading to an algorithm for solving the special Toeplitz systems defined above.

2. **Solution of the Toeplitz System.** We first present a decomposition of T_n which follows from the rationality of $T(z)$. By separately considering the upper and lower triangular parts of T_n we obtain a representation in terms of banded, triangular Toeplitz matrices. Let

(3) $$T(z) = T_+(z) + T_-(z),$$

where by rationality

(4a) $$T_+(z) = t_0/2 + \sum_{k=1}^{\infty} t_k z^k = c(z)/d(z),$$

(4b) $$T_-(z) = t_0/2 + \sum_{k=1}^{\infty} t_{-k} z^{-k} = \gamma(z)/\delta(z),$$

and $c(z)$, $d(z)$, $\gamma(z)$ and $\delta(z)$ are polynomials given by

(5a) $$c(z) = \sum_{i=0}^{p} c_i z^i, \qquad c_p \neq 0,$$

(5b) $$d(z) = 1 + \sum_{i=1}^{q} d_i z^i, \qquad d_q \neq 0,$$

(5c) $$\gamma(z) = \sum_{i=0}^{r} \gamma_i z^{-i}, \qquad \gamma_r \neq 0,$$

(5d) $$\delta(z) = 1 + \sum_{i=1}^{s} \delta_i z^{-i}, \qquad \delta_s \neq 0.$$

For notational convenience, we write $L_n(\mathbf{w})$ for the lower triangular Toeplitz matrix whose first column is the $(n + 1)$-vector $\mathbf{w}$; $U_n(\mathbf{w})$ is the upper triangular Toeplitz matrix whose first row is $\mathbf{w}'$, where prime denotes transpose. We define $(n + 1)$-vectors of the coefficients of the polynomials in (5) by

(6a) $$\mathbf{c}_n = (c_0 c_1 \cdots c_p\ 0 \cdots 0)',$$

(6b) $$\mathbf{d}_n = (1\ d_1 \cdots d_q\ 0 \cdots 0)',$$

(6c) $$\boldsymbol{\gamma}_n = (\gamma_0 \gamma_1 \cdots \gamma_r\ 0 \cdots 0)',$$

(6d) $$\boldsymbol{\delta}_n = (1\ \delta_1 \cdots \delta_s\ 0 \cdots 0)'.$$

These vectors are suitably truncated when n is less than p, q, r, or s.

The desired representation of T_n follows from the natural isomorphism between the ring of formal power series in z and the ring of semi-infinite (towards the south-east) lower triangular Toeplitz matrices; the coefficient of z^0 is associated with the diagonal element, the coefficient of z with the first subdiagonal element, etc. Polynomials in z correspond to banded matrices, and power series multiplication corresponds to matrix multiplication. Similarly, power series in z^{-1} are naturally isomorphic to semi-infinite (towards the northwest) upper triangular Toeplitz matrices with polynomials in z^{-1} corresponding to banded matrices. In both cases, a power series with a nonzero coefficient of z^0 is invertible in the ring; this corresponds to the fact that an invertible triangular Toeplitz matrix has a triangular Toeplitz inverse. (In the finite case, Traub [8] has given an expression for the (Toeplitz) inverse of a triangular Toeplitz matrix.)

Applying the isomorphisms to the power series equations (4a) and (4b) and taking the first $n + 1$ rows and columns of the corresponding matrix products, starting at the northwest and southeast corners, respectively, and combining the lower and upper triangular Toeplitz matrices gives the desired representation of T_n.

LEMMA 1. *With the notation defined above, for $n \geqslant 0$*

$$T_n = L_n^{-1}(\mathbf{d}_n)L_n(\mathbf{c}_n) + U_n(\boldsymbol{\gamma}_n)U_n^{-1}(\boldsymbol{\delta}_n). \tag{7}$$

Since power series multiplication is commutative, we have chosen a convenient ordering of the factors. Now, treating (7) simply as a matrix identity for the class of Toeplitz matrices considered here, we see that T_n can be reduced to a band matrix by cross multiplication, giving

$$R_n = L_n(\mathbf{d}_n)T_nU_n(\boldsymbol{\delta}_n) = L_n(\mathbf{c}_n)U_n(\boldsymbol{\delta}_n) + L_n(\mathbf{d}_n)U_n(\boldsymbol{\gamma}_n). \tag{8}$$

Since T_n is Toeplitz, $L_n(\mathbf{d}_n)$ is lower triangular, and $U_n(\boldsymbol{\delta}_n)$ is upper triangular, the first equality in (8) shows that for $n \geqslant 1$, R_{n-1} is the n by n principal submatrix of R_n. However, using the second equation in (8) and the Toeplitz structure of the triangular matrices, we obtain the following important structural property of R_n.

LEMMA 2. *For $n \geqslant 1$, the matrix R_n defined in* (8) *satisfies*

$$R_n - \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & R_{n-1} & \\ 0 & & \end{bmatrix} = \mathbf{c}_n\boldsymbol{\delta}_n' + \mathbf{d}_n\boldsymbol{\gamma}_n'. \tag{9}$$

From (6a)–(6d), the nonzero elements of the matrix on the right-hand side of (9) lie in the northwest corner but generally extend beyond the first row and column, so R_n differs from Toeplitz only in its northwest corner. When the degrees $p = r = 0$ so that $\mathbf{c}_n' = \boldsymbol{\gamma}_n' = (1\ 0 \cdots 0)$, R_n is Toeplitz.

A general theory for the inversion of matrices which can be expressed as sums of products of lower and upper triangular Toeplitz matrices is given by Friedlander et al. [4]. Efficient, recursive algorithms for determining the inverses of such matrices are derived, generalizing the Trench algorithm [9], [12]. The particularly simple form of (9) leads to additional simplifications of the approach in [4]. Furthermore, as in [11], the band structure of R_n may be exploited to reduce the computational complexity by an additional order of magnitude.

We propose a three step approach to solving the original system of equations (1):

(a) Compute $L_N(\mathrm{d}_N)y = \tilde{y}$.

(b) Solve $R_N\tilde{x} = \tilde{y}$.

(c) Compute $x = U_N(\delta_N)\tilde{x}$.

Thus in the following, we only describe an algorithm for the second step. Some additional notation will be required. We define for each $n \geqslant 0$ the vector

$$A_n = (a_{n0} \cdots a_{nn})'; \qquad a_{nn} = 1, \tag{10}$$

as the solution to the system

$$R_nA_n = (0 \cdots 0\ \alpha_n)', \tag{11}$$

where the scalars $\{\alpha_n; n \geqslant 0\}$ are defined recursively below. Similarly, vectors P_n and Q_n are defined by

$$R_nP_n = \mathrm{c}_n, \tag{12}$$

$$R_nQ_n = \mathrm{d}_n. \tag{13}$$

Next we let

$$\tilde{\mathrm{y}}_n = (\tilde{y}_0, \ldots, \tilde{y}_n)' \tag{14}$$

and define X_n by

$$R_nX_n = \tilde{\mathrm{y}}_n. \tag{15}$$

The matrix R_n has elements $\{r_{ij}, 0 \leqslant i, j \leqslant n\}$. Now we are ready to derive our major result. We proceed in the usual way, obtaining the quantities $A_{n+1}, P_{n+1}, Q_{n+1}$, and X_{n+1} from A_n, P_n, Q_n, and X_n. Using the structure of R_{n+1}, (9), we find

$$R_{n+1}[0\ A_n']' = (0 \cdots 0\ \alpha_n)' + \mathrm{c}_{n+1}e_n + \mathrm{d}_{n+1}f_n, \tag{16}$$

where the first term is an $(n+1)$-vector and the scalars e_n and f_n are given by

$$e_n = [0\ A_n']\delta_{n+1}, \tag{17a}$$

$$f_n = [0\ A_n']\gamma_{n+1}. \tag{17b}$$

Since R_n is a principal submatrix of R_{n+1}, we obtain

$$R_{n+1}[P_n'\ 0]' = [\mathrm{c}_n'\ g_n]', \tag{18a}$$

$$R_{n+1}[Q_n'\ 0]' = [\mathrm{d}_n'\ h_n]', \tag{18b}$$

$$R_{n+1}[X_n'\ 0]' = [\tilde{\mathrm{y}}_n'\ \Delta_n]', \tag{18}$$

where the scalars g_n, h_n, and Δ_n are given by

$$g_n = [r_{n+1,0} \cdots r_{n+1,n}]P_n, \tag{19a}$$

$$h_n = [r_{n+1,0} \cdots r_{n+1,n}]Q_n, \tag{19b}$$

(19c) $$\Delta_n = [r_{n+1,0} \cdots r_{n+1,n}]X_n.$$

We let $c_{n+1,n+1}$ and $d_{n+1,n+1}$ denote the last elements of the vectors $\mathbf{c}_{n+1}$ and $\mathbf{d}_{n+1}$, respectively. Then P_n and Q_n can be used to update A_n:

(20) $$A_{n+1} = [0\, A_n']' - [P_n'\ 0]'e_n - [Q_n'\ 0]'f_n,$$

(21) $$\alpha_{n+1} = \alpha_n + (c_{n+1,n+1} - g_n)e_n + (d_{n+1,n+1} - h_n)f_n.$$

Now that A_{n+1} is available, it may be used to update the values of X_n, P_n, and Q_n so that (12), (13), and (15) are satisfied. The required steps are given by

(22a) $$X_{n+1} = [X_n'\ 0]' - A_{n+1}(\Delta_n - \tilde{y}_{n+1})/\alpha_{n+1},$$

(22b) $$P_{n+1} = [P_n'\ 0]' - A_{n+1}(g_n - c_{n+1,n+1})/\alpha_{n+1},$$

(22c) $$Q_{n+1} = [Q_n'\ 0]' - A_{n+1}(h_n - d_{n+1,n+1})/\alpha_{n+1}.$$

This completes the updating calculations.

The initial conditions for the algorithm are quite simple:

(23) $$A_0 = 1, \quad \alpha_0 = r_{00}, \quad Q_0 = 1/r_{00}, \quad X_0 = \tilde{y}_0/r_{00}, \quad P_0 = c_0/r_{00},$$

where c_0 is obtained from (5a). In verifying the correctness of this algorithm, only the division by α_{n+1} at each stage requires additional justification. Here the assumption that T_n^{-1} exists for every $0 \leqslant n \leqslant N$ is used. From the first equation of (8), R_n^{-1} exists for every $0 \leqslant n \leqslant N$ because $L_n(\mathbf{d}_n)$ and $U_n(\delta_n)$ are unit triangular matrices. Since R_n is a principal submatrix of R_{n+1}, from Eqs. (10) and (11), $\alpha_{n+1} = \det R_{n+1}/\det R_n$; and this justifies the divisions required in the algorithm.

No use of the banded structure of R_n has yet been made; the algorithm of Theorem 1 applies to any matrix R_N having the structure in (9) and with R_n^{-1} defined for each n. This includes some Toeplitz matrices, for example. With $\gamma_n = \mathbf{c}_n = (1\ 0 \cdots 0)'$, $f_n = 0$ in (17b) for all n and Q_n in (13) is not required so the algorithm reduces to the Levinson-Trench-Zohar algorithm [13]. To exploit the banded nature of R_N, we make a minor assumption that $\rho = \max(p, q)$ is the lower bandwidth of R_N; that is we assume $r_{\rho+j,j} \neq 0$ and $r_{\rho+k,j} = 0$ for $k > j$. This is not a limitation because from (5), (6) and (8)

(24) $$r_{\rho+j,j} = \begin{cases} \gamma_0 d_q = t_0 d_q/2 \neq 0 & \text{for } q > p, \\ c_p \neq 0 & \text{for } p > q, \\ c_p + \gamma_0 d_q & \text{for } p = q, \end{cases}$$

so this condition can be assured by modifying the fraction of the constant term to that which is assigned to $T_-(z)$ in (4b) if necessary. Some observations now follow directly:

(a) Computing (17a) and (17b) requires only the first s and r components of A_n, respectively. Let $\sigma = \max(s, r)$; σ will ordinarily be the upper bandwidth of R_N.

(b) Only the last ρ elements of P_n, Q_n, and X_n are needed to compute (19a)–(19c).

(c) Consequently, in (20), (22b), (22c) only the first σ and last ρ elements of

A_{n+1}, Q_{n+1}, and P_{n+1} need to be computed for n larger than $\rho + \sigma$. In (22a) only the last ρ elements of X_{n+1} need to be computed. When $n + 1$ reaches N, the remaining elements of X_N are computed by back substitution.

We define

$$X_N = (X_{N0} \cdots X_{NN})'. \tag{25}$$

Then for $N - \rho \geqslant j \geqslant 0$ we take

$$X_{Nj} = (1/r_{j+\rho,j})\left(\widetilde{y}_{j+\rho} - \sum_{i=j+1}^{j+\rho+\sigma} r_{j+\rho,i}X_{Ni}\right), \tag{26}$$

where $X_{Ni} = 0$ for $i > N$.

Together with the algorithms of Theorem 1, these modifications provide an algorithm for solving $R_N\widetilde{x} = \widetilde{y}$; as discussed earlier, this is the only nontrivial step in the solution of (1) when T_N is rational. An operation count (of multiplications) shows that solution of (1) requires $(10\rho + 5\sigma + 6)N + O((\rho + \sigma)^2)$ operations. Notice that because R_N is Toeplitz except in its upper $(\rho + 1)$ by $(\sigma + 1)$ corner, all of its elements can be computed in $O((p + \sigma)^2)$ operations. This is still true if $T(z)$ is given in factored form

$$T(z) = (b(z)/d(z))(\beta(z)/\delta(z)) \tag{27}$$

as is often the case in applications such as the linear estimation problems considered by Trench [10].

3. **Discussion.** Our algorithm differs from Trench's [10] in the following way. By extracting triangular Toeplitz factors of known form from T_N, namely $L_N^{-1}(\mathbf{d}_N)$ and $L_N^{-1}(\boldsymbol{\delta}_n)$, we are left with a banded nearly-Toeplitz system to solve. It appears that Trench removes nearly-Toeplitz factors from T_N in order to be left with a banded Toeplitz system to solve. His motivation for so doing was the availability of an efficient algorithm for such systems. We have shown that a very similar algorithm can be used to solve the banded nearly-Toeplitz system.

If the rational power series $T(z)$ converges for some annulus centered on the origin in the complex plane, then subject to some minor assumptions, the existence of T_n^{-1} for $0 \leqslant n \leqslant N$ can be expressed as a constraint on the poles and zeros of $T(z)$. The additional assumptions are that with $T(z) = N(z)/D(z)$ for relatively prime polynomials $N(z)$ and $D(z)$, $N(0) \neq 0$ and $N(z)$ has distinct zeros. Under these circumstances, Day [2] gives an explicit formula for the determinant of T_n in terms of the zeros of $N(z)$ and $D(z)$, and a nonzero determinant is equivalent to the invertibility of T_n.

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