Corrigendum to "What Drives an Aliquot Sequence?"

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Abstract. An aliquot sequence \( n : k, k = 0, 1, 2, \ldots \), is defined by \( n : 0 = n, n : k + 1 = o(n : k) \mod n : k \), and a driver of an aliquot sequence is a number \( 2^A v \) with \( A > 0 \), \( v \) odd, \( v | 2^{A+1} - 1 \) and \( 2^{A-1} | o(v) \). Pollard has noted some errors in a proof in [1] that the drivers comprise the even perfect numbers and a finite set. These are now corrected in a revised proof.

John Pollard has observed two inaccuracies and some obscurities in a proof in [1] for which we wish to substitute the following.

**Theorem 2.** The only drivers are 2, 2\(^3\), 2\(^3\)3. 5, 2\(^5\)3. 7, 2\(^9\)3. 11. 31 and the even perfect numbers.

**Proof.** A driver is \( 2^A v \) with \( A > 0 \), \( v \) odd, \( v | 2^{A+1} - 1 \) and \( 2^{A-1} | o(v) \). If \( v = 1 \), \( 2^{A-1} | 1 \), \( A = 1 \) and we have the "downdriver" 2. If \( v = 2^{A+1} - 1 \) is a Mersenne prime, the driver is an even perfect number. Henceforth, we assume that \( v > 1 \) and that \( 2^{A-1} - 1 \) is composite.

If \( p^a \mid 2^{A+1} - 1 \), \( p \) prime, \( a > 0 \), define the deficiency, \( \delta(p) \), of \( p \) to be \( 2^d/p^a \), where \( 2^d \mid o(p^b) \) and \( p^b \mid v, 0 < b < a \). The product of all the deficiencies is greater than \( 1/4 \), since otherwise

\[
2^{A+1} > 2^{A+1} - 1 = \prod p^a > 4 \prod 2^d
\]

\( 2^{A-1} - 1 \) and \( 2^{A-1} \) would not divide \( \prod o(p^b) = o(v) \).

The power of 2 dividing \( o(p^b) \) depends only on how many factors of the product \( (p + 1)(p^2 + 1)(p^4 + 1) \ldots \) divide \( o(p^b) \), each factor other than \( p + 1 \) contributing a single 2. Hence, \( d = 0 \) if \( b \) is even and \( d = t + k - 1 \) if \( b \) is odd, where \( 2^t \mid p + 1 \), there are \( k \) such factors, and thus \( 2^k \mid b + 1 \). It then follows that

\[
\delta(p) \leq (p + 1)(b + 1)/2p^a \leq (p + 1)(a + 1)/2p^a.
\]

If \( p \) is a Mersenne prime and \( a = b = 1 \), \( \delta(p) = (p + 1)/p > 1 \). Otherwise, \( \delta(p) < 1 \). If \( p \) is not a Mersenne prime, then \( \delta(p) \leq 2/5 \) (\( \delta(5) = 2/5 \) if \( a = b = 1 \)), \( \delta(p) \leq 4/11 \) if \( p > 5 \), and \( \delta(p) \leq 2/25 \) if \( a > 2 \). If we denote by \( \Pi \delta(p) \) the product of the deficiencies of the Mersenne prime factors of \( 2^{A+1} - 1 \), it is not difficult to see that

\[
\Pi \delta(p) \leq \frac{4}{3} \cdot \frac{8}{7} \cdot \frac{32}{31} \cdot \frac{128}{127} \ldots < \frac{4}{3} \cdot \frac{8}{7} \cdot \frac{32}{31} \cdot \frac{64}{63} < \frac{8}{5}.
\]

We now note that \( 2^{A+1} - 1 \) contains at most one non-Mersenne prime factor.

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For having two such prime factors would imply that the product of the deficiencies would be less than

$$\delta(p_1)\delta(p_2) \prod \delta(p) < \frac{2}{5} \cdot \frac{4}{11} \cdot \frac{8}{5} < \frac{1}{4},$$

while $p_1^2 | 2^{A+1} - 1$ is impossible since

$$\delta(p_1) \prod \delta(p) < \frac{2}{25} \cdot \frac{8}{5} < \frac{1}{4}.$$  

For a Mersenne prime $2^a - 1 > 7, a > 1$ would imply $\delta(2^a - 1) < 32/31^2$. But $(32/31^2)(8/5) < 1/4$. For $p = 7, a > 1$ would imply

$$\delta(7) \prod_{p \neq 7} \delta(p) < \frac{8}{7^2} \cdot \frac{7}{5} < \frac{1}{4}.$$  

For $p = 3, a > 3$ would imply $\delta(3) < 8/81$. But $(8/81)(8/5) < 1/4$.

If $p^a = 3^3, 3^3 | 2^{A+1} - 1, 18 | A + 1, 19, 73 | 2^{A+1} - 1$. But neither 19 nor 73 is a Mersenne prime: contradiction. If $p^a = 3^2, 6 | A + 1$. If $A = 5$ we have the driver $2^53^7$, while for odd $A > 5, 2^{A+1} - 1$ contains a non-Mersenne prime factor $p_1$ and

$$\delta(3)\delta(p_1) \prod_{p \neq 3} \delta(p) < \frac{4}{9} \cdot \frac{2}{5} \cdot \frac{6}{5} < \frac{1}{4}.$$  

If $2 < q_1 < \cdots < q_k$, then $2^{A+1} - 1 = (2^{q_1} - 1) \cdots (2^{q_k} - 1)$ is impossible modulo $2^{q_1+1}$, and we have only to consider

$$2^{A+1} - 1 = (2^{q_1} - 1) \cdots (2^{q_k} - 1)(2^c - 1), \quad u \text{ odd, } u > 3.$$  

We know that $u = 3$ or 5, since $u > 7$ would imply

$$\delta(2^c - 1) \prod \delta(p) < \frac{2}{13} \cdot \frac{8}{5} < \frac{1}{4}.$$  

If $c = 1, u = 3$ (since $2.5 - 1$ is not prime), $2u - 1 = 5, 5 | 2^{A+1} - 1, A + 1 = 4k, 15 | 2^{A+1} - 1$. If $A = 3$, we have the drivers $2^33$ and $2^33^5$, while if $A > 7$, there is a prime $p, p | 2^{A+1} - 1, p \equiv 1 \pmod{A + 1}$, giving a second non-Mersenne prime divisor of $2^{A+1} - 1$.

So we have $c > 2, q_1 > 2, u = 3$ or 5 and

$$-1 \equiv (2^{q_1} - 1)(-1) \cdots (-1)(2^c - 1) \pmod{2^c},$$

$$-1 \equiv (-1)^{k-1}(2^{q_1} - 2^c - 1), \quad k \text{ is even and } q_1 = c. \quad \text{Now } 2^{A+1} < 2^{q_1} \cdots 2^{q_k} 2^c < 2^q - 1 \text{ divides } 2^{A+1} - 1 \text{ only if } q | A + 1 \text{ and the } q_i \text{ are distinct primes. Therefore,}$$

$$q_1 \cdots q_k | A + 1 < q_1 + \cdots + q_k + c + \log_2 u < q_1 + \cdots + q_k + q_1 + 3.$$  

If $k > 3$, this would imply $2.3q_3 < q_1q_2q_3 < 2q_1 + q_2 + q_3 + 3 < 4q_3 + 3$, a contradiction. So $k = 2, q_1q_2 < 2q_1 + q_2 + 3, (q_1 - 1)(q_2 - 2) < 5, q_1 = 2 = c$ and $q_2 = 3$ or 5. Only the latter gives a solution; $u = 3$ and $2^93.11.31$ is a driver.