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Corrigendum to "What Drives an Aliquot Sequence?"

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Abstract. An aliquot sequence $n:k$, $k = 0, 1, 2, \ldots$, is defined by $n:0 = n$, $n:k+1 = \sigma(n:k) - n:k$, and a driver of an aliquot sequence is a number $2^A v$ with $A > 0$, v odd, $v \mid 2^{A+1} - 1$ and $2^{A-1} \mid \sigma(v)$. Pollard has noted some errors in a proof in [1] that the drivers comprise the even perfect numbers and a finite set. These are now corrected in a revised proof.

John Pollard has observed two inaccuracies and some obscurities in a proof in [1] for which we wish to substitute the following.

THEOREM 2. *The only drivers are* 2, $2^3 3$, $2^3 3.5$, $2^5 3.7$, $2^9 3.11.31$ *and the even perfect numbers.*

Proof. A *driver* is $2^A v$ with $A > 0$, v odd, $v \mid 2^{A+1} - 1$ and $2^{A-1} \mid \sigma(v)$. If $v = 1$, $2^{A-1} \mid 1$, $A = 1$ and we have the "downdriver" 2. If $v = 2^{A+1} - 1$ is a Mersenne prime, the driver is an even perfect number. Henceforth, we assume that $v > 1$ and that $2^{A+1} - 1$ is composite.

If $p^a \| 2^{A+1} - 1$, p prime, $a > 0$, define the *deficiency*, $\delta(p)$, of p to be $2^d/p^a$, where $2^d \| \sigma(p^b)$ and $p^b \| v$, $0 \leqslant b \leqslant a$. The product of all the deficiencies is greater than $1/4$, since otherwise

$$2^{A+1} > 2^{A+1} - 1 = \prod_p p^a \geqslant 4 \prod_d 2^d,$$

$2^{A-1} > \Pi\, 2^d$ and 2^{A-1} would not divide $\Pi\, \sigma(p^b) = \sigma(v)$.

The power of 2 dividing $\sigma(p^b)$ depends only on how many factors of the product $(p+1)(p^2+1)(p^4+1)\ldots$ divide $\sigma(p^b)$, each factor other than $p+1$ contributing a single 2. Hence, $d = 0$ if b is even and $d = t + k - 1$ if b is odd, where $2^t \| p + 1$, there are k such factors, and thus $2^k \| b + 1$. It then follows that

$$\delta(p) \leqslant (p+1)(b+1)/2p^a \leqslant (p+1)(a+1)/2p^a.$$

If p is a Mersenne prime and $a = b = 1$, $\delta(p) = (p+1)/p > 1$. Otherwise, $\delta(p) < 1$. If p is not a Mersenne prime, then $\delta(p) \leqslant 2/5$ ($\delta(5) = 2/5$ if $a = b = 1$), $\delta(p) \leqslant 4/11$ if $p > 5$, and $\delta(p) \leqslant 2/25$ if $a \geqslant 2$. If we denote by $\Pi\, \delta(p)$ the product of the deficiencies of the Mersenne prime factors of $2^{A+1} - 1$, it is not difficult to see that

$$\prod \delta(p) \leqslant \frac{4}{3} \cdot \frac{8}{7} \cdot \frac{32}{31} \cdot \frac{128}{127} \cdots < \frac{4}{3} \cdot \frac{8}{7} \cdot \frac{32}{31} \cdot \frac{64}{63} < \frac{8}{5}.$$

We now note that $2^{A+1} - 1$ contains at most one non-Mersenne prime factor.

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For having two such prime factors would imply that the product of the deficiencies would be less than

$$\delta(p_1)\delta(p_2)\prod\delta(p) < \frac{2}{5}\cdot\frac{4}{11}\cdot\frac{8}{5} < \frac{1}{4},$$

while $p_1^2 \mid 2^{A+1} - 1$ is impossible since

$$\delta(p_1)\prod\delta(p) < \frac{2}{25}\cdot\frac{8}{5} < \frac{1}{4}.$$

For a Mersenne prime $2^q - 1 > 7$, $a > 1$ would imply $\delta(2^q - 1) \leqslant 32/31^2$. But $(32/31^2)(8/5) < 1/4$. For $p = 7$, $a > 1$ would imply

$$\delta(7)\prod_{p\neq 7}\delta(p) < \frac{8}{7^2}\cdot\frac{7}{5} < \frac{1}{4}.$$

For $p = 3$, $a > 3$ would imply $\delta(3) \leqslant 8/81$. But $(8/81)(8/5) < 1/4$.

If $p^a = 3^3$, $3^3 \mid 2^{A+1} - 1$, $18 \mid A + 1$, $19.73 \mid 2^{A+1} - 1$. But neither 19 nor 73 is a Mersenne prime: contradiction. If $p^a = 3^2$, $6 \mid A + 1$. If $A = 5$ we have the driver $2^5 3.7$, while for odd $A > 5$, $2^{A+1} - 1$ contains a non-Mersenne prime factor p_1 and

$$\delta(3)\delta(p_1)\prod_{p\neq 3}\delta(p) < \frac{4}{9}\cdot\frac{2}{5}\cdot\frac{6}{5} < \frac{1}{4}.$$

If $2 \leqslant q_1 < \cdots < q_k$, then $2^{A+1} - 1 = (2^{q_1} - 1)\cdots(2^{q_k} - 1)$ is impossible modulo 2^{q_1+1}, and we have only to consider

$$2^{A+1} - 1 = (2^{q_1} - 1)\cdots(2^{q_k} - 1)(2^c u - 1), \quad u \text{ odd}, u \geqslant 3.$$

We know that $u = 3$ or 5, since $u \geqslant 7$ would imply

$$\delta(2^c u - 1)\prod\delta(p) < \frac{2}{13}\cdot\frac{8}{5} < \frac{1}{4}.$$

If $c = 1$, $u = 3$ (since $2.5 - 1$ is not prime), $2u - 1 = 5$, $5 \mid 2^{A+1} - 1$, $A + 1 = 4k$, $15 \mid 2^{A+1} - 1$. If $A = 3$, we have the drivers $2^3 3$ and $2^3 3.5$, while if $A \geqslant 7$, there is a prime p, $p \mid 2^{A+1} - 1$, $p \equiv 1 \pmod{A + 1}$, giving a second non-Mersenne prime divisor of $2^{A+1} - 1$.

So we have $c \geqslant 2$, $q_1 \geqslant 2$, $u = 3$ or 5 and

$$-1 \equiv (2^{q_1} - 1)(-1)\cdots(-1)(2^c u - 1) \pmod{2^{\min(c,q_1)+1}},$$

$-1 \equiv (-1)^{k-1}(-2^{q_1} - 2^c u + 1)$, k is even and $q_1 = c$. Now $2^{A+1} < 2^{q_1}\cdots 2^{q_k} 2^c u$ and $2^q - 1$ divides $2^{A+1} - 1$ only if $q \mid A + 1$ and the q_i are distinct primes. Therefore,

$$q_1\cdots q_k \mid A + 1 < q_1 + \cdots + q_k + c + \log_2 u < q_1 + \cdots + q_k + q_1 + 3.$$

If $k \geqslant 3$, this would imply $2.3.q_3 \leqslant q_1 q_2 q_3 < 2q_1 + q_2 + q_3 + 3 < 4q_3 + 3$, a contradiction. So $k = 2$, $q_1 q_2 < 2q_1 + q_2 + 3$, $(q_1 - 1)(q_2 - 2) < 5$, $q_1 = 2 = c$ and $q_2 = 3$ or 5. Only the latter gives a solution; $u = 3$ and $2^9 3.11.31$ is a driver.

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1. RICHARD K. GUY & J. L. SELFRIDGE, "What drives an aliquot sequence?," *Math. Comp.*, v. 29, 1975, pp. 101–107. MR 52 #5542.