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On Stieltjes' Continued Fraction for the Gamma Function

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Abstract. The first forty-one coefficients of a continued fraction for $\ln \Gamma(z) + z - (z - \frac{1}{2}) \ln z - \ln\sqrt{2\pi}$, are given. The computation, based on Wall's algorithm for converting a function's power series representation to a continued fraction representation, was run on the algebraic manipulation system MACSYMA.**

1. Introduction. Recall Stirling's formula for the gamma function:

$$\ln \Gamma(z) = -z + (z - \tfrac{1}{2}) \ln z + \ln\sqrt{2\pi} + J(z)$$

where, for real$(z) > 0$,

$$J(z) = \frac{1}{\pi} \int_0^\infty \ln \frac{1}{1 - e^{-2\pi u}} \cdot \frac{z}{z^2 + u^2} du.$$

Furthermore, asymptotically

$$J(z) = \sum_{p=0}^{\infty} (-1)^p \frac{c_p}{z^{2p+1}}, \tag{1}$$

where

$$c_p = \frac{B_{2p+2}}{(2p + 1)(2p + 2)}, \qquad p = 0, 1, 2, \ldots,$$

and $B_2 = 1/6, B_4 = 1/30, B_6 = 1/42, \ldots$, are the Bernoulli numbers. Henrici [2] refers to $J(z)$ as the *Binet function*, and gives the details for the derivation of the above formulae.

Wall [6, pp. 192–202] gives an algorithm for constructing a continued fraction development of power series such as (1), which we summarize below:

Using the symbolic operation on polynomials of *formal integration* with respect to a variable u, and an infinite sequence of numbers $c_0, c_1, c_2, \ldots$, in which the ith

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power of u is replaced by c_i:

$$\int (k_0 + k_1 u + \cdots + k_n u^n)\, d\phi_c(u) \equiv k_0 c_0 + k_1 c_1 + \cdots + k_n c_n,$$

one computes a_i, $i = 0, 1, 2, \ldots,$ of

$$\sum_{p=0}^{\infty} \frac{c_p}{z^{p+1}} = \cfrac{a_0}{z + \cfrac{a_1}{z + \cfrac{a_2}{z + \cdots}}}$$

by defining the auxiliary polynomials

$$q_{-1}(u) = 0, \qquad q_0(u) = c_0, \tag{2}$$

initializing

$$a_0 = c_0 \tag{3}$$

and using the recurrence for $p = 1, 2, 3, \ldots$:

$$q_p(u) = u q_{p-1}(u) - a_{p-1} q_{p-2}(u), \tag{4}$$

$$a_p = \frac{\int u^p q_p(u)\, d\phi_e(u)}{\prod_{i=0}^{p-1} a_i}, \tag{5}$$

where

$$e_p = \begin{cases} \text{if } p \text{ is even} & c_{p/2}, \\ \text{if } p \text{ is odd} & 0, \end{cases} \qquad p = 1, 2, \ldots. \tag{6}$$

Stieltjes [5, pp. 520–521] gives the first five a_i for $J(z)$, noting that "Le calcul des $[a_i]$ est trés pénible . . . la loi de ces nombres paraît etrêmement compliqué." However, advances of the last decade in the power of algebraic manipulation languages and systems have made it easy to use the recurrence (2)–(6) as the basis for a computer program. The MACSYMA system [3], [4] was used to compute the first forty-one a_i coefficients.

2. The Coefficients. The first seven coefficients computed via MACSYMA agree with those given by Stieltjes, and by Wall [6, p. 365]:

$$a_0 = \frac{1}{12}, \quad a_1 = \frac{1}{30}, \quad a_2 = \frac{53}{210}, \quad a_3 = \frac{195}{371}, \quad a_4 = \frac{22999}{22737},$$

$$a_5 = \frac{29944523}{19733142}, \quad a_6 = \frac{109535241009}{48264275462}.$$

TABLE 1

i	a_i
0	8.333333333333333333333333333333333333333E-2
1	3.333333333333333333333333333333333333333E-2
2	2.523809523809523809523809523809523809524E-1
3	5.256064690026954177897574123989218328841E-1
4	1.011523068126841711747372124730615296653E0
5	1.517473649153287398428491519495476189246E0
6	2.269488974204959960909150672209877584177E0
7	3.009917383259398170073140734207715727374E0
8	4.026887192343901226168875953181442836327E0
9	5.002768080754030051688502412276188574812E0
10	6.283911370815782180072663154951647264278E0
11	7.495919122384033929752354708267505465746E0
12	9.040660234367726699531139360260481749336E0
13	1.048930365450948227718837130459262952210E1
14	1.229719361038620586398943714009191765974E1
15	1.398287695399243018825976065127873008591E1
16	1.605355141670493546971561636500626017835E1
17	1.797660739987027759256947230767155439932E1
18	2.030976202744165374380541472049489689370E1
19	2.247047163993313249551794157150792210900E1
20	2.506584654894597202916340032250630536824E1
21	2.746445182502913360917555898264622267323E1
22	3.032182123167304712688259930640578699449E1
23	3.295853392997298721999406645141208820696E1
24	3.607769893129924264515332090085545233678E1
25	3.895270668231155573454439041048104629916E1
26	4.233349004357695721138185394885609733991E1
27	4.544696085006162101442417573754145108285E1
28	4.908920312901259770816488335027508729245E1
29	5.244128875141533731256985604699610842715E1
30	5.634484534534184353842036594747611354213E1
31	5.993568390716585820785258349275211211013E1
32	6.410042275592035452790661189223791775291E1
33	6.793014078801822118636770274519853581652E1
34	7.235594055521170196968005296323621791075E1
35	7.642465462682968975258509042228752640357E1
36	8.111140323724796548481423098568346097450E1
37	8.541922127641097261458563871734868272699E1
38	9.036681472386410859551357458168337778079E1
39	9.491383710000988795307623129198692745877E1
40	1.001221784639291974889907468344668349799E2

The next few numbers in the sequence are:

$$a_7 = \frac{29404527905795295658}{9769214287853155785}$$

$$a_8 = \frac{45537703042011343221011691470 2}{113084128923675014537885725485}$$

$$a_9 = \frac{2637081256939771900193199294564557877984 9}{527124426791798080196655364914760469754 2}$$

$$a_{10} = \frac{15253749670905480988163889747298599086675385312269783 9}{24274291553105128438297398108902195365373879212227726}$$

$$a_{11} = \frac{100043420063777451042472529806266909090824649341814868347109676190691}{13346384670164266280033479022693768890138348905413621178450736182873}$$

Table 1 is a list of the a_i, $i = 0, \ldots, 40$, rounded to 40 significant digits, computed from the exact rational coefficients using MACSYMA's "bigfloat" facilities (see [1]).

TABLE 2

i	b_i	c_i
0	.4000000000000000000000000000E0	
1	9.335849056603773584905660377 36E0	1.211428571428571428571428571 43E0
2	3.034796060736154932211036373 07E1	7.655948181402079586486848857 80E1
3	6.335287628957229757178749687 70E1	4.959201190175930198012731830 99E2
4	1.083558632771751753342885403 86E2	1.745366077535117757619059316 74E3
5	1.653579659183977933179002143 59E2	4.526920976861447519967723398 16E3
6	2.343595666665265077206341279 78E2	9.758600341887450877062551736 97E3
7	3.153608467725436326269903734 96E2	1.857441569437939445625405361 35E4
8	4.083619057989025567474210640 72E2	3.232437611535013974272020050 54E4
9	5.133628040084974348718802755 04E2	5.257448904270412865298085575 43E4
10	6.303635804877012676600675116 62E2	8.110676073729689939046113370 66E4
11	7.593642619397524121625199890 94E2	1.199191965788779129521326217 80E5
12	9.003648673633295805563725357 36E2	1.712258012653309827013301548 74E5
13	1.053365410723662938709672356 24E3	2.374565789528654189666941851 38E5
14	1.218365902565135220248816872 77E3	3.212575333581539453319148834 08E5
15	1.395366351030092420955275393 28E3	4.254906678348833346879164760 37E5
16	1.584366762527262908571291775 65E3	5.532339854324943018853742608 91E5
17	1.785367142184496700667181720 63E3	7.077814889421611826753306967 83E5
18	1.998367494163907257192798436 44E3	8.926431809334035617413846745 44E5
19	2.223367821886487958583077670 48E3	1.111545063783674123903065444 96E6
20		1.368429139702583147492112697 28E6

Table 2 above is a similar table of coefficients for an alternative representation of $J(z)$

$$J(z) = \cfrac{z}{12z^2 + b_0 - \cfrac{c_1}{12z^2 + b_1 - \cfrac{c_2}{}}}$$

where

$$b_0 = .4, \quad b_i = 12(a_{2i+1} + a_{2i}), \quad c_i = 144(a_{2i}a_{2i-1}), \qquad i = 1, 2, 3, \ldots .$$

3. Details of the Computation. The computation was carried out on the MACSYMA Consortium Decsystem 10 (KL model) computer, located at the Massachusetts Institute of Technology's Laboratory for Computer Science in Cambridge, Massachusetts. The programming was done in the MACSYMA language [4], whose mathematical features allowed a few lines of code to completely specify the computation. The computation of the coefficients a_0 through a_{40}, as well as their 40-digit approximations, took approximately twenty minutes of central processing unit time (including the time spent on list "garbage collection"). The limiting constraint to continuing the computation is that the space requirements of the simple data representations used in the program exhaust available memory (approximately 1 million bytes) after the computation of a_{40}.

A listing of the MACSYMA program and output is available upon request from the author.

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