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A Cardinal Function Method of Solution of the Equation $\Delta u = u - u^3$

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Abstract. The steady-state form of the Klein-Gordon equation is given by

$$(*) \qquad \Delta u = u - u^3, \quad u = u(X), \quad X \in R^3.$$

For solutions which are spherically symmetric, (*) takes the form $\ddot{u} + 2\dot{u}/r = u - u^3$, $u = u(r)$, where r is the distance from the origin in R^3. The function $y = ru$ satisfies

$$(**) \qquad \ddot{y} = y - y^3/r^2.$$

It is known that (**) has solutions $\{y_n\}_{n=0}^{\infty}$, where y_n has exactly n zeros in $(0, \infty)$, and where $y(0) = y(\infty) = 0$.

In this paper, an approximation is obtained for the solution y_0 by minimizing a certain functional over a class of functions of the form

$$\sum_{k=-m}^{m} a_k \operatorname{sinc}\left[\frac{r - kh_m}{h_m}\right].$$

It is shown that the norm of the error is $O(m^{3/8}\exp(-\alpha m^{1/2}))$ as $m \longrightarrow \infty$, where α is positive.

1. Introduction. The Klein-Gordon equation

$$\Box\phi = (u^2 - \lambda\phi^*\phi)\phi, \quad \phi = \phi(X, t), \quad X = (x_1, x_2, x_3), \tag{1.1}$$

is an equation arising in spinor particle theory. Three steady-state forms of (1.1) are

$$\Delta u = u - u^3, \qquad u = u(X), \tag{1.2}$$

$$\ddot{u} + 2\dot{u}/r = u - u^3, \qquad u = u(r), \tag{1.3}$$

and

$$\ddot{y} = y - y^3/r^2, \qquad y = y(r). \tag{1.4}$$

Equation (1.1) reduces to (1.2) once the following substitutions are made: $\phi = v\exp[i(k \cdot Y - wt)]$, $\beta Y = X$, $\beta^2 = u^2 + k^2 - w^2c^{-2}$, $u = \beta^{-1}\lambda^{1/2}v$. For the radially symmetric case, (1.2) reduces to (1.3), and (1.4) is obtained by setting $y = ru$ in (1.3).

Here, we shall be interested in the solution of the problem

$$\ddot{y} = y - y^3/r^2, \qquad y(0) = y(\infty) = 0, \tag{1.5}$$

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which corresponds to

$$\ddot{u} + 2\dot{u}/r = u - u^3, \quad \dot{u}(0) = u(\infty) = 0. \tag{1.6}$$

The problem (1.5) has been considered by many authors [1], [2], [4], [5], [10], [11], [12], [14], [18]. Nehari [10] proved the existence of solutions to (1.5), and Ryder [13] showed that solutions y_n exist, for $0 \leqslant n < \infty$, such that y_n has exactly n zeros in $(0, \infty)$. It is these solutions, y_0 in particular, which this paper concerns.

Problem (1.5) contains obvious numerical difficulties: a nonlinear boundary value problem over an unbounded interval. Initial attempts to use initial value problem techniques are discouraging. Each solution of (1.4) is asymptotic to one of $y = -r$, $y = 0$ or $y = r$, as $r \longrightarrow \infty$, and apparently (see [2]) no solution of (1.5) is stable. In fact, the evidence indicates that for the collection of initial value problems

$$\ddot{y} - y - y^3/r^2, \quad y(0) = 0, \quad y'(0) = \alpha, \tag{1.7}$$

there is a countable collection $\{\alpha_j\}$ with no cluster point in R such that if $\alpha \notin \{\alpha_j\}$, then $|y/r| \longrightarrow 1$ as $r \longrightarrow \infty$.

Standard proofs of Galerkin methods fail for (1.5), since the derivative of the Lagrangian of (1.4) is neither positive nor negative at a solution to (1.5). However, Chauvette and Stenger [2] do succeed in a difficult proof of their application of the Bubnov-Galerkin method.

Here we consider a variational approach based on the approximation theory developed in [8]. The results of [17] suggests that the rate of convergence, $O(\exp(-cm^{1/2}))$ where m is the number of unknowns, is best possible. Several proofs, which have been deleted in the exposition, are available on request.

2. Important Results. In what follows, we shall make use of the functionals $J(a, b)$ and $G(a, b)$ defined by

$$J(a, b)y = \int_a^b (\dot{y}^2 + y^2)\, dr, \tag{2.1}$$

and

$$G(a, b)y = \int_a^b y^4 r^{-2}\, dr. \tag{2.2}$$

For simplicity, we write J and G for $J(0, \infty)$ and $G(0, \infty)$, respectively. We also consider the set $S(a, b)$ consisting of all functions $y \in H_0^1(a, b)$ such that $y(c_y) > 0$ for some $c_y \in (a, b)$ and

$$J(a, b)y = G(a, b)y. \tag{2.3}$$

THEOREM 2.1 [13]. *Let $n \geqslant 0$, and let S_n be the collection of all functions y for which the following is satisfied: there exist $\{r_k\}_{k=1}^n \subset (0, \infty)$ ($\{r_k\}_{k=1}^n = \phi$ if $n = 0$) such that $0 = r_0 < r_1 < \cdots < r_n < r_{n+1} = \infty$, and $(-1)^k y \in S(r_k, r_{k+1})$, for $0 \leqslant k \leqslant n$. Then J is minimized over S_n by the solution y_n of problem (1.5). Further, for each $n \geqslant 0$, $y_n \in C^1[0, \infty)$, and y_0 is the unique minimum of J over S_0.*

Coffman [1] provides further results on uniqueness and existence, including the following two lemmas.

LEMMA 2.2. *For each* $\beta > 0$, (1.4) *has a unique solution* $y = y(r, \beta)$ *which satisfies* $y \in C^2(0, \infty)$ *and* $\lim_{r\to 0} r^{-1}y = \beta$. *Further, there is at most one* $\beta_0 \in (0, \infty)$ *for which* $y(r, \beta_0) > 0$ *on* $(0, \infty)$ *and* $\lim_{r\to\infty} y(r, \beta_0) = 0$.

LEMMA 2.3. *Let* $n \geqslant 0$, *and let* b *be the first zero of* y_n *in* $(0, \infty]$. *Then* $\delta \equiv 0$ *is the unique solution of*

$$\delta'' - \delta + 3r^{-2}y_n\delta = 0, \quad \delta(0) = \delta(b) = 0. \tag{2.4}$$

The theorem which follows is a consequence of Lemma 2.3. Its proof, which employs the Sturm-Liouville theory and Courant's maximum-minimum principle, is deleted.

THEOREM 2.4. *Let* b *and* y_n *be as in Lemma* 2.3. *Then there exists a constant* $\kappa \in (0, 2)$ *such that for all* $\epsilon \in H_0^1(0, b)$,

$$6\int_0^b y_n^2\epsilon^2 r^{-2}\,dr < (2-\kappa)\int_0^b (\dot\epsilon^2 + \epsilon^2)\,dr. \tag{2.5}$$

3. **Whittaker's Cardinal Function.** Let f be defined on the real line R, and let h be a positive constant. Then the function $C(f, h)$ is defined on the complex plane $\mathbf{C}$ by

$$C(f, h)(z) = \sum_{k=-\infty}^{\infty} f(kh)S(k, h)(z), \tag{3.1}$$

where the function $S(k, h)$ is given by

$$S(k, h)(z) = \left[\pi\,\frac{z-kh}{h}\right]^{-1} \sin\left[\pi\,\frac{z-kh}{h}\right]. \tag{3.2}$$

The function $C(f, h)$ was introduced by E. T. Whittaker [21], and has been studied further by several authors (see [8], [9], [16], [22]). In this paper, we approximate the solution y_0 of (1.5) by a function of the form

$$C_m(f, h) = \sum_{k=-m}^{m} f(kh)S(k,h). \tag{3.3}$$

The foundation for such approximations is given in [8].

The results given in [2] may be used to show that for $\alpha = 1$, y_0 satisfies the conditions on f in the theorem which follows.

THEOREM 3.1 [8]. *Let* $d, \alpha > 0$ *and assume* f *satisfies*

(i) f *is analytic on* $\mathcal{D}_d \equiv \{x + iy\colon |y| < d\}$;

(ii) $\int_{-d}^{d} |f(x+iy)|\,dy \longrightarrow 0$ *as* $x \longrightarrow \pm\infty$;

(iii) $\lim_{y\to d^-} \int_R |f(x \pm iy)|^2\,dx < \infty$;

(iv) *there is a positive constant* C *such that* $|f(x)| \leqslant Ce^{-\alpha|x|}$, *for all* $x \in R$.

Let γ *be positive and for each positive integer* m *let* $h_m = \gamma m^{-1/2}$, *and let*

$\epsilon_m = C(f, h_m) - f$. *Then there exist positive constants* $K_{1,2}$ *and* K_∞, *independent of* m, *such that*

$$\|\epsilon_m(f, h_m)\|_2,\ m^{-3/4}\|\epsilon_m(f, h_m)\|_{1,2} \leqslant K_{1,2}\exp(-\beta m^{1/2}), \tag{3.4}$$

and

$$\|\epsilon_m(f, h_m)\|_\infty,\ m\|\epsilon_m(f, h_m)'\|_\infty \leqslant K_\infty m^{1/2}\exp(-\beta m^{1/2}), \tag{3.5}$$

where $\beta = \min(\pi d/\gamma, \alpha\gamma)$.

It is important to recognize the implications of (3.4) and (3.5). For example, (3.4) says that

$$\|\epsilon_m(f, h_m)\|_{1,2} = O(h_m^{-3/2}e^{-\gamma/h_m}) \quad \text{as } h_m \longrightarrow 0\ (m \longrightarrow \infty),$$

and so for every positive integer p,

$$\|\epsilon_m(f, h_m)\|_{1,2} = o(h_m^p) \quad \text{as } h_m \longrightarrow 0.$$

We complete this section with the presentation of two results which are needed in the computations and in the proof of validity of the approximation scheme. The equalities (3.6)–(3.8) are given in [8], and (3.10) is obtained quite easily from the theory in [8].

THEOREM 3.2. *Let* h *be positive, let* m *be a positive integer and let*

$$g = \sum_{k=-m}^{m} a_k S(k, h). \tag{3.6}$$

Then

$$\int_R g^2(x)\,dx = h\sum_{k=-m}^{m} a_k^2, \tag{3.7}$$

and

$$\int_R [g'(x)]^2\,dx = h^{-1}\sum_{k=-m}^{m}\sum_{j=-m}^{m} c_{k,j}a_k a_j, \tag{3.8}$$

where

$$c_{k,j} = \begin{cases} \pi^2/3 & \text{if } k = j, \\ 2(-1)^{k+j}(j-k)^{-2} & \text{if } k \neq j. \end{cases} \tag{3.9}$$

If $a_0 = 0$, *then*

$$\int_R g^4(x)x^{-2}\,dx = 2h^{-1}\sum_{k=-\infty}^{\infty} k^{-2}g^4(kh/2). \tag{3.10}$$

THEOREM 3.3. *Let* f, m, h_m *and* β *be as in Theorem* 3.1, *and assume* $f(0) = 0$. *Then there exists a positive constant* L, *independent of* m, *such that*

(3.11) $$\left|\int_R f^4(x)x^{-2}\,dx - \int_R C_m(f, h_m)x^{-2}\,dx\right| \leqslant L\exp(-\beta m^{1/2}).$$

Proof. We have

$$\left|\int_R f^4(x)x^{-2}\,dx - \int_R C_m(f, h_m)x^{-2}\,dx\right|$$

$$\leqslant \left[\sup_{x\in R\setminus\{0\}} \frac{f^2(x) + C_m(f, h_m)^2(x)}{x^2}\right]\int_R |f^2(x) - C_m(f, h_m)^2|\,dx$$

$$\leqslant \left[\sup_{x\in R\setminus\{0\}} \frac{f^2(x) + C_m(f, h_m)^2(x)}{x^2}\right]\|f + C_m(f, h_m)\|_2\|f - C_m(f, h_m)\|_2.$$

Now, $C_m(f, h_m)(0) = f(0) = 0$, and so the supremum above is finite for each m since each of f and $C_m(f, h_m)$ is analytic at $z = 0$. Further, (3.4) and (3.5) imply that the supremum and $\|f + C_m(f, h_m)\|_2$ are bounded as functions of m. So, (3.11) follows from (3.4).

4. Approximating y_0. The following definitions are suggested since y_0 is an odd function satisfying Theorems 3.1 and 3.3.

Definition 4.1. For each positive integer m, let $h_m = \gamma m^{-1/2}$, where γ is a fixed positive number. Then $\mathcal{W}_m$ is the collection of all functions w of the form

(4.1) $$w(r) = \sum_{k=-m}^{m} a_k \operatorname{sinc}\left[\frac{r - kh_m}{h_m}\right],$$

which satisfy

(4.2) $$a_{-k} = -a_k \quad \text{for } 0 \leqslant k \leqslant m,$$

(4.3) $$Jw = Gw, \quad \text{where } J, G \text{ are as defined in Section 2,}$$

(4.4) $$a_k \geqslant 0 \quad \text{for } 1 \leqslant k \leqslant m, \quad \text{and} \quad a_j \neq 0 \quad \text{for some } j.$$

Definition 4.2. For each positive integer m, let

(4.5) $$z_m = C_m(y_0, h_m),$$

(4.6) $$\tilde{z}_m = [Jz_m/Gz_m]^{1/2}z_m,$$

and let w_m be an element of $\mathcal{W}_m$ for which

(4.7) $$Jw_m = \min_{w\in\mathcal{W}_m} Jw.$$

The function w_m is our mth approximation to y_0.

We note that $\tilde{z}_m \in \mathcal{W}_m$, and so $\mathcal{W}_m \neq \emptyset$. Thus, obtaining w_m is equivalent to minimizing a continuous function on a nonempty compact subset of R^m. Further, with S_0 defined as in Theorem 2.1, we have $\mathcal{W}_m \subset S_0$, and so $J\tilde{z}_m > Jy_0$.

LEMMA 4.1. *There is a positive constant K such that for all $m \geqslant 1$,*

(4.8) $$0 < Jw_m - Jy_0 \leqslant J\tilde{z}_m - Jy_0 \leqslant Km^{3/4}\exp(-\beta m^{1/2}),$$

where $\beta = \min(\pi d/\gamma, \gamma)$ and d is as in Theorem 3.1.

Proof. We have $w_m \in S$, and $w_m((m+1)h) = 0$. Thus, $w_m \not\equiv y_0$ and so $0 < Jw_m - Jy_0$. Since $\tilde{z}_m \in \mathcal{W}_m$, $Jw_m - Jy_0 \leqslant J\tilde{z}_m - Jy_0$ follows from (4.7).

To verify the last inequality in (4.8), we first note that

$$(4.9)\quad J\tilde{z}_m - Jy_0 = |\|\tilde{z}_m\|_{1,2}^2 - \|y_0\|_{1,2}^2| \leqslant (\|\tilde{z}_m\|_{1,2} + \|y_0\|_{1,2})\|\tilde{z}_m - y_0\|_{1,2}.$$

Also,

$$(4.10)\quad \|\tilde{z}_m - y_0\|_{1,2} \leqslant \|\tilde{z}_m - z_m\|_{1,2} + \|z_m - y_0\|_{1,2}.$$

We recall from (3.4) that

$$(4.11)\quad \|z_m - y_0\|_{1,2} \leqslant K_1 m^{3/4}\exp(-\beta m^{1/2}),$$

for some positive constant K_1. Further,

$$\begin{aligned}\|\tilde{z}_m - z_m\|_{1,2} &= |[Jz_m/Gz_m]^{1/2} - 1|\|z_m\|_{1,2}\\ &= \|z_m\|_{1,2}|(Jz_m)^{1/2} - (Gz_m)^{1/2}|(Gz_m)^{-1/2}\\ &\leqslant \|z_m\|_{1,2}(Gz_m)^{-1/2}\{|\|z_m\|_{1,2} - \|y_0\|_{1,2}| + |(Gy_0)^{1/2} - (Gz_m)^{1/2}|\}\\ &\leqslant \|z_m\|_{1,2}(Gz_m)^{-1/2}\{\|z_m - y_0\|_{1,2} + |(Gy_0)^{1/2} - (Gz_m)^{1/2}|\}.\end{aligned}$$

We know the sequence $\{Gz_m\}_{m=1}^{\infty}$ is bounded, since Theorem 3.3 implies $Gz_m \longrightarrow Gy_0$ as $m \longrightarrow \infty$. Furthermore, Theorem 3.3 implies

$$(4.12)\quad |(Gz_m)^{1/2} - (Gy_0)^{1/2}| \leqslant K_2 \exp(-\beta m^{1/2}),$$

where K_2 is some positive constant. Thus, there exists a constant K_3 such that

$$(4.13)\quad \|\tilde{z}_m - z_m\|_{1,2} \leqslant K_3 m^{3/4} \exp(-\beta m^{1/2}).$$

Using (4.11) and (4.13) in (4.10), we arrive at (4.8), since $\|\tilde{z}_m\|_{1,2} \longrightarrow \|y_0\|_{1,2}$ in (4.9).

Our next result is now obvious. Its nontrivial proof, which draws on results in functional analysis, is deleted.

THEOREM 4.2. *The sequence $\{w_m\}_{m=1}^{\infty}$ converges in $H_0^1(0, \infty)$ to y_0. Hence, $\|w_m - y_0\|_{1,2}$, $\|w_m - y_0\|_{\infty} \longrightarrow 0$ as $m \longrightarrow \infty$.*

Next, we wish to determine the rate of convergence of w_m to y_0. We note that on S_0, $F \equiv J$, where

$$(4.14)\quad F = J^2/G.$$

Also, since $\{y \in H_0^1(0, \infty): y = \beta y_0, \beta \text{ a scalar}\}$ is a closed subspace of $H_0^1(0, \infty)$, Theorem 4.2 implies that for m large enough there exists a nonzero scalar β_m and a function $\epsilon_m \in H_0^1(0, \infty)$ such that

$$(4.15)\quad w_m = \beta_m(y_0 + \epsilon_m),$$

and

$$(4.16)\quad \langle y_0, \epsilon_m\rangle = 0,$$

where $\langle\cdot, *\rangle$ is the inner product in H_0^1.

Since y_0 and ϵ_m satisfy the properties of y_n and ϵ of Theorem 2.4, we have

$$6\int_0^\infty y_0^2\epsilon_m^2 r^{-2}\,dr < (2-\kappa)\|\epsilon_m\|_{1,2}^2, \tag{4.17}$$

where κ is as in (2.5). The proof of the lemma which follows is deleted.

LEMMA 4.3. *Let* $\{\epsilon_m\}$ *be as in* (4.15)–(4.16). *Then*

$$\int_0^\infty y_0\epsilon_m^3 r^{-2}\,dr, \int_0^\infty \epsilon_m^4 r^{-2}\,dr = O(\|\epsilon_m\|_{1,2}^3) \quad \textit{as } m \longrightarrow \infty. \tag{4.18}$$

THEOREM 4.4. *There exists a positive constant C independent of m such that*

$$\|w_m - y_0\|_{1,2} \leqslant Cm^{3/8}\exp(-.5\beta m^{1/2}) \quad \textit{for all } m \geqslant 1, \tag{4.19}$$

where β *is as in Lemma* 4.1.

Proof. For ϵ_m as in (4.15),

$$\begin{aligned} G(y_0+\epsilon_m) &= \int_0^\infty (y_0+\epsilon_m)^4 r^{-2}\,dr \\ &= \int_0^\infty y_0^4 r^{-2}\,dr + 4\int_0^\infty y_0^3\epsilon_m r^{-2}\,dr + 6\int_0^\infty y_0^2\epsilon_m^2 r^{-2}\,dr \\ &\quad + 4\int_0^\infty y_0\epsilon_m^3 r^{-2}\,dr + \int_0^\infty \epsilon_m^4 r^{-2}\,dr. \end{aligned}$$

Hence, applying (4.17), and (4.18), we see that for large m,

$$G(y_0+\epsilon_m) < Jy_0 + 4\langle y_0, \epsilon_m\rangle + (2-\kappa)\|\epsilon_m\|_{1,2} + O(\|\epsilon_m\|_{1,2}^3).$$

So evaluation of F gives us

$$\begin{aligned} F(y_0+\epsilon_m) &= [J(y_0+\epsilon_m)]^2/G(y_0+\epsilon_m) \\ &> [(Jy_0)^2 + 2\|\epsilon_m\|_{1,2}^2 Jy_0 + \|\epsilon_m\|_{1,2}^4][Jy_0 + (2-\kappa)\|\epsilon_m\|_{1,2}^2 \\ &\qquad + O(\|\epsilon_m\|_{1,2}^3)]^{-1} \\ &= Jy_0 + \kappa\|\epsilon_m\|_{1,2}^2 + O(\|\epsilon_m\|_{1,2}^3), \end{aligned}$$

for m large enough. Thus, we have for all large m,

$$Jw_m - Jy_0 = F(y_0+\epsilon_m) - Jy_0 > \kappa\|\epsilon_m\|_{1,2}^2 + O(\|\epsilon_m\|_{1,2}^3).$$

From this, it follows

$$\|\epsilon_m\|_{1,2}^2 = O(Jw_m - Jy_0) \quad \text{as } m \longrightarrow \infty. \tag{4.20}$$

Now, we set $\delta_m = \beta_m\epsilon_m$ and note that $\beta_m \longrightarrow 1$ as $m \longrightarrow \infty$. Then

$$\|\delta_m\|_{1,2}^2 = O(Jw_m - Jy_0) \quad \text{as } m \longrightarrow \infty. \tag{4.21}$$

Next, we have

$$Jw_m - Jy_0 = (\beta_m^2 - 1)\|y_0\|_{1,2}^2 + \|\delta_m\|_{1,2}^2,$$

and so $|\beta_m - 1| = O(Jw_m - Jy_0)$, as $m \longrightarrow \infty$. Hence,

$$(\beta_m - 1)^2 = O(Jw_m - Jy_0) \quad \text{as } m \longrightarrow \infty. \tag{4.22}$$

Finally, since

$$\|w_m - y_0\|_{1,2}^2 = (\beta_m - 1)^2 \|y_0\|_{1,2}^2 + \|\delta_m\|_{1,2}^2,$$

we obtain

$$\|w_m - y_0\|_{1,2}^2 = O(Jw_m - Jy_0) \quad \text{as } m \longrightarrow \infty. \tag{4.23}$$

Applying Lemma 4.1, we obtain (4.19).

5. **Numerical Results.** Define J_m and G_m on R_n by

$$J_m(a_1, a_2, \ldots, a_m) = 2h_m \sum_{k=1}^{m} a_k^2 + h_m^{-1} \sum_{k=1}^{m} \sum_{j=1}^{m} s_{k,j} a_k a_j \tag{5.1}$$

and

$$G_m(a_1, a_2, \ldots, a_m) = 2h_m^{-1} \sum_{k=1}^{\infty} k^{-2} w^4(kh_m/2), \tag{5.2}$$

where

$$s_{k,j} = \begin{cases} 2\pi^2/3 - k^{-2} & \text{if } k = j, \\ 4(-1)^{k+j}[(k-j)^{-2} - (k+j)^{-2}] & \text{if } k \neq j, \end{cases} \tag{5.3}$$

and

$$w(r) = \sum_{k=1}^{m} a_k [S(k, h)(r) - S(-k, h)(r)]. \tag{5.4}$$

Then Theorem 3.2 implies that

$$w_m = \sum_{k=1}^{m} a_k^* [S(k, h) - S(-k, h)], \tag{5.5}$$

where $(a_1^*, a_2^*, \ldots, a_m^*)$ is that element of

$$A_m \equiv \{(a_1, a_2, \ldots, a_m): a_k \geqslant 0, 1 \leqslant k \leqslant m; a_k \neq 0 \text{ for some } k; J_m = G_m\}$$

for which J_m is minimized.

This minimization problem was solved using Newton's method with a Lagrange multiplier. The error bound of Theorem 4.4 depends on $\beta = \min(\pi d/\gamma, \alpha\gamma)$. In Section 3 it was noted that $\alpha = 1$, and the proof of Theorem 3.1 (which was deleted) indicates that $d \geqslant \sqrt{6}/[1 + \dot{y}_0(0)]$. In [2], $\dot{y}_0(0)$ is estimated to be 4.2. Taking $\alpha = 1$, $\dot{y}_0(0) = 4$ and $\gamma = 1.37$, we obtain $d = .59$ and $\beta = 1.37$. These estimates yield the following values of $m^{3/8}\exp(-.5\beta m^{1/2})$:

m	$m^{3/8}\exp(-.69m^{1/2})$
50	.0330
60	.0222
70	.0153

The approximation w_m was obtained for each of the three cases listed above. The results for the case $m = 70$ are given in Table 1. The accuracy was estimated by comparing successive approximations for $m = 50$ and $m = 60$. For $m = 60$, C was then estimated to be .00217 in the bound $Cm^{3/8}\exp(-.69m^{1/2})$; this value of C was used to obtain the "estimated accuracy" .000033 for $m = 70$.

TABLE 1
Results for $m = 70$, $h_{70} = 1.37(70)^{-1/2}$

J = 6.01518220

$\hat{w}_{70}(0)$ = 4.33676333

Estimated accuracy: 3.3×10^{-5}

k	a_k	k	a_k
1	.06576128	36	.00746708
2	1.07191767	37	.00634406
3	1.21711402	38	.00538107
4	1.19252416	39	.00457287
5	1.09031867	40	.00387762
6	.96315625	41	.00329627
7	.83589519	42	.00279404
8	.71855787	43	.00237613
9	.61450145	44	.00201305
10	.52392830	45	.00171289
11	.44596881	46	.00145012
12	.37919875	47	.00123478
13	.32225491	48	.00104434
14	.27373763	49	.00089008
15	.23249396	50	.00075178
16	.19741569	51	.00064148
17	.16763458	52	.00054078
18	.14231951	53	.00046211
19	.12084058	54	.00038849
20	.10258413	55	.00032354
21	.08710050	56	.00027843
22	.07393793	57	.00023878
23	.06277867	58	.00019868
24	.05328942	59	.00017068
25	.04524783	60	.00014061
26	.03840675	61	.00012087
27	.03261239	62	.00009792
28	.02768017	63	.00008397
29	.02350548	64	.00006599
30	.01994918	65	.00005600
31	.01694179	66	.00004137
32	.01437724	67	.00003394
33	.01221109	68	.00002135
34	.01036140	69	.00001545
35	.00880150	70	.00000363

6. Concluding Remarks. We cannot close without reiterating that the order of convergence obtained in Section 4 is the best that can be obtained with known approximation techniques. If one were to use splines or other standard approximating functions, the rate of convergence would be $O(m^{-p})$, for some integer p. Our rate of convergence is faster than $O(m^{-p})$, for every positive p. For example, if 8-place accuracy were required, over 11,000 unknowns might be needed on the interval [0, 10] if p were 3; at most 500 unknowns would be needed in our approach.

The method presented here for y_0 could be perturbed slightly to obtain y_1. However, for $n \geqslant 1$ it would be more natural to use an approach based on the results given in [8]. That is, on each interval (r_j, r_{j+1}) of Theorem 2.1, use the composition of a cardinal function and a conformal mapping to approximate y_n.

Finally, Eq. (1.4) is one of many equations of the form $\ddot{y} - y + yF(y^2, r) = 0$. In [13], Ryder gives results for the more general problem. Our method may be employed once analyticity, uniqueness and exponential decay of solutions has been verified.

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1. C. V. COFFMAN, "Uniqueness of the ground state solution for $\Delta u - u + u^3 = 0$ and a variational characterization of other solutions," *Arch. Rational Mech. Anal.*, v. 46, 1972, pp. 81–92.

2. J. CHAUVETTE & F. STENGER, "The approximate solution of the nonlinear equation $\Delta u = u - u^3$," *J. Math. Anal. Appl.*, v. 51. 1975, pp. 229–242.

3. L. COLLATZ, *Functional Analysis and Numerical Mathematics*, Academic Press, New York, 1966.

4. G. W. DAREWICH & H. SCHIFF, "Particle solutions of a class of nonlinear field equations," *J. Mathematical Phys.*, v. 8, 1967, pp. 1479–1482.

5. R. FINKELSTEIN, R. LE LEVIER & M. RUDERMAN, "Nonlinear spinor fields," *Phys. Rev.*, v. 83, 1950, pp. 326–332.

6. E. HILLE, *Analytic Function Theory*, Vol. 2, Blaisdell, Waltham, Mass., 1962.

7. IMSL Library 2, Edition 4 (Fortran V), International Mathematical and Statistical Libraries, Inc., Houston, Texas, 1974.

8. L. LUNDIN & F. STENGER, "Cardinal type approximation of a function and its derivatives," *SIAM J. Math. Anal.*, v. 10, 1979, pp. 139–160.

9. J. McNAMEE, F. STENGER & E. L. WHITNEY, "Whittaker's cardinal function in retrospect," *Math. Comp.*, v. 25, 1963, pp. 141–154.

10. Z. NEHARI, "On a nonlinear differential equation arising in nuclear physics," *Proc. Roy. Irish Acad. Sect. A*, v. 62, 1963, pp. 117–135.

11. P. D. ROBINSON, "Extremum principles for the equation $\nabla \phi = \phi - \phi^3$," *J. Mathematical Phys.*, v. 12, 1971, pp. 23–28.

12. W. RUDIN, *Real and Complex Analysis*, McGraw-Hill, New York, 1966.

13. G. H. RYDER, "Boundary value problems for a class of nonlinear differential equations," *Pacific J. Math.*, v. 22, 1967, pp. 477–503.

14. H. SCHIFF, "A classical theory of bosons," *Proc. Roy. Soc. Ser. A*, v. 269, 1962, pp. 277–286.

15. F. STENGER, *Convergence and Error of the Bubnov-Galerkin Method*, SIAM Conf. on Ordinary Differential Equations, Fall 1972.

16. F. STENGER, "Approximation via Whittaker's cardinal function," *J. Approximation Theory*, v. 17, 1976, pp. 222–240.

17. F. STENGER, "Optimal convergence of minimum norm approximations in H_p." (Submitted.)

18. J. L. SYNGE, "On a certain nonlinear differential equation," *Proc. Roy. Irish Acad. Sect. A*, v. 62, 1961, pp. 17–41.

19. P. G. CIARLET, M. H. SCHULTZ & R. S. VARGA, "Numerical methods of high-order accuracy for nonlinear problems, III. Eigenvalue problems," *Numer. Math.*, v. 12, 1968, pp. 120–133.

20. K. YOSIDA, *Functional Analysis*, Springer-Verlag, New York, 1966.

21. E. T. WHITTAKER, "On the functions which are represented by the expansions of the interpolation theory," *Proc. Roy. Soc. Edinburgh Sect. A*, v. 35, 1915, pp. 181–194.

22. J. M. WHITTAKER, "On the cardinal function of interpolation theory," *Proc. Edinburgh Math. Soc. Ser.* I (2), 1927, pp. 41–46.