REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

The numbers in brackets are assigned according to the revised indexing system printed in Volume 28, Number 128, October 1974, pages 1191–1194.

10[3.10, 3.20].—J. J. DONGARRA, J. R. BUNCH, C. B. MOLER & G. W. STEWART, *LINPACK User's Guide*, SIAM, Philadelphia, Pa., 1979. Price \$14.00, \$11.20 to SIAM members.

This book is more than a guide for users of LINPACK subroutines. LINPACK is a package of well-structured, portable, Fortran subroutines that solve linear systems of equations and some related problems. The related problems include ordinary least-squares, inverses, determinants, and the condition estimator for the solution of linear systems of equations. The software, and thus this guide for users, addresses computations for dense matrices.

The preface presents the background and the goals of LINPACK, a three-year project sponsored jointly by the National Science Foundation and the Department of Energy. The introduction gives an overview of the linear systems problems in LINPACK, the structure of the subroutines, the naming conventions and the software design. General numerical properties having to do with rounding errors, accuracy of computed results, and detection of singularity are discussed. The introduction contains an especially good discussion on the need and strategies for scaling; a section that should be studied by originators of linear systems problems.

The book contains eleven chapters, references, and four appendices. Each chapter addresses a particular decomposition, and most of the chapters contain background numerical information, algorithmic details, directions for use of the subroutines, and programming and performance details. LINPACK includes subroutines for LU decomposition, QR orthogonal factorizations by Householder transformations, Cholesky decomposition, and the singular value decomposition. Matrices of special structure are treated separately and include positive definite symmetric, symmetric indefinite, positive definite band, triangular, and tridiagonal matrices. A separate chapter treats updating QR and Cholesky decompositions. The notation used in the chapters on QR and the singular value decompositions is taken from the statistical model, $y = X\beta + e$. The notation in other chapters follows that of standard numerical linear algebra, $Ax = b$.

The four appendices include the Basic Linear Algebra Subprograms, timing data, program listings for the real single precision subroutines in LINPACK, and the BLA listings. LINPACK, itself, contains single and double precision real and complex versions of the software, which can be obtained on tape from the National Energy Software Center or from International Mathematical and Statistical Libraries, Inc.

The subroutines in LINPACK are well-documented. The LINPACK *User's Guide* will be useful to the naive or sophisticated user of LINPACK. Furthermore, the book will be extremely useful to teachers and students of linear algebra. Students in computer

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science can gain a lot of information about numerical computation and mathematical software by reading this book. Potential users of software for linear systems are well-advised to use the LINPACK subroutines rather than attempting to write software for linear systems in specific applications.

We offer a few criticisms of LINPACK and the *User's Guide.* Figures 1–7 are awkward to read because of the condensed form in which they are printed. The subroutines in LINPACK use the Basic Linear Algebra Subroutines, and the overhead of their subroutine calls degrades the computation time for matrices of reasonably low order. The preprocessor, TAMPR, used to generate type and precision for LINPACK could have generated code without the Basic Linear Algebra Subroutines. Such code would be useful in practice.

Finally, strict portability requires that machine-dependent constants not be used. Practice shows, however, that the skewness of the range of arithmetic (given side-effects of underflow) on many machines presents problems in orthogonal factorizations used in iteratively reweighted and nonlinear least-squares. Perhaps subsequent editions of *LINPACK User's Guide* will indicate how LINPACK subroutines can be modified to avoid such side-effects in applications subsystems.

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11[2.05.5].—J. GILEWICZ, *Approximants de Padé,* Lecture Notes in Math., Vol. 667, Springer-Verlag, New York, xiv + 511 pp. Price $21.00.

The importance of Padé approximants in various fields of mathematics, physics and chemistry is now well known. The literature on the subject is quite extensive and, although several books recently appeared, it was necessary to write a synthesis including topics connected with the subject such as totally monotonic functions, to make the definitions precise and to study in detail some questions such as the block structure of the Padé table and algorithms for computing its elements.

Gilewicz's book is a very dense one which covers all these problems and also many others. It is certainly the most extensive treatise on the subject now available (a book by G. A. Baker, Jr. and P. R. Graves-Morris will appear in the near future).

The first four chapters deal with the mathematical concepts which are useful for the remainder of the book. Chapter 1 is an introduction to sequences and series, extrapolation processes, convergence acceleration, difference operators, Hankel determinants and the c-table. Chapter 2 is a complete study of totally monotonic functions and sequences. It contains very nice results, obtained with M. Froissart, on necessary and sufficient conditions for a function to be totally monotonic and on infinite interpolation. A correct proof of a theorem by S. Bernstein is given. The connection with totally monotonic sequences is extensively studied, which leads to new results for such sequences. Chapter 3 deals with the properties of two classes of functions for which the convergence of sequences of Padé approximants has been proved and also with the famous moment problem. Chapter 4 gives the necessary account of continued fractions for a better understanding of Padé approximants.

Chapters 5, 6 and 7 form the heart of the book since they deal, respectively, with the algebraic theory of Padé approximants, their convergence, and their practical computation. The reader is assumed to be familiar with basic complex analysis.

The beginning of Chapter 5 is a review and a synthesis on the various definitions of Padé approximants, Padé forms, Padé quotients, . . . depending on the solubility of the linear system defining the approximants, that is to say, on the block structure of the Padé and c-tables. The connection with continued fractions is given and the theory of compatible transformations is developed. The chapter ends on the subject of lacunary series. In the whole chapter, the emphasis is placed on the possible presence of square blocks in the Padé table, and all the theorems take this possibility into account. This is one of the important features of the book.

Chapter 6 deals with the convergence theory of sequences of Padé approximants. The first section presents orthogonal polynomials as extracted from Padé approximants for some special series. I think that the reverse presentation would have been better; it is possible to construct general orthogonal polynomials from a given sequence of moments and then to use their properties to derive results on Padé approximants. This would lead to a simplification and a better understanding of many results, and would have been much more general. Section 2 deals with Stieltjes series for which a complete convergence theory exists. Section 3 is devoted to the various convergence theorems in the general case. The fourth section treats the case of functions with a natural boundary for which very curious numerical results had been obtained by M. Froissart. The last section derives new convergence results for the Shanks transformation of sequences, (or the epsilon algorithm of Wynn), from its connection with Padé approximants.

Chapter 7 deals with the computational algorithms for sequences of Padé approximants. The various algorithms are carefully studied as to the number of arithmetic operations and the possible presence of blocks. In particular, the computation of the c-table in the presence of blocks is completely treated.

The last part of the book is devoted to some numerical problems posed by the practical use of Padé approximants. The examples treated are from physical situations which were personally encountered by the author.

Chapter 8 deals with the problem of finding the best Padé approximant (in the sense of the maximum norm over some region of the complex plane) in a finite set of approximants. Gilewicz proposes an empirical method, the ρ-method, for handling this problem and uses it to explain some results in fluid mechanics. Some other empirical methods based on valleys in the c-table are also presented.

Chapter 9 is devoted to the application of the ρ-method to the approximate evaluation of the missing coefficients of a Fourier series and to the numerical computation of a conformal mapping, arising in a diffraction problem for electromagnetic waves. Extensive numerical results are given as well as a complete discussion of them.

The entire book is well presented and easy to read (specially for French reading people). It ends with a good bibliography and a detailed index.

One can object that the book contains too many results for a beginner in the subject, who will certainly be unable to find his way without a guide. A good idea would have been to indicate the sections which could be skipped during a first reading.

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12[9.10].—R. J. EVANS, *Table of Cyclotomic Numbers of Order Twenty-Four,* 98 pp. of computer printout, deposited in the UMT file, 1979.

Let $P = 24F + 1$ be a prime with fixed primitive root g. For integers I, J (mod 24), define the cyclotomic number (I, J) of order 24 to be the number of integers N (mod P) for which N/g^I and $(1 + N)/g^J$ are (nonzero) 24th power residues (mod P). Let $\beta = \exp(2\pi i/24)$ and fix a character χ (mod P) of order 24 such that $\chi(g) = \beta$. For characters λ, ψ (mod P), define the Jacobi sums

$$J(\lambda, \psi) = \sum_{N \pmod P} \lambda(N)\psi(1 - N), \qquad K(\lambda) = \lambda(4)J(\lambda, \lambda).$$

It is known [1] that there exist integers X, Y, A, B, C, D, U, V such that

$$K(\chi^6) = -X + 2Yi \quad (P = X^2 + 4Y^2,\ X \equiv 1 \pmod 4)),$$
$$K(\chi^4) = -A + Bi\sqrt{3} \quad (P = A^2 + 3B^2,\ A \equiv 1 \pmod 6)),$$
$$K(\chi^3) = -C + Di\sqrt{2} \quad (P = C^2 + 2D^2,\ C \equiv 1 \pmod 4)),$$

and

$$K(\chi) = U + 2Vi\sqrt{6} \quad (P = U^2 + 24V^2,\ U \equiv -C \pmod 3)).$$

Since $J(\chi, \chi^2) \in \mathbf{Z}[\beta]$, there are integers $D_0, D_1, \ldots, D_7$, such that

$$J(\chi, \chi^2) = \sum_{i=0}^{7} D_i\beta^i.$$

Each number $576(I, J)$ can be expressed as an integer linear combination of $P, 1, X, Y, A, B, C, D, U, V, D_0, \ldots, D_7$. Such formulas are presented in tables of cyclotomic numbers of order twenty-four here deposited in the UMT files. (For references to tables of cyclotomic numbers of other orders, see [2].) The tables were constructed by evaluating the right side of [4, (2.7)] using relations between Jacobi sums of orders dividing 24, particularly the relations in [3, p. 496].

There are 48 tables, corresponding to the 48 choices of the 4-tuples (F', V', Z, T) with $F' = F \pmod 2 \in \{0, 1\}$, $V' = V \pmod 2 \in \{0, 1\}$, $Z = \text{ind } 2 \pmod{12} \in \{0, 2, 4, 6\}$, $T = \text{ind } 3 \pmod 8 \in \{0, 2, 4\}$, where the indices of 2 and 3 are taken with respect to g. Each of the 48 tables contains 109 formulas. The first formula in Table 1, for example, is:

$$576(0, 0) = P - 71 - 114X - 80A - 24C + 96U + 192D_0 + 96D_4.$$

To obtain a formula for a number (I, J) not listed in a table, one makes use of the facts [3, p. 489] that $(I, J) = (-I, J - I)$ and

$$(I, J) = \begin{cases} (J, I), & \text{if } 2 \mid F, \\ (J + 12, I + 12), & \text{if } 2 \nmid F. \end{cases}$$

For example, to find the formula for 576(20, 4) in Table 1 ($F' = V' = Z = T = 0$), one observes that $(20, 4) = (4, 8) = (8, 4)$. To obtain a formula for (I, J) where $Z = Z_0$, $T = T_0$ with either

(i) $Z_0 \in \{8, 10\}, T_0 = 6$,

(ii) $Z_0 \in \{8, 10\}, T_0 \in \{0, 2, 4\}$,

or

(iii) $Z_0 \in \{0, 2, 4, 6\}, T_0 = 6$,

one uses the fact that the transformation $g \longrightarrow g^M$ yields the transformation $(I, J) \longrightarrow (MI, MJ)$, where $M = -1, 5, 7$ in cases (i), (ii), (iii), respectively. Illustrations are now given for a fixed choice of F' and V'.

Case (i). To find $576(I, J)$, one would find $576(-I, -J)$ in the table for $Z = 12 - Z_0$, $T = 2$ modified so that the heading

$$P\ 1\ X\ Y\ A\ B\ C\ D\ U\ V\ D_0 \cdots D_7$$

is replaced by

$$P\ 1\ X\ {-Y}\ A\ {-B}\ C\ {-D}\ U\ {-V}\ E_0 \cdots E_7,$$

where

$$E_0 = D_0 + D_4, \quad E_1 = D_3, \quad E_2 = D_2, \quad E_3 = D_1, \quad E_4 = -D_4,$$
$$E_5 = -D_3 - D_7, \quad E_6 = -D_2 - D_6, \quad E_7 = -D_1 - D_5.$$

For example, to determine $\alpha = 576(0, 23)$ when $F' = V' = 1, Z_0 = 8, T_0 = 6$, note that the table for $F' = V' = 1, Z = 4, T = 2$ (Table 44) gives

$$576(0, 1) = P + 1 - 26X + 48Y - 4A - 4C - 8U$$
$$+ 96V + 8D_0 - 40D_2 - 8D_4 - 8D_6 - 24D_7.$$

Thus,

$$\alpha = P + 1 - 26X - 48Y - 4A - 4C - 8U - 96V$$
$$+ 8(D_0 + D_4) - 40D_2 + 8D_4 + 8(D_2 + D_6) + 24(D_1 + D_5).$$

Case (ii). To find $576(I, J)$, one would find $576(5I, 5J)$ in the table for $Z = 12 - Z_0$, $T = T_0$ modified so that its heading is replaced by

$$P\ 1\ X\ Y\ A\ {-B}\ C\ {-D}\ U\ V\ E_0 \cdots E_7$$

where

$$E_0 = D_0 + D_4, \quad E_1 = D_5, \quad E_2 = -D_2, \quad E_3 = -D_3 - D_7,$$
$$E_4 = -D_4, \quad E_5 = D_1, \quad E_6 = D_2 + D_6, \quad E_7 = D_7.$$

Case (iii). To find $576(I, J)$, one would find $576(7I, 7J)$ in the table for $Z = Z_0$, $T_0 = 2$ modified so that its heading is replaced by

$$P\ 1\ X\ -Y\ A\ B\ C\ -D\ U\ V\ E_0 \cdots E_7$$

where

$$E_0 = D_0, \quad E_1 = D_3 + D_7, \quad E_2 = -D_2, \quad E_3 = -D_5, \quad E_4 = D_4,$$
$$E_5 = -D_3, \quad E_6 = -D_6, \quad E_7 = D_1 + D_5.$$

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