Five-Diagonal Sixth Order Methods for Two-Point Boundary Value Problems Involving Fourth Order Differential Equations

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Abstract. We present a sixth order finite difference method for the two-point boundary value problem $y^{(4)} + f(x, y) = 0$, $y(a) = A_0$, $y(b) = B_0$, $y'(a) = A_1$, $y'(b) = B_1$. In the case of linear differential equations, our difference scheme leads to five-diagonal linear systems.

Consider the two-point boundary value problem

\begin{equation}
    y^{(4)} + f(x, y) = 0, \quad y(a) = A_0, \quad y(b) = B_0, \quad y'(a) = A_1, \quad y'(b) = B_1.
\end{equation}

In a recent paper Usmani [1] has given finite difference methods of orders two, four and six for the boundary value problem (1) in the case when $f(x, y)$ is linear. While his methods of orders two and four can be easily adapted for nonlinear $f(x, y)$ and lead to five-diagonal linear systems when $f(x, y)$ is linear, the sixth order method given by Usmani leads to a nine-diagonal linear system. In the following we present a sixth order method for the nonlinear boundary value problem (1) which, in the case of linear $f(x, y)$, leads to five-diagonal linear systems.

At the grid points $x_k$, $k = 2(1)N - 1$, where $x_k = a + kh$, $k = 0(1)N + 1$, $N \geq 5$, the differential equation in (1) can be discretized by

\begin{equation}
    \delta^4 y_k + h^4 [2a_0 f_k + a_1 (f_{k+1} + f_{k-1}) + a_2 (f_{k+2} + f_{k-2})] + T_k(h) = 0,
\end{equation}

where we have set $y_k = y(x_k)$ and $f_k = f(x_k, y_k)$.

In order that $T_k(h) = O(h^{10})$, we find that

$$(a_0, a_1, a_2) = (1/720) (237, 124, -1).$$

Note that $y_0 = A_0$, $y_{N+1} = B_0$. Let $y'_k = y'(x_k)$, $k = 0, N + 1$. The discretizations for the boundary conditions $y'_0 = A_1$, $y'_{N+1} = B_1$ can be obtained following Chawla and Katti [2]. Now, for the boundary conditions $y'_0 = A_1$ and $y'_{N+1} = B_1$, consider the discretizations

\begin{equation}
    \sum_{k=0}^{3} b_k y_k + c h y'_k + h^4 \left( \sum_{k=0}^{3} d_k f_k + \sum_{k=0}^{1} d'_k f_{k+1/2} \right) + T_1(h) = 0,
\end{equation}

Received August 20, 1979.

1980 Mathematics Subject Classification. Primary 65L10.

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0025-5718/80/0000-0158/$01.75

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and
\[
\sum_{k=0}^{3} b_k y_{N+1-k} - ch y_N' + h^4 \left( \sum_{k=0}^{3} d_k f_{N+1-k} + \sum_{k=0}^{1} d_k^* \bar{f}_{N-k+1/2} \right) + T_N(h) = 0,
\]
(3b)

where we have set \( \bar{f}_{k+1/2} = f(x_{k+1/2}, \bar{y}_{k+1/2}), x_{k+1/2} = x_k + h/2, k = 0(1)N, \) and where
\[
\bar{y}_{k+1/2} = \sum_{m=0}^{3} u_{k,m} y_{m} + h^4 \sum_{m=0}^{3} w_{k,m} f_{m}, \quad k = 0, 1,
\]
(4a)

and
\[
\bar{y}_{N-k+1/2} = \sum_{m=0}^{3} u_{k,m} y_{N+1-m} + h^4 \sum_{m=0}^{3} w_{k,m} f_{N+1-m}, \quad k = 0, 1.
\]
(4b)

In order that \( T_1(h) \) and \( T_N(h) = O(h^{10}) \), we find the following values for the parameters in (3) and (4):

\[
(b_0, b_1, b_2, b_3) = \left( -\frac{11}{2}, 9, -\frac{9}{2}, 1 \right), \quad c = -3,
\]

\[
(d_0, d_1, d_2, d_3) = \left( \frac{1}{4200} \right) (20, 1335, 460, 7),
\]

\[
(d_0^*, d_1^*) = \left( \frac{1}{4200} \right) (488, 840),
\]

\[
(u_{0,0}, u_{0,1}, u_{0,2}, u_{0,3}) = \left( \frac{1}{16} \right) (5, 15, -5, 1),
\]

\[
(w_{0,0}, w_{0,1}, w_{0,2}, w_{0,3}) = \left( \frac{1}{256} \right) (-3, 13, 0, 0),
\]

\[
(u_{1,0}, u_{1,1}, u_{1,2}, u_{1,3}) = \left( \frac{1}{16} \right) (-1, 9, 9, -1),
\]

\[
(w_{1,0}, w_{1,1}, w_{1,2}, w_{1,3}) = \left( \frac{3}{256} \right) (0, -1, -1, 0),
\]

While determining these parameters, we have set the free parameters \( w_{0,2}, w_{0,3}, w_{1,0}, \)
\( w_{1,1} = 0 \) for simplicity, and we have fixed \( b_3 = 1 \) for the reason that when the discretization is written in a matrix form the inverse of the coefficient matrix \( (D) \) would be available in [1]. We also note that then

\[
T_k(h) = -\frac{h^{10}}{3024} y_k^{(10)} + O(h^{12}), \quad k = 2(1)N - 1,
\]

and

\[
T_1(h) = h^{10} \left[ \frac{1}{38400} y_0^{(10)} - \left\{ \frac{(61F_{1/2} + 9F_{3/2})}{46080} \right\} y_0^{(6)} \right] + O(h^{11}),
\]
\[ T_N(h) = h^{10} \left[ \frac{1}{38400} y_{N+1}^{(10)} - \left( \frac{61F_{N+1/2} + 9F_{N-1/2}}{46080} \right) y_{N+1}^{(6)} \right] + O(h^{11}), \]

where \( F = \frac{\partial f}{\partial y} \).

Now, a method based on the discretizations (3a), (2) and (3b) can be expressed in the matrix form as

\[ (5) \quad D\tilde{Y} + G(\tilde{Y}) = 0. \]

For the derivation of the above difference scheme, we have assumed that \( y \in C^1[a, b] \), and for \( x \in [a, b], -\infty < y < \infty \), \( f \) is six times continuously differentiable and that \( \frac{\partial f}{\partial y} \) exists and is continuous.

Following arguments given in Usmani [1], we can show that if \( E = \tilde{Y} - Y \), then in the uniform norm, for sufficiently small \( h \),

\[ \|E\| = O(h^6), \]

provided \( U^* < 2592/(7K(b - a)^4) \), where

\[ K = \frac{181}{180} \quad \text{and} \quad U^* = \max \left| \frac{\partial f}{\partial y} \right|. \]

**Acknowledgement.** I would like to thank Professor M. M. Chawla for his helpful comments.

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