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On Determination of Best-Possible Constants in Integral Inequalities Involving Derivatives

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Abstract. This paper is concerned with the numerical approximation of the best possible constants $\gamma_{n,k}$ in the inequality

$$\|F^{(k)}\|^2 \leqslant \gamma_{n,k}^{-1}\{\|F\|^2 + \|F^{(n)}\|^2\},$$

where

$$\|F\|^2 = \int_0^\infty |F(x)|^2\,dx.$$

A list of all constants $\gamma_{n,k}$ for $n \leqslant 10$ is given.

1. Introduction. This paper utilizes the algorithm given in [1] to numerically approximate the best possible constants $\gamma_{n,k}$, $1 \leqslant k < n$, for $n \leqslant 10$ in the inequality:

$$\|F^{(k)}\|^2 \leqslant \gamma_{n,k}^{-1}\{\|F\|^2 + \|F^{(n)}\|^2\}, \tag{1}$$

where $\|\cdot\|$ denotes the $L_2[0, \infty)$ norm. The function F has a locally absolutely continuous $(n-1)$st derivative. The inequality (1) is equivalent to

$$\|F^{(k)}\| \leqslant M_{n,k}\|F\|^{(n-k)/n}\|F^{(n)}\|^{k/n}, \tag{2}$$

where

$$M_{n,k}^2 = \gamma_{n,k}^{-1}\left(\frac{n-k}{k}\right)^{k/n} + \left(\frac{k}{n-k}\right)^{(n-k)/n}; \tag{3}$$

see [1].

Interest in inequalities (1) and (2) increased because of their close connection with problems of best approximation of the differentiation operator by bounded operators; see [2], [3], [4], [5], and with the problem of best approximation of one class of functions by another; see [4], [6], [7].

In the next section we shall give lower and upper bounds for the best possible constants $\gamma_{n,k}$ and $M_{n,k}$ for $n \leqslant 10$.

2. Numerical Results. In this section the best possible constants $\gamma_{n,k}$ and $M_{n,k}$ are listed.

$$\gamma_{21} = 1, \quad \text{see [1].}$$

$$\gamma_{31} = \gamma_{32} = \sqrt[3]{3 - 2\sqrt{2}} = .555669, \quad \text{see [1].}$$

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In [1], γ_{41} is characterized as the smallest positive zero of the polynomial $Z^8 - 6Z^4 - 8Z^2 + 1$, and γ_{42} is the smallest positive zero of the polynomial $Z^4 - 2Z^2 - 4Z + 1$. Using Müller's method [8], we obtain $\gamma_{41} = \gamma_{43} = .339246$, $\gamma_{42} = .225270$.

Remark. It is known, see [1], that

$$\gamma_{n,n-k} = \gamma_{n,k} \quad \text{for all } n, k. \tag{4}$$

Using the algorithm in [1], one has the following table of lower and upper bounds on $\gamma_{n,k}$ for $2 \leqslant n \leqslant 10$ and $1 \leqslant k \leqslant [n/2]$. For other values of k, use (4).

TABLE 1

$\gamma_{n,k}$ *for* $2 \leqslant n \leqslant 10$, $1 \leqslant k \leqslant [n/2]$

n\k	1	2	3	4	5
2	1.				
3	.555669				
4	.339246	.225271			
5	(.225837, .2258375)	(.102266, .102268)			
6	(.160328, .160338)	(.051986, .05199)	(.0361167, .0361177)		
7	(.11936, .11943)	(.028924, .02895)	(.014698, .0147)		
8	(.09128, .09129)	(.0172, .01723)	(.0068112, .00681124)	(.005014, .0050145)	
9	(.07593, .07594)	(.010795, .0108)	(.00345, .0036)	(.00193, .001938)	
10	(.0479, .048)	(.0068, .007)	(.0014163, .0014165)	(.000681505, .00068151)	(.000642565,.00064257)

Using (3) and the values listed in Table 1, one has the following table of lower and upper bounds on $M_{n,k}$ for $2 \leqslant n \leqslant 10$ and $1 \leqslant k \leqslant [n/2]$. For other values of k, use $M_{n,n-k} = M_{n,k}$ for all n, k.

TABLE 2

$M_{n,k}$ *for* $2 \leqslant n \leqslant 10$, $1 \leqslant k \leqslant [n/2]$

n\k	1	2	3	4	5
2	1.41421				
3	2.07005				
4	2.27432	2.97963			
5	(2.70248, 2.70249)	(4.37797, 4.37801)			
6	(3.12838, 3.12848)	(6.02917, 6.02940)	(7.44141, 7.44151)		
7	(3.55221, 3.55325)	(7.92662, 7.93019)	(11.60467,11.60546)		
8	(3.99579, 3.99601)	(10.09176,10.10056)	(16.86722,16.86727)	(19.97106,19.97206)	
9	(4.32029, 4.32057)	(12.54043,12.54333)	(23.07295,23.23717)	(32.02543,32.09173)	
10	(5.36995, 5.37555)	(15.35013,15.57423)	(36.06112,36.06367)	(53.62984,53.63004)	(55.78980,55.79001)

Remarks. 1. The lower and upper bounds for each n and k are given in parentheses and separated by a comma, for example, $.11936 \leqslant \gamma_{7,1} \leqslant .11943$.

2. The number $M_{4,2}$ in Table 2 agrees with that obtained by Bradley and Everitt [7].

3. The number $M_{6,3}$ in this table agrees with a result of Dawson and Everitt [9].

Conjecture. For fixed k the $\gamma_{n,k}$ are decreasing functions of n. For fixed n the $\gamma_{n,k}$ are decreasing functions of k up to $k = [n/2]$.

Thus the initial value of $\gamma_{n,k}$ may be taken in the interval

$$I_{n,k}^* = (0, \gamma_{n-1,k}) \quad \text{for } n > 2$$

rather than the interval suggested by Kupcov, namely

$$I_{n,k} = (0, g_{n,k}),$$

where

$$g_{n,k} = \frac{n}{k^{k/n}(n-k)^{(n-k)/n}}.$$

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