The Exact Degree of Precision of Generalized Gauss-Kronrod Integration Rules

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Abstract. It is shown that the Kronrod extension to the n-point Gauss integration rule, with respect to the weight function \((1 - x^2)^{\mu - \frac{1}{2}}\), \(0 < \mu < 2\), \(\mu \neq 1\), is of exact precision \(3n + 1\) for \(n\) even and \(3n + 2\) for \(n\) odd. Similarly, for the \((n+1)\)-point Lobatto rule, with \(-\frac{1}{2} < \mu < 1\), \(\mu \neq 0\), the exact precision is \(3n\) for \(n\) odd and \(3n + 1\) for \(n\) even.

1. Introduction. In this paper we shall consider the Kronrod extensions (KE) to the Gauss-Gegenbauer integration rules (GGIR) and the Lobatto-Gegenbauer rules (LGIR). The Gegenbauer polynomials, \(C_n^\mu(x)\), \(\mu > -\frac{1}{2}\), are those polynomials which are orthogonal with respect to the weight function \(w(x; \mu) \equiv (1 - x^2)^{\mu - \frac{1}{2}}\) and have the following normalization [4, p. 174]

\[
\int_{-1}^{1} w(x; \mu) C_n^\mu(x) C_m^\mu(x) \, dx = \delta_{nm} h_{nm}^\mu,
\]

where

\[
h_{nm}^\mu = \pi^{1/2} \Gamma(n + 2\mu) \Gamma(\mu + \frac{1}{2}) / ((n + \mu)n! \Gamma(\mu) \Gamma(2\mu)),
\]

which implies that \(C_n^\mu(x) = k_{nm} x^n + \ldots\), where

\[
k_{nm} = 2^n \Gamma(n + \mu) / (n! \Gamma(\mu)).
\]

\(C_n^\mu(x)\) is even (odd) if \(n\) is even (odd). Special cases of \(C_n^\mu(x)\), perhaps with a different normalization, are \(T_n(x)\), the Chebyshev polynomials of the first kind \((\mu = 0)\), \(P_n(x)\), the Legendre polynomials \((\mu = \frac{1}{2})\), and \(U_n(x)\), the Chebyshev polynomials of the second kind \((\mu = 1)\).

The \(n\)-point GGIR is given by

\[
I(f) \equiv \int_{-1}^{1} w(x; \mu) f(x) \, dx = \sum_{i=1}^{n} w_i f(x_i) + c_{nm} M_2n(f),
\]

where we have omitted the dependence of \(w_i\) and \(x_i\) on \(\mu\) and \(n\), \(x_i\) are the zeros of \(C_n^\mu(x)\),

\[
c_{nm} = 2^{2n} h_{nm} / k_{nm}^2,
\]

and \(M_2(f)\) is defined to be equal to \(f^{(0)}(\xi)/2\) for some \(\xi \in (-1, 1)\). The corresponding LGIR has \(n + 1\) points and is given by

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where the $x_i$ are the zeros of $(1 - x^2)C_n^{u+1}(x)$, and
\begin{equation}
\bar{c}_{n\mu} = -\frac{2^{2n}R_{n-1,\mu+1}}{k_{n-1,\mu+1}^2} = -4c_{n-1,\mu+1}.
\end{equation}

Since the weights of the integration rules considered do not play a part in the discussion, we shall not treat them here except to remark that Monegato [9], [10] has shown that the weights $u_i$ in (8) below are positive for $0 \leq \mu \leq 1$ and the $v_i$, for $0 \leq \mu \leq 2$.

The KE of the $n$-point GGIR is given by
\begin{equation}
If = \sum_{i=1}^{n+1} w_i f(x_i) + \sum_{i=1}^{n+1} v_i f(y_i) + E_{p_n}(f),
\end{equation}
where $E_s(f) = 0$ if $f$ is a polynomial of degree $< s$ and $p_n = 2[(3n + 3)/2]$. The $y_i$ are the zeros of a certain polynomial $E_{n+1,\mu}(x)$ which we shall study in the next section. For the moment we state a result of Szegö [16] that for $0 \leq \mu \leq 2$, the $y_i$ are real, lie in $[-1, 1]$, and are separated by the $x_i$. (For $\mu \neq 0$, the $y_i$ lie in $(-1, 1)$.)

The corresponding KE of the $n$-point LGIR is given by
\begin{equation}
If = \sum_{i=1}^{n+1} \bar{u}_i f(x_i) + \sum_{i=1}^{n+1} \bar{v}_i f(y_i) + E_{q_n}(f),
\end{equation}
where $q_n = 2[(3n + 2)/2]$, and the $\bar{y}_i$ are the zeros of $E_{n,\mu+1}(x)$. Thus, taking into account that $\mu > -\frac{1}{2}$, we see that practical KE of KE's exist for $0 \leq \mu \leq 2$ and KE's, for $-\frac{1}{2} < \mu \leq 1$.

The first one to discover a KE was Kronrod [7] who dealt with the case $\mu = \frac{1}{2}$, the Gauss-Legendre or standard Gauss rule. Subsequently, Patterson [13], Piessens and Branders [14], and Monegato [11] improved on Kronrod's original work and extended his results to the usual Lobatto case ($\mu = \frac{1}{2}$). Barrucand [2] was the first to point out the connection between the KE's and the Szegö polynomials $E_{n+1,\mu}(x)$. KE's to other integration rules are discussed by Baratella [1], Kahaner and Monegato [5], Monegato [9], [12], and Ramskiil [15].

In the entire literature on this subject, it is stated that the KE's have error terms which vanish for polynomials of degree less than $p_n$ (Gauss) or $q_n$ (Lobatto), and in Kronrod's tables, he gives the error in the integration of $x^{p_n}$ by the KE with $\mu = \frac{1}{2}$. However, nowhere is it proved that these KE's are of exact degree $p_n - 1$ or $q_n - 1$, as the case may be, that is, that there exists a polynomial of degree $p_n$ or $q_n$ for which the corresponding KE is not exact. Indeed, such a statement is not true for all $\mu$. Thus, as Monegato [9] points out, the KE of the $n$-point GGIR with $\mu = 0$, the first Gauss-Chebyshev rule, is exact for polynomials of degree $\leq 4n - 1$ and in fact is identical with the KE of the corresponding $(n + 1)$-point LGIR, being the $(2n + 1)$-point LGIR, the first Lobatto-Chebyshev rule. Furthermore, the KE of the $n$-point GGIR with $\mu = 1$, the second Gauss-Chebyshev rule, is exact for polynomials of degree...
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< 4n + 1 and, in fact, is identical with the (2n + 1)-point GGIR. In the present work, we shall show that, except for $\mu = 0, 1$ in the GGIR case and $\mu = 0$ in the LGIR case, we have the result that the exact precision of the KEGLGR is $p_n - 1$ while that of the KELGIR is $q_n - 1$. Furthermore, if these rules are of simplex type, i.e., if we can express the error term in the form $K_{p_n} f(\eta_n) (\xi)$ or $K_{p_n} f(\eta_n) (\xi)$, which we have not been able to prove, then we have the following result:

\[
I f = \sum_{i=1}^{n} u_i f(x_i) + \sum_{i=1}^{n+1} v_i f(y_i) + d_{n\mu} c_{n\mu} M_{p_n}(f),
\]

\[
I f = \sum_{i=1}^{n+1} \tilde{u}_i f(\tilde{x}_i) + \sum_{i=1}^{n} \tilde{v}_i f(\tilde{y}_i) + d_{n-1,\mu+1} c_{n\mu} M_{q_n}(f),
\]

where $d_{n\mu}$ is easily computable and does not vanish for $0 < \mu \leq 2$, $\mu \neq 1$, and all $n \geq 2$. For $\mu = 2$ we have the explicit expression

\[
d_{n\mu} = \begin{cases} 
\dfrac{2}{n+3} \left( \dfrac{n+1}{n+3} \right)^m, & \text{n even,} \\
4(n+2)(n+1)^{m-1}/(n+3)^{m+1}, & \text{n odd,}
\end{cases}
\]

where $m = \lfloor (n+1)/2 \rfloor$.

2. The Szegö Polynomials $E_{n+1,\mu}$. We give here the main results of Szegö with some minor modification of his notation and refer to [16] for details. See also Davis and Rabinowitz [3, pp, 82–89] with Monegato [11].

The Gegenbauer function of the second kind, $Q_n^{\mu}(z)$, defined by

\[
Q_n^{\mu}(z) = \frac{\Gamma(2\mu)}{2\Gamma(\mu + \frac{1}{2})} \int_{-1}^{1} w(t; \mu) \frac{C_n^{\mu}(t)}{z - t} dt
\]

\[
= \frac{\Gamma(2\mu)}{2\Gamma(\mu + \frac{1}{2})} z^{-n-1} \sum_{i=0}^{\infty} \beta_i z^{-2i},
\]

where

\[
\beta_i = \int_{-1}^{1} w(t; \mu) C_n^{\mu}(t) t^{i+2i} dt, \quad i = 0, 1, \ldots,
\]

is analytic in the entire complex plane with a slit on the closed interval $[-1, 1]$. Hence

\[
\frac{1}{Q_n^{\mu}(z)} = z^{n+1} \sum_{i=0}^{\infty} \gamma_i z^{-2i} = E_{n+1,\mu}(z) + \delta_1 z^{-1} + \delta_2 z^{-2} + \ldots,
\]

defining the polynomial $E_{n+1,\mu}(z)$ which is even (odd) for $n$ odd (even). Thus,

\[
Q_n^{\mu}(z) E_{n+1,\mu}(z) = 1 + b_1 z^{-n-2} + b_2 z^{-n-3} + \ldots,
\]

and by the argument given in [16] or [3]

\[
Q_n^{\mu}(z) E_{n+1,\mu}(z) = 1 + \sum_{i=0}^{n} c_i Q_n^{\mu}(z),
\]

for certain constants $c_0, \ldots, c_n$ depending on $\mu$ and $n$. Since $Q_n^{\mu}(z)$ is an odd (even)
function if \( n \) is even (odd), we have that \( Q^\mu_n(x)E_{n+1,\mu}(x) \) is always an odd function which implies that \( c_0 = 0 \) if \( n \) is odd.

Now, the functions of the second kind satisfy the following relations:

1. \[
\lim_{\epsilon \to 0} (Q^\mu_n(x + i\epsilon) - Q^\mu_n(x - i\epsilon)) = -i\pi \frac{\Gamma(2\mu)}{\Gamma(\mu + \frac{1}{2})} w(x; \mu) C^\mu_n(x),
\]

2. \[
\lim_{\epsilon \to 0} (Q^\mu_n(x + i\epsilon) + Q^\mu_n(x - i\epsilon)) = 2\tilde{Q}^\mu_n(x),
\]

where \( \tilde{Q}^\mu_n(x) \) is defined on the segment \([-1, 1]\). Hence

\[
C^\mu_n(x)E_{n+1,\mu}(x) = \sum_{i=0}^{n} c_i C^\mu_{n+1+i}(x),
\]

and

\[
\tilde{Q}^\mu_n(x)E_{n+1,\mu}(x) = 1 + \sum_{i=0}^{n} c_i \tilde{Q}^\mu_{n+1+i}(x).
\]

From (20) it follows that

\[
\int_{-1}^{1} w(x; \mu) C^\mu_n(x)E_{n+1,\mu}(x)x^k \, dx = 0, \quad k = 0, 1, 2, \ldots, n,
\]

so that, by the theorem in [3, p. 77], an interpolatory integration rule based on the zeros of \( C^\mu_n(x) \) and \( E_{n+1,\mu}(x) \) is exact for all polynomials of degree \( \leq 3n + 1 \) which forms the basis for KEGGIR's.

Now, it can be shown that

\[
Q^\mu_n(z) = \gamma_{n\mu} w^{-n-1} F(1 - \mu, n + 1; n + \mu + 1; w^{-2})
\]

\[
= \gamma_{n\mu} \sum_{j=0}^{\infty} f_{j\mu} w^{-n-1-2j},
\]

where \( z = \frac{1}{2}(w + w^{-1}), \gamma_{n\mu} = \sqrt{\pi} \Gamma(n + 2\mu)/\Gamma(n + \mu + 1), F(a, b; c; z) \) is the usual hypergeometric function, \( f_{0\mu} = 1, \)

\[
f_{j\mu} = (1 - \mu/j)(1 - \mu/(n + \mu + j))f_{j-1,\mu},
\]

and we have not shown the dependence on \( n \) of the \( f_{j\mu} \).

Setting \( w = e^{-\theta} \) and \( x = \cos \theta \), we get that

\[
\tilde{Q}^\mu_n(x) = \gamma_{n\mu} \sum_{j=0}^{\infty} f_{j\mu} T_{n+1+2j}(x).
\]

Since \( E_{n+1,\mu}(x) \) contains only even or odd powers of \( x \), we can write \( E_{n+1,\mu}(x) \) in the form

\[
E_{n+1,\mu}(x) = \sum_{i=0}^{m-1} \lambda_{i\mu} T_{n+1-2i}(x) + \begin{cases} \lambda_{\mu} T_1(x), & n \text{ even,} \\ \frac{1}{2} \lambda_{\mu}, & n \text{ odd,}
\end{cases}
\]

where \( m = [(n + 1)/2] \).

To determine the coefficients \( \lambda_{i\mu} \), we equate, in view of (21) and (25), the coefficients of \( T_k(x), k = 1, \ldots, n + 1, \) in the product
(27) \[ \tilde{Q}^\mu_n(x) E^\alpha_{n+1,\mu} (x) = \gamma_{\mu} \left( \sum_{j=0}^{\infty} f_{\mu} T_{n+1+2j}(x) \right) \left( \sum_{i=0}^{m} \lambda_{\mu} T_{n+1-2i}(x) \right) \]

to zero and the coefficient of \( T_0(x) \) to unity. Here the prime means that if \( n \) is odd, we replace \( \lambda_{\mu n} \) by \( \frac{1}{2} \lambda_{\mu (n+1)} \). Since \( T_r(x) T_s(x) = \frac{1}{2}(T_{r+s}(x) + T_{|r-s|}(x)) \), we see that the \( \lambda_{\mu i} \) must satisfy the following equations

(28) \[ \lambda_{0\mu} = 2\gamma_{\mu}^{-1}, \quad \sum_{i=0}^{k} f_{\mu} \lambda_{k-i,\mu} = 0, \quad k = 1, \ldots, m. \]

Following Monegato [11], we define \( \alpha_{i\mu} = \lambda_{i\mu}/\lambda_{0\mu} \) so that \( \alpha_{0\mu} = 1, \alpha_{1\mu} = -f_{1\mu} \), and

(29) \[ \alpha_{k\mu} = -f_{k\mu} - \sum_{i=1}^{k-1} f_{\mu} \alpha_{k-i,\mu}, \quad k = 2, \ldots, m. \]

From this, we see that the \( \alpha_{i\mu} \) are the first \( m + 1 \) coefficients in the series

(30) \[ \sum_{i=0}^{\infty} \alpha_{i\mu} u^i = \left\{ \sum_{i=0}^{\infty} f_{\mu} u^i \right\}^{-1}, \]

so that we can also use (29) for indices \( k > m \). Here also we have not indicated the dependence on \( n \) of the \( \lambda_{i\mu} \) and \( \alpha_{i\mu} \).

3. The Exact Degree of Precision of KEGGIR's and KELGIR's. Let us define

(31) \[ f_k(x) = C_n^\mu(x) E_{n+1,\mu}(x) C_n^{\mu+1+k}(x), \quad k = 0, \ldots, n. \]

Then from (20) it follows that \( I f_k = c_k h_{n+1+k,\mu} \). Since the KEGGIR applied to \( f_k(x) \) vanishes, we have from (8) that \( E_{p_k}(f_k) = c_k h_{n+1+k,\mu} \) so that the exact precision of the KEGGIR is determined by the first index \( k \), say \( k_0 \), for which \( c_{k_0} \neq 0 \). Indeed, \( p_n = 3n + 1 + k_0 \). We now show that for \( 0 < \mu < 2, \mu \neq 1, c_0 \neq 0 \) for \( n \) even and \( c_1 \neq 0 \) for \( n \) odd.

Consider first the case \( n \) even. Substituting (25) and (27) into (21) and equating the coefficients of \( T_{n+2}(x) \), we find that

(32) \[ c_0 \gamma_{n+1,\mu} = \frac{\gamma_{n\mu}}{2} \left( \lambda_{\mu f_0} + \lambda_{\mu f_1} + \lambda_{m-1,\mu f_2} + \ldots + \lambda_{0\mu f_{m+1,\mu}} \right) \]

\[ = \alpha_{m\mu} + \alpha_{m\mu} f_{1\mu} + \alpha_{m-1,\mu} f_{2\mu} + \ldots + \alpha_{1\mu} f_{m\mu} + f_{m+1,\mu} \]

\[ = \alpha_{m\mu} - \alpha_{m+1,\mu}. \]

Thus, it suffices to show that \( \alpha_{m\mu} - \alpha_{m+1,\mu} \) does not vanish. In fact, we shall show that the \( \alpha_{i\mu} \) are strictly monotonic. For \( 0 < \mu < 1 \), the sequence \( \{ f_{\mu j} \} \) is completely monotonic, i.e., \( (-1)^k \Delta^k f_{\mu j} > 0 \) for all \( j \) and \( k \) [17, p. 137]. Hence, by a theorem of Kaluza [6], the sequence \( \{-\alpha_{i+1,\mu}\} \) is also completely monotonic and hence strictly monotonic. For \( 1 < \mu < 2 \), the sequence \( \{-f_{i+1,\mu}\} \) is completely monotonic. From this it follows, by some results in [6], that

\[ \frac{\alpha_{i-1,\mu}}{\alpha_{i\mu}} > \frac{\alpha_{i\mu}}{\alpha_{i+1,\mu}}, \quad i = 1, 2, \ldots. \]
Since \( \sum_{i=0}^{\infty} \alpha_{i\mu} \) converges, and in fact equals \( \{F(1 - \mu, n + 1; n + \mu + 1; 1)\}^{-1} \), it follows that the sequence \( \{\alpha_{i\mu}\} \) is strictly monotonic. For \( \mu = 2 \), Szegö [16] gives an explicit expression for the \( \lambda_{i\mu} \),

\[
\lambda_{i2} = \frac{2}{\sqrt{\pi}} \frac{1}{n + 3} \left( \frac{n + 1}{n + 3} \right)^{i}, \quad i = 0, 1, \ldots,
\]

which again shows that the \( \alpha_{i2} \) are strictly monotonic.

We now consider the case \( n \) odd. Proceeding as before, this time equating the coefficients of \( T_{n+3}(x) \), we find that

\[
c_1\gamma_{n+2,\mu} = \frac{\gamma_{n\mu}}{2} \left( \lambda_{m\mu}f_{1\mu} + \lambda_{m-1,\mu}f_{0\mu} + \lambda_{m-1,\mu}f_{2\mu} + \lambda_{m-2,\mu}f_{3\mu} + \cdots + \lambda_{0\mu}f_{m+1,\mu} \right)
\]

\[
= \alpha_{m-1,\mu} + \alpha_{m,\mu}f_{1\mu} + \alpha_{m-1,\mu}f_{2\mu} + \cdots + \alpha_{1\mu}f_{m\mu} + f_{m+1,\mu}
\]

\[
= \alpha_{m-1,\mu} - \alpha_{m+1,\mu}.
\]

Since the \( \alpha_{i\mu} \) are strictly monotonic, it follows that \( c_1 \neq 0 \).

For \( \mu = 0 \), \( f_{j0} = 1 \), \( i = 0, 1, 2, \ldots \), so that \( \lambda_{00} = -\lambda_{10} = 2n/\pi^{1/2}, \lambda_{00} = 0 \), \( \lambda_{i1} = 0 \), \( i > 1 \) and \( E_{n+1,0} = (2n/\pi^{1/2}) \{T_{n+1}(x) - T_{n-1}(x)\}, n \geq 2 \). Hence

\[
C_0^0(x)E_{n+1,0}(x) = k_1 T_n(T_{n+1} - T_{n-1}) = k_1 \frac{1}{2} \{T_{2n+1} - T_{2n-1}\}
\]

\[
= k_2 (1 - x^2)U_{2n-1} = k_3 (1 - x^2)C_{2n-1}(x),
\]

and the zeros of \( C_0^0(x)E_{n+1,0}(x) \) are the abscissas of the \((2n + 1)\)-point LGIR for the weight \( w(x; 0) \) which is of exact precision \( 4n - 1 \), as can also be seen from the fact that \( c_{n-2} \) is the first \( c_k \) which does not vanish.

For \( \mu = 1, f_{j1} = 1 \), \( i > 0 \) so that \( \lambda_{01} = \lambda_{11} = 0, i > 0 \), and \( E_{n+1,1}(x) = (2/\sqrt{\pi})T_{n+1}(x) \). Hence

\[
C_1^1(x)E_{n+1,1}(x) = k_1^1 U_n(x)T_{n+1}(x) = k_2^1 C_{2n+1}(x),
\]

and the zeros of \( C_1^1(x)E_{n+1,1}(x) \) are the abscissas of the \((2n + 1)\)-point GGIR for the weight \( w(x; 1) \) which is of exact precision \( 4n + 1 \) and which also follows from the fact that \( c_n \) is the first \( c_k \) which does not vanish.

In the case of the KELGIR, we define

\[
\tilde{f}_k(x) = (1 - x^2)C_{n-1}^{\mu+1}(x)E_{\nu+1}(x)C_{n+1}^{\mu+1}(x), \quad k = 0, 1, \ldots, n - 1,
\]

so that \( \tilde{f}_k = c_k h_{n+k,\mu+1} \). Hence, since \( c_0 = c_0(n - 1, \mu + 1) \neq 0 \) for \( n - 1 \) even, i.e., for \( n \) odd, while \( c_1 \neq 0 \) for \( n - 1 \) odd, we have that the \((2n + 1)\)-point KELGIR is of exact precision \( 3n + 1 \), for \( n \) even, and \( 3n \), for \( n \) odd, provided that \( \mu \neq 0 \). For \( \mu = 0 \), we have as before that \( E_{n+1}(x) = (2/\pi^{1/2})T_n(x) \), so that

\[
(1 - x^2)C_{n-1}^{1}(x)E_{n1}(x) = \tilde{k}_1 (1 - x^2)C_{2n-1}(x),
\]

whose zeros are again the abscissas of the \((2n + 1)\)-point LGIR for the weight \( w(x; 0) \).
If we now define
\begin{equation}
 d_{n\mu} = \begin{cases}
 \alpha_{m\mu} - \alpha_{m+1,\mu}, & n \text{ even}, \\
 \alpha_{m-1,\mu} - \alpha_{m+1,\mu}, & n \text{ odd},
\end{cases}
\end{equation}
we have that for the Gauss case
\begin{equation}
 d_{n\mu} = \begin{cases}
 c_0 \gamma_{n+1,\mu}, & n \text{ even}, \\
 c_1 \gamma_{n+2,\mu}, & n \text{ odd},
\end{cases}
\end{equation}
while for the Lobatto case
\begin{equation}
 d_{n-1,\mu+1} = \begin{cases}
 c_0 \gamma_{n,\mu+1}, & n \text{ even}, \\
 c_1 \gamma_{n+1,\mu+1}, & n \text{ odd},
\end{cases}
\end{equation}
where we have suppressed the dependence of \(c_0\) and \(c_1\) on \(n\) and \(\mu\). This leads us immediately to formulas (10) and (11). For example, applying (8) with \(n\) even to \(f_0(x)\), we have that
\begin{equation}
 c_0 h_{n+1,\mu} = K_{n\mu} k_{n\mu} 2^{\gamma_{n\mu}} 12^n k_{n+1,\mu}^{3n+2}!,
\end{equation}
so that
\begin{equation}
 K_{n\mu} = \frac{d_{n\mu}}{\gamma_{n+1,\mu}} \frac{h_{n+1,\mu} \gamma_{n\mu}}{2^{n+1} k_{n\mu} k_{n+1,\mu}^{3n+2}!} = \frac{d_{n\mu} c_{n\mu}}{2^p n p_n^\mu}.
\end{equation}
For \(n\) odd, we consider \(f_1(x)\) while in the Lobatto case we work with \(f_0(x)\) and \(f_1(x)\).

4. Remarks. a. Monegato [11] gives an error bound for KEGGIR’s with \(0 < \mu < 1\). We shall show how to improve this bound slightly and extend it to the case \(1 < \mu < 2\), as well as to KELGIR’s with \(-\frac{1}{2} < \mu \leq 1, \mu \neq 0\).

For \(n\) even, Monegato writes the error \(E_{pn}(f)\) for \(f \in C^{3n+2}[-1, 1]\) in the form
\begin{equation}
 E_{pn}(f) = \frac{2^{-2n}}{k_{n\mu}^{3n+2}!} \int_{-1}^{1} w(x; \mu) C_n^\mu(x)(\overline{E}_{n+1,\mu}(x))^{2} f^{3n+2}(\xi_x) dx,
\end{equation}
where
\begin{equation}
 \overline{E}_{n+1,\mu}(x) = E_{n+1,\mu}(x)/\lambda_{0\mu} = \sum_{i=0}^{m_{\mu}} a_{i\mu} T_{n+1-2i}(x).
\end{equation}
Hence
\begin{equation}
 |E_{pn}(f)| \leq \frac{\pi n^{(n+2\mu)} B^2_{n+1,\mu}}{2^{3n+2\mu-1} p_n! n^{(\mu+1)} n^{(n+\mu)} M_{p\mu}}.
\end{equation}
where
\begin{equation}
 M_s = \max_{-1 \leq x \leq 1} |f^{(s)}(x)| \quad \text{and} \quad B_{n+1,\mu} = \max_{-1 \leq x \leq 1} |\overline{E}_{n+1,\mu}(x)|.
\end{equation}
For $0 < \mu < 1$, Monegato states that $B_{n+1,\mu} < 2$ and replaces $B_{n+1,\mu}$ by 2 in (45). Now, while this bound is the best available for $0 < \mu \leq \frac{1}{2}$, we can improve on it for $\frac{1}{2} < \mu < 1$. In addition, a bound on $B_{n+1,\mu}$ is also available for $1 < \mu \leq 2$. This follows from our observation above that

$$\sum_{i=0}^{\infty} a_{i\mu} = \{F(1 - \mu, n + 1; n + \mu + 1, 1)\}^{-1} = T_{n\mu} \quad \text{(46)}$$

Now for $\frac{1}{2} < \mu < 1$, $a_{0\mu} = 1, a_{i\mu} < 0, i > 0$. Since

$$B_{n+1,\mu} \leq \sum_{i=0}^{m} |a_{i\mu}| = 1 - \sum_{i=1}^{m} a_{i\mu} < 1 - \sum_{i=1}^{\infty} a_{i\mu},$$

it follows that $B_{n+1,\mu} < 2 - T_{n\mu} < 2$. For $1 < \mu < 2$, we have that $a_{i\mu} > 0$, for all $i$. Hence $B_{n+1,\mu} \leq \sum_{i=0}^{m} a_{i\mu} < T_{n\mu}$. For $\mu = 2$,

$$\sum_{i=0}^{\infty} a_{i2} = \left(1 - \frac{n + 1}{n + 3}\right)^{-1} = \frac{n + 3}{2} > B_{n+1,2}. \quad \text{(47)}$$

For $n$ odd, using classical arguments, we have the same bound.

In the Lobatto case, we have, similarly for $n$ odd, that

$$E_{q_{n}}(x) = \frac{2^{2-2n}}{k_{n-1,\mu + 1}(3n + 1)!} \int_{-1}^{1} w(x; \mu + 1) C_{n-1}^{\mu+1}(x) \left(\bar{E}_{n,\mu+1}(x)\right)^{2} f^{(3n+1)}(\xi_{x}) \, dx, \quad \text{(47)}$$

whence

$$|E_{q_{n}}(f)| \leq \frac{\pi \Gamma(n + 2\mu + 1)B_{n,\mu + 1}^{2}}{2^{3n+2\mu-2}q_{n}!\Gamma(n + \mu + 2)} M_{q_{n}}, \quad \text{(48)}$$

where for $-\frac{1}{2} < \mu < 0, B_{n,\mu+1} < 2 - T_{n-1,\mu+1}$ and for $0 < \mu < 1, B_{n,\mu+1} < T_{n-1,\mu+1}$. For $\mu = 1, B_{n2} < (n + 2)/2$. As before, the same bound holds for $n$ even.

b. The Fourier-Gegenbauer coefficients of a function $f(x)$ are defined by

$$FG_{n\mu}(f) = h_{n\mu}^{-1} \int_{-1}^{1} w(x; \mu) C_{n}^{\mu}(x) f(x) \, dx, \quad n = 0, 1, \ldots. \quad \text{(49)}$$

As Barrucand [2] points out, the integral is most efficiently evaluated by a $(2n + 1)$-point KEGGIR applied to the function $C_{n}^{\mu}(x)f(x)$ which reduces to the $(n + 1)$-point formula

$$FG_{n\mu}(f) \approx h_{n\mu}^{-1} \sum_{i=1}^{n+1} \vec{v}_{i} C_{n}^{\mu}(y_{i}) f(y_{i}) = \sum_{i=1}^{n+1} \vec{v}_{i} f(y_{i}). \quad \text{(50)}$$

For $\mu \neq 0, 1$, we get a rule which is exact for polynomials of degree $< p_{n} - n$, which is the best possible. For assume that there existed an $(n + 1)$-point rule, say

$$FG_{n}(f) \approx \sum_{i=1}^{n+1} \vec{v}_{i} f(y_{i}). \quad \text{(51)}$$

exact for polynomials of degree $p_{n} - n, n$ even. This would imply that
\[(52) \quad \int_{-1}^{1} w(x; \mu) C_n^\mu(x) E_{n+1, \mu}(x) \prod_{i=1}^{n+1} (x - y_i) \, dx = 0,\]

which contradicts our results above. Similarly for \(n\) odd.

For \(\mu = 0\), the rule (50) is exact for polynomials of degree \(\leq 3n - 1\), a result which has already been reported in [8]. For \(\mu = 1\), (50) is exact for polynomials of degree \(\leq 3n + 1\) which is the best possible result, so that the highest precision is achieved for Fourier-Chebyshev coefficients of the second kind. However, we should warn the user that the weights \(\tilde{v}_i\) in (50) alternate in sign inasmuch as the \(v_i\) are positive and the zeros of \(C_n^\mu(x)\) separate those of \(E_{n+1, \mu}(x)\), so that the \(C_n^\mu(y_i)\) alternate in sign.

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