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The Determination of all Imaginary, Quartic, Abelian Number Fields With Class Number 1

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Abstract. In this paper, it is proved that there are just seven imaginary number fields, quartic cyclic over the rational field, and having class number 1. These are the quartic, cyclic imaginary subfields of the cyclotomic fields generated by the fth roots of unity, where f is 16 or is a prime less than 100. This completes the list of imaginary, quartic, abelian number fields with class number 1. There are 54 such fields, with maximal conductor 67.163.

In [5], Uchida proves the following about imaginary, quartic abelian number fields of class number 1:

(A) If the field is bicyclic, there are 47 such fields, (with maximal conductor 67.163) with possibly one more.

(B) If the field is cyclic, the conductor must be less than 50000.

The existence of another field in (A), depends, however, on there being an imaginary quadratic field with class number 2 and discriminant < -427. Stark [4], and Montgomery-Weinberger [3], have shown there is no such quadratic field, so Uchida's list of bicyclic fields is complete. Brown and Parry [1], using the results of Stark, have also given a complete list of imaginary bicyclic quartic fields of class number 1.

This paper describes the computations that were carried out to show that there are just 7 imaginary, quartic cyclic number fields with class number 1.

THEOREM. *There are exactly* 54 *imaginary, quartic, abelian number fields with class number* 1:

(a) 47 *bicyclic fields, with maximal conductor* 67.163; *and*

(b) 7 *cyclic fields, with maximal conductor* 61.

As will be seen below, the cyclic fields are contained in the cyclotomic fields Ω_{16}, Ω_5, Ω_{13}, Ω_{29}, Ω_{37}, Ω_{53}, and Ω_{61}. (Ω_N is the field generated by the Nth roots of unity.) Each is the unique imaginary, cyclic, quartic subfield of the corresponding cyclotomic field.

The theory of the relative class number developed by Hasse [2], was used to eliminate all possible fields up to the 50000 limit, except for the above seven. We describe the details relevant to the present problem. Let k be an imaginary, cyclic quartic field. There is, associated to k, a pair of quartic Dirichlet characters ψ and $\overline{\psi}$ of conductor $f =$ conductor of k. These are, essentially, the characters of order 4 on $\mathrm{Gal}(\Omega_f/Q) \cong (Z/f)^*$, which are orthogonal to $\mathrm{Gal}(\Omega_f/k)$. Since k is totally complex,

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$\psi(-1) = \bar{\psi}(-1) = -1$. Now, also, k contains a unique real quadratic subfield, k_1. Let h = class number of k, h_1 = class number of k_1. Then $h = h_1 h^*$, where h^* is an integer, the relative class number. Hasse (p. 79 of [2]), gives a formula for h^*, which in the present case reduces to

$$h^* = w\Theta(\psi)\Theta(\bar{\psi}),$$

where w = number of roots of unity in k and

$$\Theta(\psi) = -\frac{1}{2f}\sum_{x=1}^{f-1} x\psi(x).$$

(Note that it is easy to verify that a fundamental unit of k_1 is also a fundamental unit of k, so the factor Q in Hasse's original formula is 1 in the present case.)

The search for k of class number 1 was first reduced by the following

PROPOSITION 1. *If $h = 1$, then $f = 16$ or f is a prime, $f \equiv 5 \pmod 8$.*

Proof. Since $h \geqslant h^* \geqslant 2\Theta(\psi)\Theta(\bar{\psi})$, if h is to be 1, then $\Theta(\psi)$ cannot be integral. But, $\Theta(\psi)$ is integral if f is divisible by two distinct primes; Section 28 of [2]. If $f = p^e$ is a power of an odd prime, then $\Theta(\psi)$ is integral if $e > 1$; Satz 32, p. 93 of [2]. For there to be a quartic character (mod f), f an odd prime, we must have $f \equiv 1 \pmod 4$. If the character is to be odd, then $f \not\equiv 1 \pmod 8$. Finally, if $f = 2^e$, then $f = 16$, since the only odd quartic characters (mod 2^e) have conductor 16.

In each case $f = 16$, f a prime $\equiv 5 \pmod 8$, the field k is a uniquely determined subfield of Ω_f, and the character pair is the unique pair of odd, quartic characters (mod f). For $f = 16$, $k_1 = Q[\sqrt{2}]$ so $h_1 = 1$. Also, $w = 2$. It is easily checked that $\Theta(\psi)\Theta(\bar{\psi}) = ½$. So, in the case $f = 16$, $h = 1$. Now assume that f is a prime $\equiv 5$ (mod 8). The computation of $\Theta(\psi)$ can be done by counting quartic residues (mod f).

PROPOSITION 2. *Let f be a prime $\equiv 5 \pmod 8$ and ψ a quartic character (mod f), where $\psi(2) = i$, $i^2 = -1$. Then*

$$\Theta(\psi) = ((3(\alpha_0 - \delta_0) + (\beta_0 - \gamma_0)) + ((\alpha_0 - \delta_0) - 3(\beta_0 - \gamma_0))i)/10,$$

and

$$\Theta(\psi)\Theta(\bar{\psi}) = ((\alpha_0 - \delta_0)^2 + (\beta_0 - \gamma_0)^2)/10,$$

where α_0 is the number of quartic residues (mod f) *contained in the interval* $[1, (f-1)/4]$; β_0, *in* $[(f+3)/4, (f-1)/2]$; γ_0, *in* $[(f+1)/2, (3f-3)/4]$; δ_0, *in* $[(3f+1)/4, (f-1)]$.

Note that, since $f \equiv 5 \pmod 8$ is assumed, 2 is not a quadratic residue (mod f), so $(\psi(2))^2 = -1$.

Note also that the definition of $\Theta(\psi)$ and Proposition 2 give $\Theta(\psi)$ as members of the Gaussian field ($Q[i]$) with denominators $2f$ and 10, respectively. If $f > 5$, the true denominator is at worst 2, so $3(\alpha_0 - \delta_0) + (\beta_0 - \gamma_0) \equiv 0 \pmod 5$. Also, $\alpha_0 + \beta_0 + \gamma_0 + \delta_0 = (f-1)/4 \equiv 1 \pmod 2$, so $\Theta(\psi)$ and $\Theta(\psi) \cdot \Theta(\bar{\psi})$ have exact denominator 2. This also follows from Satz 32 of [2].

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Proof. We first derive the formula

$$\Theta(\psi) = \psi(2) \sum_{x=1}^{(f-1)/2} \psi(x)/(4\psi(2) - 2). \tag{1}$$

Let

$$G_1 = \sum_{x=1}^{(f-1)/2} x\psi(x), \qquad G_2 = \sum_{x=(f+1)/2}^{f-1} x\psi(x).$$

Then

$$2\psi(2)G_1 = \sum_{x=1}^{(f-1)/2} 2x\psi(2x),$$

and

$$2\psi(2)G_2 = \sum_{x=(f+1)/2}^{f-1} (2x - f)\psi(2x - f) - f\psi(2) \sum_{x=1}^{(f-1)/2} \psi(x).$$

Since f is odd,

$$2\psi(2)(G_1 + G_2) = G_1 + G_2 - f\psi(2) \sum_{x=1}^{(f-1)/2} \psi(x).$$

As $\Theta(\psi) = (G_1 + G_2)/(-2f)$, (1) is proved.

Now, the subgroup of quartic residues (mod f) is index 4 in $(Z/fZ)^*$, with coset representatives 1, 2, 4, 8. Let α_j be the number of elements in 2^j-coset contained in the interval $[1, (f-1)/4]$, for $0 \leqslant j \leqslant 3$. ($\alpha_0$ is the same as above.) Define $\beta_j, \gamma_j, \delta_j$ similarly for the successive quarter intervals of $[1, f-1]$. Then

$$\sum_{x=1}^{(f-1)/2} \psi(x) = (\alpha_0 + \beta_0) - (\alpha_2 + \beta_2) + (\alpha_1 + \beta_1)i - (\alpha_3 + \beta_3)i. \tag{2}$$

Since -1 is a quadratic, but not quartic, residue (mod f), multiplication by -1 gives the following relations:

$$\begin{aligned} &\alpha_0 = \delta_2, \quad \alpha_1 = \delta_3, \quad \alpha_2 = \delta_0, \quad \alpha_3 = \delta_1, \\ &\beta_0 = \gamma_2, \quad \beta_1 = \gamma_3, \quad \beta_2 = \gamma_0, \quad \beta_3 = \gamma_1. \end{aligned} \tag{3}$$

Also, the quartic residues in $[1, (f-1)/4]$ and $[(f+1)/2, (3f-3)/4]$, when multiplied by 2, map one to one onto the 2-coset elements in $[1, (f-1)/2]$. Thus $\alpha_0 + \gamma_0 = \alpha_1 + \beta_1$. Similarly:

$$\begin{aligned} &\alpha_0 + \gamma_0 = \alpha_1 + \beta_1, \quad \beta_0 + \delta_0 = \gamma_1 + \delta_1, \\ &\alpha_1 + \gamma_1 = \alpha_2 + \beta_2, \quad \beta_1 + \delta_1 = \gamma_2 + \delta_2, \\ &\alpha_2 + \gamma_2 = \alpha_3 + \beta_3, \quad \beta_2 + \delta_2 = \gamma_3 + \delta_3, \\ &\alpha_3 + \gamma_3 = \alpha_0 + \beta_0, \quad \beta_3 + \delta_3 = \gamma_0 + \delta_0. \end{aligned} \tag{4}$$

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(3) and (4) imply

$$\alpha_0 = \beta_1 = \gamma_3 = \delta_2, \quad \alpha_2 = \beta_3 = \gamma_1 = \delta_0,$$
$$\alpha_1 = \beta_2 = \gamma_0 = \delta_3, \quad \alpha_3 = \beta_0 = \gamma_2 = \delta_1. \tag{5}$$

Combining (5), (2) and (1), then rationalizing the denominator ($4\psi(2) - 2 = 4i - 2$), proves the proposition. □

A program was written to compute $\Theta(\psi)$ for prime $f \equiv 5 \pmod 8$, $f \leqslant 50021$. The computation was done on an IBM 370. A machine language subroutine was used to compute, for each f, $x^4 \pmod f$ for $1 \leqslant x \leqslant (f-1)/2$. The count of $\alpha_0, \beta_0, \gamma_0, \delta_0$ was made from this. The whole computation took less than 10 minutes of CPU time. For $h = 1$, $\Theta(\psi) \cdot \Theta(\bar{\psi})$ must be ½ or less. This occurred only for $f = 5$ (for which $\Theta(\psi)\Theta(\bar{\psi}) = 1/10$) and $f = 13, 29, 37, 53, 61$ (for which $\Theta(\psi)\Theta(\bar{\psi}) = ½$).

Proof of Theorem. By Uchida's result, it only remains to show that the seven cyclic fields do have class number 1. The case $f = 16$ was considered before. For $f = 5$, $k = \Omega_5$ and $k_1 = Q[\sqrt{5}]$. So, $w = 10$, $h_1 = 1$ and $\Theta(\psi)\Theta(\bar{\psi}) = 1/10$, that is, $h = 1$. In the remaining five cases, $k_1 = Q[\sqrt{f}]$ and $h_1 = 1$. Also, $w = 2$, $\Theta(\psi)\Theta(\bar{\psi}) = ½$ so $h = 1$ in these cases. □

Generating equations for these fields can be obtained by observing that $k = k_1[\sqrt{\pi}]$, where $\pi = -\sqrt{f}\epsilon$ ($-\sqrt{2}\epsilon$ for $f = 16$) and $\epsilon > 0$ is a fundamental unit for k_1 (norm of ϵ is -1, necessarily). Then one has

$$\begin{aligned}
f &= 5, & x^4 + 5x^2 + 5 &= 0,\\
f &= 13, & x^4 + 13x^2 + 13 &= 0,\\
f &= 29, & x^4 + 29x^2 + 29 &= 0,\\
f &= 37, & x^4 + 74x^2 + 37 &= 0,\\
f &= 53, & x^4 + 53x^2 + 53 &= 0,\\
f &= 61, & x^4 + 305x^2 + 61 &= 0,\\
f &= 16, & x^4 + 4x^2 + 2 &= 0.
\end{aligned}$$

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