# REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

The numbers in brackets are assigned according to the revised indexing system printed in Volume 28, Number 128, October 1974, pages 1191–1194.

13[7.30].—SALVADOR CONDE, *Raices de Ecuaciones Trascendentes con Productos Cruzados de Funciones de Bessel de Diferentes Ordenes,* Facultad de Ingenieria Universidad del Zulia, Maracaibo, Venezuela, 1977; a report of 78 pages with 67 pages of computer printed tables, deposited in the UMT file.

Let

$$B_\nu(x) \equiv J_\nu(x)Y_{\nu-1}(\beta x) - J_{\nu-1}(\beta x)Y_\nu(x) = 0,$$

$$C_\nu(x) = J_\nu(\beta x)Y_{\nu-1}(x) - J_{\nu-1}(x)Y_\nu(\beta x) = 0.$$

This report gives the first 28 positive zeros of these functions for $\nu = 0(0.1)1.0$ and $\beta = 0.1(0.1)0.9$, $1.1(0.1)2.0$, to 12S. If $\beta = 1$, there are no zeros.

If $\nu = 1$, some scattered and low accuracy tables are reported in the FMRC index [1]. Again, if $\nu = 1$, the first 10 positive zeros for a large number of $\beta$ values are given to 10D in reports by Fettis and Caslin [2], [3]. If $\nu = 1/2$, the positive zeros are $(2m - 1)\pi/2(1 - \beta)$, $m$ a positive integer. Thus, the tables under review are the most extensive known to me.

The method of computation is described in the introduction. The zeros were determined to a precision of order $10^{-13}$. I am not sure what this means, but I should think the data were intended to be correct to within a half unit of the last place recorded.

I have compared the common entries of the report under review with those of Fettis and Caslin. According to the method of computation used by the latter authors, their entries should be correct to within a half unit of the last place recorded. In the comparison, we found numerous discrepancies of as many as 6 units in the last place, where, if necessary, the Conde tables are rounded to 10D. We also found entries which differed by 10, 11, 14 and 31 units in the last place. We have not attempted to determine which are correct. Neither do we record the errata more precisely since the data is believed to be more than of sufficient accuracy for most applications.

The question of interpolation is not considered. For interpolation in the $\beta$ direction, the technique discussed by Fettis and Caslin should prove helpful.

Y. L. L.


1. A. FLETCHER, J. C. P. MILLER, L. ROSENHEAD & L. J. COMRIE, *An Index of Mathematical Tables,* 2nd ed., published for Scientific Computing Service Ltd. by Addison-Wesley, Reading, Mass., 1962, pp. 414, 415.

2. HENRY E. FETTIS & JAMES C. CASLIN, *An Extended Table of Zeros of Cross Products of Bessel Functions,* Report ARL 66-0023, Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force, Wright-Patterson Air Force Base, Ohio, February 1966, v + 129 pp., 28 cm. (Copies obtainable from the Defense Documentation Center, Cameron Station, Alexandria, Virginia.) See also *Math. Comp.*, v. 21, 1967, pp. 507, 508.




0025-5718/80/0000-0186/$02.75

3. HENRY E. FETTIS & JAMES C. CASLIN, *More Zeros of Bessel Function Cross Products*, Report ARL 68-0209, Aerospace Research Laboratories, Office of Aerospace Research, United States Air Force, Wright-Patterson Air Force Base, Ohio, December 1968, v + 56 pp., 28 cm. (Released to the Clearinghouse, U. S. Department of Commerce, Springfield, Virginia 22151.) See also *Math. Comp.*, v. 23, 1969, p. 884.

14[3.05].—PHILIP J. DAVIS, *Circulant Matrices*, Wiley, New York, 1979, xv + 250 pp. Price $18.95.

A circulant matrix of order $n$ is a square matrix of the form

$$C = \begin{pmatrix} c_1 & c_2 & \cdots\cdots & c_n \\ c_n & c_1 & \cdots\cdots & c_{n-1} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ c_2 & c_3 & \cdots\cdots & c_1 \end{pmatrix} = \operatorname{circ}(c_1, c_2, \ldots, c_n).$$

This special structure affords an intimate relationship between the study of circulants and Fourier analysis; this relationship provides the backbone for the elegant analysis of circulants presented in this compact, well-organized book by the well-known numerical analyst Philip J. Davis.

Circulant matrices and their generalizations have important applications in physics, image processing, probability and statistics, number theory, geometry, and numerical analysis, e.g. in the numerical solution of periodic boundary value problems. Although the book does not actively pursue any of these applications, it does fulfill the author's avowed purpose of serving as a general reference on circulants, so that the basic facts need not be "rediscovered over and over again" by someone researching a specific application.

Chapter 1 motivates the study of circulants with a geometric example of nested polygons, the vertices of successive polygons being related by a circulant transformation. In fact, those with a taste for geometry will find throughout the book a host of connections between circulant matrix properties and geometrical results (cf. Chapter 4).

Chapter 2 is packed full of useful information on general matrix theory and is intended to provide background for the more specific study of circulants to follow in Chapter 3. In particular, the reader will find an interesting self-contained presentation of material on least squares, the singular value decomposition, and generalized inverses, although computational aspects are not addressed.

In Chapter 3, a central result is the representation of all circulants as polynomials in the special circulant $\pi = \operatorname{circ}(0, 1, 0, \ldots, 0)$, from which many interesting properties of circulants may be obtained. Generalizations of circulants (e.g. skew-circulants, block circulants) appear here and in the fifth chapter. Finally, the study of centralizers in Chapter 6 is used by Davis "to encompass and unify a number of results previously obtained as well as to point us in several new directions."

In order to appreciate the material in the book, the reader should have a reasonable background in matrix theory and some knowledge of abstract algebra. As such, and with the nice selection of problems included, the book can serve as a text for a very interesting secondary excursion into the realm of matrix theory.

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15[3, 3.25].—I. S. DUFF & G. W. STEWART (Editors), *Sparse Matrix Proceedings,* SIAM, Philadelphia, Pa., 1978, xvi + 334 pp., 24 cm. Price $21.50.

The papers in this book were presented at the Symposium on Sparse Matrix Computations held in Knoxville, Tennessee on November 2–3, 1978. Fourteen papers were presented on applications, software, and algorithms. The programming committee has tried to present an up-to-date account of developments in the area of sparse matrix computations.

J. H. B.

16[13.05].—CLAUDE JABLON & JEAN CLAUDE SIMON, *Applications des Modèles Numérique en Physique,* Interdisciplinary System Research 53, Birkhäuser Verlag, Basel, Stuttgart, 1978, 283 pp., 23 cm. Price Fr. 48.–.

This volume is written, in particular with physicists in mind, to explain the rules and help avoid pitfalls in numerical computation. The introduction is unusual in that it includes a discussion of the representation of mathematical models by computer programs from the aspects of linguistics and the theory of computation. The book then explains the basic concepts of numerical computation and goes on to treat numerical methods for a nonlinear equation, interpolation and approximation, and differential equations. The depth of treatment varies considerably and, as is reasonable in a book aimed at physicists, parabolic and elliptic partial differential equations are given relatively much space. The final chapter contains a useful discussion of the role of numerical models in physics and gives some hints on how to structure and document a Fortran program.

This book does not attempt to give a complete coverage of numerical methods, but even so the topics could have been better chosen. The most striking omission is that there is no systematic treatment of numerical methods of linear algebra. For example, eigenvalue problems, which certainly arise very frequently in physical applications, are not at all treated here. I also think that some space should have been devoted to methods for solving minimization problems and systems of nonlinear equations. The hope, expressed by the authors on page 13, that ideas in the (very short) section on solving a single nonlinear equation should enable a reader to tackle systems of nonlinear equations, seems to me to be very optimistic. I would have much preferred a modern treatment on spline approximation to the long section devoted to approximation by sums of translated functions. In the last chapter I missed a comment on the importance of portability.

The fact that descriptions in this book are very much from a user physicist's point of view might attract readers. I liked the introduction and the final chapter best.

However, the presentation is often rather superficial and to a large extent the book does not present state of the art methods. Therefore, there are several other books (in the English language) which I would rather recommend to a physicist or engineer. Unfortunately, these books are not even referenced in this text.

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17[5.00].—W. L. Wendland, *Elliptic Systems in the Plane*, Fearon-Pitman, Belmont, California, 1979, xi + 404 pp., 24 cm. Price $64.95.

The numerical solution of elliptic boundary value problems is an active area of research, as readers of this journal know. Most of the computational effort in this subject is devoted to two-dimensional problems, if only because of the cost of higher-dimensional calculations. On the other hand, the theory of elliptic boundary value problems is well developed, and the theory has simplifications for problems of two independent variables. This book gives, in the first part, a treatment of boundary value problems for elliptic systems in two independent variables, and in the second part, a treatment of the numerical solution of these problems. The result is an unusual collection of material that is worthy of consideration by those that are seriously interested in the subject.

The first part of the book is devoted to the solvability and regularity theory for elliptic systems in two independent variables. Various normal forms for these systems are given, and the Fredholm character of the problem is established by a homotopy argument. Formulas for the index of the problem are given. There is a brief discussion of nonlinear problems. Several integral equation formulations of these problems are also given. There is no discussion of piecewise smooth coefficients or domains, or of systems that are elliptic in the more general sense of Agmon-Douglis-Nirenberg, topics of current interest to numerical analysts.

The second part of the book is devoted to a theoretical analysis of numerical methods for the solution of elliptic boundary value problems. There are chapters on integral equation methods, finite difference methods, and a final chapter on least squares and Galerkin methods. In each case, the treatment includes the formulation of the method and an error analysis. The chapter on integral equations includes a section on numerical methods for conformal mapping. The chapter on difference equations starts with matrices of positive type and bounds for the discrete Green's function for the Laplace equation, as found in the classical papers of Bramble, Hubbard, and Thomée. The discrete Green's function is used to cast the discrete boundary value problem into a system of discrete integral equations, and error estimates are obtained for the solution of this system. The final chapter develops the theory of weighted least squares methods of a first order elliptic system. A final section relates this to Galerkin methods for second order systems, and gives an error analysis, including $L_\infty$ error estimates.

This book contains a lot of mathematics. It is written in a somewhat compressed style, but each chapter contains an introduction which carefully explains the material that follows. There are many references to the literature. The character of the book is that of a research monograph rather than a textbook. The more practical aspects of numerical analysis are not to be found here, but it is a valuable exposition of the related topics of existence theory and error estimates for elliptic boundary value problems.

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18 [5.00].—SEYMOUR V. PARTER, Editor, *Numerical Methods for Partial Differential Equations,* Academic Press, New York, 1979, ix + 332 pp., 23 cm. Price $14.50. (Publication of the Mathematics Research Center, the University of Wisconsin-Madison, no. 42.)

The book is the proceedings of a seminar held in Madison in October 1978. A person contemplating whether to order the book might be served by a list of the contributions:



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**19[7.15].**—H. J. J. TE RIELE, *Tables of the First* 15,000 *Zeros of the Riemann Zeta Function to* 28 *Significiant Figures, and Related Quantities,* Report NW 67/79, Stichting Mathematisch Centrum, Amsterdam, June 1979, 5 pp. text + 154 pp. tables.

Table 1 of this report is a table of the imaginary parts of the first 15,000 zeros $\rho_n$ of the Riemann zeta function $\zeta(s)$ in the upper half of the critical strip. Table 2 gives $|\rho_n\zeta'(\rho_n)|^{-1}$, and Table 3 gives $\arg(\rho_n\zeta'(\rho_n))$.

The computation of Table 1 was performed using (mainly) double-precision arithmetic on a Cyber 73/173 and required about 21 hours of CPU time. The author claims an accuracy of "about 28 digits" and the table gives each $\rho_n$ to 28 significant digits.

The reviewer checked the accuracy of a few entries in Table 1, using 40S computation and his multiple-precision arithmetic package [1], and both the Euler-Maclaurin formula for $\zeta(s)$ (as used by te Riele) and the Riemann-Siegel formula with a sufficient number of terms [2]. The largest error found was 66 units in the last place ($\rho_{10142} =$ 9998.850397089674049057631757 is correct, te Riele gives 9998.850. . . 631691). Thus, the final two digits of entries in the table should be regarded with suspicion. Despite this, the table is a significant advance over the 9S table of $\rho_1, \ldots, \rho_{1600}$ given in [3] and other tables known to the reviewer.

Tables 2 and 3 are given to 10S, although computed to "about 14 significant digits". The reviewer has not found any errors exceeding 0.5 units in the last place in these tables.

The tabulated quantities were used by the author in computations concerning Mertens' conjecture [4]. However, the reviewer wonders what point there is in publishing *all* of them in this manner. Anyone wishing to continue the Mertens conjecture computation would hardly use the values of $\rho_n$ given in Table 1; a horrifying amount of keypunching and verifying would be required. Instead, he would either recompute them, using Table 1 for checking purposes, or obtain them in machine-readable form, e.g. on magnetic tape. Unfortunately, te Riele does not say if his tables are available on magnetic tape, nor does he reproduce the program that generated them, although he does clearly describe the computational method.

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1. R. P. BRENT, "A Fortran multiple-precision arithmetic package," *ACM Trans. Math. Software,* v. 4, 1978, pp. 57–70.

2. R. P. BRENT, *Numerical Investigation of the Riemann Siegel Approximation,* Tech. Report, Dept. of Computer Science, Australian National University. (To appear.)

3. C. B. HASELGROVE in collaboration with J. P. C. MILLER, *Tables of the Riemann Zeta Function,* Royal Soc. Math. Tables, Vol. 6, Cambridge, 1960.

4. H. J. J. TE RIELE, "Computations concerning the conjecture of Mertens," *J. Reine Angew. Math.,* v. 311/312, 1979, pp. 356–360.

**20[9.10].**—SIRPA MÄKI, *The Determination of Units in Real Cyclic Sextic Fields,* Computer Table, 122 pages, University of Turku, Finland, 1979. Reference [3] subsequently appeared as Lecture Notes No. 797, 1980 and contains a photographic copy of this table.

There are 1337 real cyclic sextic fields having conductor less than 2022. Fundamental units and class numbers have been computed for all but 12 of these fields. In six of these cases the necessary information on the cubic subfield was unavailable and for the remaining six fields the numbers became too large to be handled by the program.

For any real sextic field there exist two units $\epsilon_1$ and $\epsilon_2$ which together with the units of the proper subfields generate the whole unit group. Although the table does not list values for $\epsilon_1$ and $\epsilon_2$, they can be expressed as products and quotients of units which are listed.

The class number $h_6$ of the sextic field can be expressed in the form

$$h_6 = h_2 h_3 h_R,$$

where $h_2$ and $h_3$ are the class numbers of the quadratic and cubic subfields and $h_R$ is the so-called relative class number. In fact $h_R = (U_6 \colon U_6')$ where $U_6$ is the unit group of the sextic field and $U_6'$ is a subgroup of $U_6$ which is generated by the units of the proper subfields together with the so-called cyclotomic units of the sextic fields. For approximately 90 percent of the fields listed $h_R = 1$. Moreover $h_R$ assumes the values 3, 4, and 7 for approximately 3, 4.5, and 1.7 percent of the fields respectively. In addition $h_R$ assumes the values 9, 12, and 16 each one time.

The relatively large number of occurrences of 3 and 4 as relative class numbers can be partially explained by the fact that there exists a subgroup $U_6^*$ of $U_6$ containing $U_6'$ such that $(U_6 \colon U_6^*) = 1, 3, 4$ or 12. In 70 percent of the cases where 3 divides $h_R$, this index is also divisible by 3. When 3 is replaced with 4, the percentage becomes 85.

The table tempts one to conjecture that $h_R$ never assumes the values 2, 5, 6 or 8. This, in fact, is true. Let $H_2'$, $H_3'$ and $H_6'$ denote the 3-complements of the class groups of the quadratic, cubic, and sextic fields and denote the orders of these groups by $h_2'$, $h_3'$, and $h_6'$. An automorphism of the sextic field (and hence of its cubic subfield) of order 3 will be denoted by $\sigma$. By decomposing $H_3'$ into orbits under $\sigma$ it is seen that

$$h_3' \equiv 1 \pmod 3.$$

A similar decomposition of $H_6'$ leads to

$$h_6'/h_2' \equiv 1 \pmod 3.$$

But

$$h_6' = h_2' h_3' h_R',$$

where $h_R'$ is the largest factor of $h_R$ which is prime to 3. Thus $h_R' \equiv 1 \pmod 3$.

The method of computation which is described in an accompanying manuscript [3] is essentially a refinement of the method of Leopoldt [1] and [2].

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1. H. W. LEOPOLDT, "Über Einheitengruppe und Klassenzahl reeler abelscher Zahlkörper," *Abh. Deutsch. Akad. Wiss. Berlin Math.-Nat. Kl.*, **1953**, No. 2, Berlin, 1954.

2. H. W. LEOPOLDT' "Über ein Fundamental-problem der Theorie der Einheiten algebraischer Zahlkörper," *Sitz. Ber. Bayr. Akad. Wiss. Math.-Nat. Kl.*, **1956**, No. 5.

3. S. MÄKI, *The Determination of Units in Real Cyclic Sextic Fields*, Lecture Notes in Math., Springer-Verlag, Berlin, Heidelberg and New York, 1980.