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Stability of Numerical Schemes Solving Quasi-Linear Wave Equations

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Abstract. A generalization of the Riemann invariants for quasi-linear wave equations of the type $\partial^2 w/\partial t^2 = \partial f(\partial w/\partial x)/\partial x$, which includes the shock curves, is proposed and is used to solve the Riemann problem. Three numerical schemes, whose accuracy is of order one (the Lax-Friedrichs scheme and two extensions of the upstreaming scheme), are constructed by L^2-projection, onto piecewise constant functions, of the solutions of a set of Riemann problems. They are stable in the L^∞-norm for a class of wave equations, including a nonlinear model of extensible strings, which are not genuinely nonlinear. The problem with boundary conditions is detailed, as is its treatment, by the numerical schemes.

I. Introduction. Let $T > 0$. We consider the quasi-linear wave equation

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left(f\left(\frac{\partial w}{\partial x}\right)\right), \tag{1}$$

on $Q =]0, 1[\times]0, T[$. The function $f \in C^1(\mathbf{R})$ is increasing, to make this a problem of hyperbolic type, and such that $f(0) = 0$. The initial conditions are

$$w(x, 0) = w_0(x), \quad \frac{\partial w}{\partial t}(x, 0) = v_0(x) \quad \text{for } x \in]0, 1[, \tag{2}$$

with w_0 in $W^{1,\infty}(]0, 1[)$ and v_0 in $L^\infty(]0, 1[)$. The boundary conditions are

$$w(0, t) = w(1, t) = 0 \quad \text{for } t \in]0, 1[. \tag{3}$$

For compatibility between (2) and (3), w_0 is assumed to be zero at $x = 0$ and $x = 1$. Equation (1) may be written as a system of two equations involving $u = \partial w/\partial x$ and $v = \partial w/\partial t$. We get, for $(x, t) \in Q$,

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial x}, \tag{4}$$

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x} f(u). \tag{5}$$

The initial and boundary conditions become, with $u_0 = dw_0/dx$,

$$u(x, 0) = u_0(x) \quad \text{for } x \in]0, 1[, \tag{6}$$

$$v(x, 0) = v_0(x) \quad \text{for } w \in]0, 1[, \tag{7}$$

$$v(0, t) = v(1, t) = 0 \quad \text{for } t \in]0, T[. \tag{8}$$

Such a system is often called a p-system; see [11], [12], [13]. The Riemann problem, the solution of which is shaped with shocks, rarefaction waves, or constant states, is solved in Section II. How to build this solution is detailed, and

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then is improved for the approximation. This is set up in Section III for the Lax-Friedrichs scheme and two extensions of the upstreaming scheme. An application to a model of extensible strings, for which $|f|$ is a convex function, is proposed in Section IV. This model presents the property to have bounded convex invariant sets in the plane (u, v), and, since mainly Riemann problems and L^2-projections are used, we derive, in Section V, the L^∞ stability of the schemes introduced in Section III, provided that a stability condition is satisfied.

Such a property, to have bounded convex invariant sets in the phase plane, is also true for other important applications and implies the stability of the same numerical schemes in the L^∞-norm. This is proved for the Shallow Water model, in [5] and [6], and for the isentropic gas dynamics equations, with a wide class of pressure laws, and for some supersonic models in [7]. When source terms appear, the increase of these sets may be estimated rather easily, and the stablity condition does not become too restrictive for computations with large values of the time.

II. The Riemann Problem. We consider, in this section, the problem (4), (5) on $\mathbf{R} \times]0, T[$, with the initial data

$$(u(x, 0), v(x, 0)) = \begin{cases} (u_+, v_+) & \text{if } x > 0, \\ (u_-, v_-) & \text{if } x < 0. \end{cases} \tag{9}$$

For a fixed t in $]0, T[$, the solution u of (4), (5), (9) will present mainly three kinds of shape. Indeed, shocks and rarefaction waves can occur, separated by constant states.

Since, for a rarefaction wave, both u and v are locally monotonic functions of x, we can express v as a function of u by eliminating x. We introduce this function $v(u)$ into (4) and (5) and obtain two quasi-linear equations on u

$$\frac{\partial u}{\partial t} = v'(u)\frac{\partial u}{\partial x}, \quad v'(u)\frac{\partial u}{\partial t} = f'(u)\frac{\partial u}{\partial x}. \tag{10}$$

Next, we claim that these equations give the same speed of propagation for the data, that is $v'(u) = \pm f'(u)^{1/2}$.

By integrating this differential equation, we get the Riemann invariants

$$v(u) = \pm g(u) + C \quad \text{with } g(u) = \int_0^u \sqrt{f'(y)}\, dy, \tag{11}$$

where C is some constant. The sign in (11) is imposed by the sense of the propagation. Knowing this, the first equation in (10) permits us now to get u, when the scalar equation has a regular solution; that is if the convexity of $v(u)$ does not require a solution with shock satisfying an entropy condition.

Similar arguments are developed to analyze a shock, which is a discontinuity of the first kind for u. Writing the Rankine-Hugoniot relations, we obtain the speed dx/dt of this shock and a compatibility condition, which shows that the same speed is given by both equations. Denoting by u_1 and u_2 the values of u on each side of the curve, along which the discontinuity travels, of equation $x = x(t)$, and, similarly, by v_1 and v_2 the values of v, we have

$$-\frac{dx}{dt} = \frac{f(u_1) - f(u_2)}{v_1 - v_2} = \frac{v_1 - v_2}{u_1 - u_2}. \tag{12}$$

The last equality gives

$$v_2 - v_1 = (u_2 - u_1)\left[\frac{f(u_1) - f(u_2)}{u_1 - u_2}\right]^{1/2}, \tag{13}$$

which is to be compared with (11). Then, at each point of the phase plane, two tangent increasing curves can be drawn, with the same sense of convexity; their equations are given by (11), (13). Two similar decreasing curves can be drawn at the same point. Using, now, the entropy condition (see [14], [3]), for the scalar equation, we know when a shock must appear. We can then give the same equation for shock or rarefaction curves. This is detailed in the following; see also [17].

Starting from a point (u_0, v_0) of the phase plane, we look at the solutions which have a positive speed of propagation. Let $\tilde{f}(y; u_0, u)$ be the value of the convex hull of f on $[u_0, u]$, if $u > u_0$, or of the concave hull of f on $[u, u_0]$, if $u < u_0$, at the point y of this interval. We define

$$G(u, u_0) = \int_{u_0}^{u} \sqrt{\frac{d}{dy}\tilde{f}(y; u_0, u)}\, dy. \tag{14}$$

Note that for $u > u_0$, we get

$$G(u, u_0) = \begin{cases} g(u) - g(u_0) & \text{if } f \text{ is convex,} \\ \sqrt{(f(u) - f(u_0))(u - u_0)} & \text{if } f \text{ is concave.} \end{cases}$$

For a general functon f, $G(u, u_0)$ becomes complicated. The generalized Riemann invariant with a positive propagation and starting from (u_0, v_0) is defined by

$$v = v_0 - G(u, u_0). \tag{15}$$

For a negative propagation, we obtain, similarly, the invariant

$$v = v_0 + G(u, u_0), \tag{16}$$

with G defined as in (14).

We now solve the Riemann problem (4), (5), (9). Writing (15) for $(u, v) = (u_+, v_+)$ and (16) for $(u, v) = (u_-, v_-)$, we find (u_0, v_0) at the intersection of these two curves, if it does exist. The curve connecting (u_0, v_0) to (u_+, v_+) describes the wave which has a positive speed. It is the solution of a scalar first order equation, given by putting (15) into the first equation of (10), for a rarefaction wave, or its speed is obtained obviously for a shock. The wave with a negative speed, which connects (u_0, v_0) to (u_-, v_-), is similarly given by (16). The solution is equal to (u_0, v_0) for $x = 0$ and $t > 0$. This technique builds the solution of the Riemann problem when the curves intersect; a sufficient condition to ensure it is the following

$$\lim_{R \to \pm\infty} |g(R)| = +\infty. \tag{17}$$

Moreover, the entropy condition proposed by T. P. Liu in [8] (see also [14]) is verified by this solution, and that implies uniqueness. Writing at a point (x, t) of discontinuity of the solution

$$\sigma(k, l; u_1, v_1) = \frac{f(k) - f(u_1)}{l - u_1}, \tag{18}$$

with $u_1 = u(x-0, t)$, $v_1 = v(x-0, t)$, and (k, l) satisfying the shock condition

$$(l - v_1)^2 = (f(u_1) - f(k))(u_1 - k), \tag{19}$$

this condition is the following, with $u_2 = u(x+0, t)$, $v_2 = v(x+0, t)$

$$\forall\, k \in I(u_1, u_2) \quad \sigma(k, l; u_1, v_1) \leqslant \sigma(u_2, v_2; u_1, v_1) \quad \left(= -\frac{dx}{dt}\right). \tag{20}$$

Here, $I(\alpha, \beta)$ is a notation for the interval $[\mathrm{Inf}(\alpha, \beta), \mathrm{Sup}(\alpha, \beta)]$.

For instance, we get, when the propagation has a positive speed, that (20) implies

$$\frac{f(u_2) - f(u_1)}{u_2 - u_1} = \operatorname*{Min}_{k \in I(u_2, u_1)} \left\{ \frac{f(k) - f(u_1)}{k - u_1} \right\}, \tag{21}$$

which is exactly the entropy condition (E) of O. A. Oleinik in [14]; (20) is its generalization for the system (4), (5). A maximum appears in (21) for a shock with a negative speed.

From this construction of the solution of the Riemann problem, we are able to propose a construction of some numerical schemes. Their stability, which depends on the stability of the Riemann problem and is ensured when $|f|$ is a convex function, will be stated later, in Sections IV and V.

III. Approximation by Numerical Schemes. For $I \in \mathbf{N}$ and $q > 0$, let $h = 1/I$ be the space meshsize and qh the time meshsize. We introduce the intervals

$$I_i = \left[\left(i - \tfrac{1}{2}\right)h, \left(i + \tfrac{1}{2}\right)h\right[\quad \text{for } i = 0, 1, \ldots, I,$$
$$J_n = [nqh, (n+1)qh[\quad \text{for } n = 0, 1, \ldots, N = 1 + [T/qh].$$

The solution of (4), (5), (6), (7), (8) will be approached by (u_h, v_h), which has a constant value (u_i^n, v_i^n) on each $I_i \times J_n$. The initial condition is introduced by

$$u_i^0 = \frac{1}{h} \int_{I_i} u_0(x)\, dx, \quad v_i^0 = \frac{1}{h} \int_{I_i} v_0(x)\, dx, \qquad i \in \{1, \ldots, I - i\}, \tag{22}$$

and the boundary conditions by

$$v_0^n = v_I^n = 0, \qquad n = 0, \ldots, N. \tag{23}$$

Let $n \geqslant 0$. We suppose that all the values (u_i^n, v_i^n) for $i = 1, \ldots, I-1$ are known, and we propose to build the (u_i^{n+1}, v_i^{n+1}). Solving the Riemann problem (4), (5), (9), with the initial condition translated of $(i + \frac{1}{2})h$ given by

$$(u_+, v_+) = (u_{i+1}^n, v_{i+1}^n), \quad (u_-, v_-) = (u_i^n, v_i^n),$$

for $x = (i + \frac{1}{2})h$ and $t > nqh$, $i \in \{1, \ldots, I-2\}$, we find that the solution is equal to a constant $(u_{i+1/2}^n, v_{i+1/2}^n)$ at these points, which is defined by

$$v_{i+1/2}^n = v_{i+1}^n + G(u_{i+1}^n, u_{i+1/2}^n) = v_i^n - G(u_i^n, u_{i+1/2}^n), \tag{24}$$

from (15) and (16). This is also true for the solution of the problem (4), (5) with the Cauchy data $u_h(x, nqh)$ at the time $t = nqh$, when two adjacent Riemann problems cannot interfere with themselves; that is when $(t - nqh)$ is small enough. This will correspond to a stability condition which requires that the speed is still less than $1/2q$; that is more restrictive than the well-known Courant-Friedrichs-Lewy condition, for which we have $1/q$.

For $x = h/2$, we consider only the wave with a positive speed which gives a value $(u_{1/2}^n, v_{1/2}^n)$ defined by

$$v_1^n = -G(u_1^n, u_{1/2}^n), \quad v_{1/2}^n = 0. \tag{25}$$

Similarly for $x = (I - 1/2)h$, a value $(u_{I-1/2}^n, v_{I-1/2}^n)$ is defined by

$$v_{I-1}^n = G(u_{I-1}^n, u_{I-1/2}^n), \quad v_{I-1/2}^n = 0. \tag{26}$$

Next, for $i = 1, \ldots, I - 1$, we integrate by parts the equations (4), (5), on the set $I_i \times J_n$, on three sides of which u and v are known. This gives the L^2-projection (u_i^{n+1}, v_i^{n+1}) on the fourth side; we get

$$u_i^{n+1} = u_i^n + q(v_{i+1/2}^n - v_{i-1/2}^n), \quad v_i^{n+1} = v_i^n + q(f(u_{i+1/2}^n) - f(u_{i-1/2}^n)). \tag{27}$$

The scheme (24), (27) is a generalization of the Godunov scheme. For the linear problem, that is for $f(u) = k^2u$ with $k > 0$, (24) and (27) are equivalent to the upstreaming scheme on each Riemann invariant (see [4]),

$$\begin{aligned} (v + ku)_i^{n+1} &= (v + ku)_i^n + qk\{(v + ku)_{i+1}^n - (v + ku)_i^n\}, \\ (v - ku)_i^{n+1} &= (v - ku)_i^n - qk\{(v - ku)_i^n - (v - ku)_{i-1}^n\}. \end{aligned} \tag{28}$$

To compute $u_{i+1/2}^n$ from (24) is often hard when shocks occur. A small error is made by replacing $G(u, u_0)$ by $g(u) - g(u_0)$, even when u_i^n is not close to u_{i+1}^n. That is similar to write that only rarefaction waves can happen. Nevertheless, we get another scheme, which is easier to compute and gives relatively good profiles for shocks. This is the following

$$\begin{aligned} v_{i+1/2}^n &= \frac{1}{2}\{v_{i+1}^n + v_i^n + g(u_{i+1}^n) - g(u_i^n)\}, \\ g(u_{i+1/2}^n) &= \frac{1}{2}\{v_{i+1}^n - v_i^n + g(u_{i+1}^n) + g(u_i^n)\}, \end{aligned} \tag{29}$$

and (27) as above. Near the boundaries, we set

$$v_{1/2}^n = v_{I-1/2}^n = 0, \quad g(u_{1/2}^n) = v_1^n + g(u_1^n), \quad g(u_{I-1/2}^n) = -v_{I-1}^n + g(u_{I-1}^n). \tag{30}$$

This scheme is also the same scheme as (28) for the linear problem.

The Lax-Friedrichs scheme may be fashioned the same way. We suppose that I is even and take only the intervals I_j with a length equal to $2h$, at time nqh, with $j + n$ odd. Solving the Riemann problem (4), (5), (9), with

$$(u_+, v_+) = (u_{i+1}^n, v_{i+1}^n), \quad (u_-, v_-) = (u_{i-1}^n, v_{i-1}^n),$$

and integrating (4) and (5) by parts on $I_i \times J_n$, we get, as above, the L^2-projection of its solution on I_i at time $(n + 1)qh$. This is the value (u_i^{n+1}, v_i^{n+1}) given by

$$\begin{aligned} u_i^{n+1} &= \frac{1}{2}\{u_{i+1}^n + u_{i-1}^n\} + \frac{q}{2}\{v_{i+1}^n - v_{i-1}^n\}, \\ v_i^{n+1} &= \frac{1}{2}\{v_{i+1}^n + v_{i-1}^n\} + \frac{q}{2}\{f(u_{i+1}^n) - f(u_{i-1}^n)\}, \end{aligned} \tag{31}$$

which is the Lax-Friedrichs scheme. This is available for two adjacent Riemann problems provided that the solution is constant at $x = ih$, for $t \in J_n$, which requires the Courant-Friedrichs-Lewy condition.

The boundary conditions are introduced as follows, for n even,

$$v_0^{n+1} = v_I^{n+1} = 0, \quad u_0^{n+1} = u_1^n + qv_1^n, \quad u_I^{n+1} = u_{I-1}^n - qv_{I-1}^n. \tag{32}$$

No conditions are needed for n odd in (31). We have, from (31) and (32), for n odd

$$\sum_{1<i<I-1} u_i^{n+1} = \frac{1}{2}(u_0^n + u_I^n) + \sum_{2<j<I-2} u_j^n = \sum_{1<i<I-1} u_i^{n-1}.$$

Since the restrictions of I_0 and I_I on [0, 1] have a length equal to h, this gives the conservation of the integral of u. We have for all n

$$\int_0^1 u_h(x, (n+1)qh)\, dx = \int_0^1 u_h(x, nqh)\, dx. \tag{33}$$

From (27), this conservation is obviously realized by the two other schemes.

IV. A Model of Extensible Strings. A string of length l is initially put to the length $l(1 + \lambda)$, where λ is a nonnegative constant which is proportional to the tension. Next, initial position and speed are prescribed at each point of this string, which is fastened at each bound. Neglecting the gravity, the transversal displacement $w(x, t)$ of the point $x \in]0, l[$ of the string, in its steady state, follows the wave nonlinear equation

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial}{\partial x}\left[f\left(\frac{\partial w}{\partial x}\right)\right],$$

where f is given by

$$f(u) = cu\left[1 - \frac{1}{1+\lambda}\left\{1 + \frac{u^2}{(1+\lambda)^2}\right\}^{-1/2}\right],$$

with c depending on the mechanical properties of the string. This function f is odd and strictly increasing; it admits the two asymptotes $c(u \pm 1)$, and f' is bounded by c.

Moreover $|f|$ is a convex function. For u close to zero, we have, by a Taylor expansion up to the order 3 (a greater order does not give an increasing function),

$$f(u) \sim c\frac{\lambda}{1+\lambda}u + \frac{c}{2}\frac{u^3}{(1+\lambda)^3}.$$

The linear model corresponds to the first term for $\lambda \neq 0$. If λ is equal to zero, that is when the string is submitted to no initial tension, the linear model has no meaning, while a nonlinear model with f or its equivalent $cu^3/2$ may be used. The function g corresponding to f by (11) to solve (29) is not easy to compute, but this step may be realized by an approximation technique. Solving (24) is still harder and also needs a numerical method; for example, a few iterations of the Newton method (the derivative is known) may be performed.

When f is supposed to be an increasing function, convex for $u \geqslant 0$ and concave for $u \leqslant 0$, which is equivalent to say that $|f|$ is a convex function if $f(0)$ is zero, the stability of the Riemann problem (4), (5), (9) is ensured.

THEOREM 1. *For $|f|$ convex, the Riemann problem* (4), (5), (9) *has a unique weak solution, whose values lie in the bounded convex set of the phase plane*

$$K_0 = \{(u, v) \in \mathbf{R}^2, |v| + |g(u)| \leqslant M_0\}, \tag{34}$$

with $M_0 = \operatorname{Max}\{|v_-| + |g(u_-)|, |v_+| + |g(u_+)|\}$.

This result is similar to those of T. P. Liu, J. Smoller, or B. L. Keyfitz and H. C. Kranzer. See [2], [8], [9], [10], [15], [16].

Proof. Since $|g|$ is convex, (17) is satisfied and K_0 is obviously a bounded convex set. The existence and the uniqueness follow from the construction which is developed above, and from T. P. Liu [8], provided that the entropy condition (20) is verified.

The stationary value (u_0, v_0) is given, as previously, from (14) by

$$v_+ = v_0 - G(u_+, u_0), \quad v_- = v_0 + G(u_-, u_0).$$

Two main cases can occur to prove that the solution has all its values in K_0.

Case 1. $u_0 \notin I(u_-, u_+)$. If $u_0 \geqslant \operatorname{Max}(u_-, u_+)$, then $v_- \leqslant v_0 \leqslant v_+$, and the solution presents two shocks, with speeds of different sign. Let us look at the wave which has a positive speed. A unique value $\bar{u}$ exists such that the concave hull of f on $[u_+, u_0]$ coincides with f on $[u_+, \bar{u}]$ and is a straight line of slope

$$(f(u_0) - f(\bar{u}))/(u_0 - \bar{u})$$

on $[\bar{u}, u_0]$. For $u_+ < 0$, $\bar{u}$ is negative and not equal to u_+; a rarefaction wave appears, which is ruled by the scalar equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}[g(u)] = 0,$$

with $u \in [u_+, \bar{u}]$ and $v = v_+ + g(u_+) - g(u)$.

Since $v_- \leqslant v_0 \leqslant v \leqslant v_+$, $u_0 \geqslant \bar{u} \geqslant u \geqslant u_+$, we have

$$|v + g(u)| = |v_+ + g(u_+)| \leqslant M_0,$$

and

$$v - g(u) \leqslant v_+ - g(u_+) \leqslant M_0, \qquad v - g(u) \geqslant v_0 - g(u_0).$$

The values $\bar{u}$ and u_0 are those on both sides of a shock, which appears when u_0 is positive. By the Cauchy-Schwarz inequality, we get

$$g(u_0) - g(\bar{u}) = \int_{\bar{u}}^{u_0} \sqrt{f'(y)\, dy} \leqslant \{[f(u_0) - q(\bar{u})][u_0 - \bar{u}]\}^{1/2},$$

thus, we have

$$g(u_0) - g(\bar{u}) \leqslant -G(\bar{u}, u_0).$$

Since

$$g(u_+, u_0) = g(u_+) - g(\bar{u}) + G(\bar{u}, u_0),$$

that is, exactly,

$$v_0 + g(u_0) \leqslant \bar{v} + g(\bar{u}) = v_+ + g(u_+) \leqslant M_0,$$

we have also

$$v_0 - g(u_0) \leqslant v_+ - g(u_+) \leqslant M_0.$$

For the wave with a negative speed, a value $\underline{u}$, which is negative or equal to u_-, exists such that a rarefaction wave with values in $[u_-, \underline{u}]$ and a shock appears. We get similar estimates as above for the rarefaction wave

$$|v - g(u)| = |v_- - g(u_-)| \leqslant M_0, \quad v + g(u) \geqslant v_- + g(u_-) \geqslant -M_0,$$
$$v + g(u) \leqslant v_0 + g(u_0),$$

which is less than $v_+ + g(u_+)$, as proved above. By the Cauchy-Schwarz inequality, we get

$$g(u_0) - g(u_-) \leqslant -G(u_-, u_0),$$

which gives

$$v_0 - g(u_0) \geqslant v_- - g(u_-) \geqslant -M_0.$$

For the rarefaction wave with a positive speed, we have now $v - g(u) \geqslant -M_0$. We have also

$$v_0 + g(u_0) \geqslant v_- + g(u_-) \geqslant -M_0,$$

and then all the values of the solution belong to K_0.

When a shock with a positive speed appears, we have

$$\operatorname*{Min}_{k \in [\bar{u}, u_0]} \frac{f(k) - f(u_0)}{k - u_0} = \frac{f(\bar{u}) - f(u_0)}{\bar{u} - u_0},$$

from a property of the concave hull. This is the entropy condition (21). A similar condition is obtained in the same way for a shock with a negative speed. The same estimates may be derived when $u_0 < \operatorname{Min}(u_-, u_+)$ since $v_- \geqslant v_0 \geqslant v_+$, and a convex hull arises in this case.

Case 2. $u_0 \in I(u_-, u_+)$. If $u_+ \leqslant u_-$, then $v_0 \leqslant \operatorname{Max}(v_-, v_+)$, and if $u_0 > 0$, a rarefaction wave with a negative speed appears such that

$$v_- - g(u_-) = v - g(u), \quad u_0 \leqslant u \leqslant u_-, \quad v_0 \leqslant v \leqslant v_-.$$

For $v > 0$, we have

$$|v| + |g(u)| = v + g(u) \leqslant v_- + g(u_-) \leqslant M_0,$$

and, for $v > 0$,

$$|v| + |g(u)| = -v + g(u) = -(v_- - g(u_-)) \leqslant M_0.$$

The wave with a positive speed may contain a rarefaction wave and a shock, whose values go from u_0 to $\bar{u}$, defined as above. We prove that the values of the solution lie in K_0 and that the entropy condition is verified by using the same arguments as in the first case, since $u_0 > u_+$. For $u_0 < 0$ or for $u_- \leqslant u_+$, we have a similar proof. Theorem 1 is proved.

V. Stability of the Numerical Schemes. We suppose that $|f|$ is a convex function and consider the problem (4), (5) with the initial and boundary conditions (6), (7), (8). We recall that f is an increasing function and that u_0 and v_0 belong to $L^\infty(]0, 1[)$. A convex bounded set of the phase plane K_0 is defined as in (34), with

$$M_0 = |v_0|_{L^\infty(0,1)} + |g(u_0)|_{L^\infty(0,1)}.$$

We have the following

THEOREM 2. *For any $h > 0$, the approximate solution, built by the Lax-Friedrichs scheme* (31) *or by one of the two schemes* (24), (27) *and* (29), (27), *has all its values in K_0 if the stability condition*

$$q \sup_{|g(k)| \leqslant M_0} \sqrt{f'(k)} \leqslant \alpha, \tag{35}$$

is verified with $\alpha = 1$ for (31) *and $\alpha = 1/2$ for* (24), (27) *or* (29), (27).

Note that (35) is not really harder for (24), (27) or (29), (27) than for (31) since I_i has a length equal to $2h$ for this last scheme.

Proof. We first consider those schemes at an interior point (ih, nqh). The boundary points will be treated later. For the Lax-Friedrichs scheme (31), or for the scheme (24), (27), the value (u_i^{n+1}, v_i^{n+1}) is the L^2 projection of the solution of a Riemann problem, whose values lie in the convex set K_0. Obviously this projection also belongs to K_0. The stability condition (35) implies that two adjacent Riemann problems cannot produce two waves which meet each other in the time interval J_n.

The scheme (29), (27) does not correspond to the solution of a Riemann problem, and we have to consider several cases. In order to limit the number of these, we give the following form to (27)

$$u_{i,1}^{n+1} = u_i^n + 2q(v_{i+1/2}^n - v_i^n), \quad v_{i,1}^{n+1} = v_i^n + 2q\big(f(u_{i+1/2}^n) - f(u_i^n)\big), \tag{36}$$

$$u_{i,2}^{n+1} = u_i^n + 2q(v_i^n - v_{i-1/2}^n), \quad v_{i,2}^{n+1} = v_i^n + 2q\big(f(u_i^n) - f(u_{i-1/2}^n)\big). \tag{37}$$

We obviously have

$$u_i^{n+1} = \tfrac{1}{2}\big[u_{i,1}^{n+1} + u_{i,2}^{n+1}\big], \quad v_i^{n+1} = \tfrac{1}{2}\big[v_{i,1}^{n+1} + v_{i,2}^{n+1}\big].$$

Then (u_i^{n+1}, v_i^{n+1}) belongs to K_0 if the values defined by (36), (37) are in K_0.

We suppose that all the (u_j^n, v_j^n) are in K_0, for $j = 1, \ldots, I-1$. By (29), we get that $(u_{j+1/2}^n, v_{j+1/2}^n)$ belongs to K_0 for $j = 1, \ldots, I-2$.

Let us consider (36) now. Three cases may appear:

Case 1. $0 \leqslant u_i^n \leqslant u_{i+1/2}^n$. We have by (29)

$$v_{i+1/2}^n - v_i^n = g(u_{i+1/2}^n) - g(u_i^n) \geqslant 0,$$

which gives

$$u_{i,1}^{n+1} = u_i^n + 2q\big(g(u_{i+1/2}^n) - g(u_i^n)\big).$$

Introducing $\eta \in [u_i^n, u_{i+1/2}^n]$ such that

$$g(u_{i+1/2}^n) - g(u_i^n) = \sqrt{f'(\eta)}\,(u_{i+1/2}^n - u_i^n),$$

we get

$$u_{i,1}^{n+1} = \big(1 - 2q\sqrt{f'(\eta)}\,\big)u_i^n + 2q\sqrt{f'(\eta)}\,u_{i+1/2}^n, \tag{38}$$

where both coefficients of u_i^n and $u_{i+1/2}^n$ are nonnegative by (35). Thus, we have $0 \leqslant u_i^n \leqslant u_{i,1}^{n+1} \leqslant u_{i+1/2}^n$.

On the other hand, we introduce $\xi \in [u_i^n, u_{i+1/2}^n]$ such that

$$f(u_{i+1/2}^n) - f(u_i^n) = \sqrt{f'(\xi)}\,\left(g(u_{i+1/2}^n) - f(u_i^n)\right),$$

by using the average formula

$$\int_{u_i^n}^{u_{i+1/2}^n} f'(y)\,dy = \sqrt{f'(\xi)} \int_{u_i^n}^{u_{i+1/2}^n} \sqrt{f'(y)}\,dy.$$

From (29) we get

$$v_{i,1}^{n+1} = v_i^n + 2q\sqrt{f'(\xi)}\,(v_{i+1/2}^n - v_i^n), \tag{39}$$

and by (35) $v_{i,1}^{n+1}$ belongs to $[v_i^n, v_{i+1/2}^n]$, as above for $u_{i,1}^{n+1}$.

Now, we have, since g is an increasing function,

$$v_{i+1/2}^n + g(u_{i+1/2}^n) \geqslant v_{i,1}^{n+1} + g(u_{i,1}^{n+1}) \geqslant v_i^n + g(u_i^n), \tag{40}$$

and from $g(u_{i,1}^{n+1}) \geqslant 0$ we get

$$v_{i,1}^{n+1} - g(u_{i,1}^{n+1}) \leqslant v_{i,1}^{n+1} + g(u_{i,1}^{n+1}). \tag{41}$$

We have, for some ζ in $[u_i^n, u_{i,1}^{n+1}]$,

$$g(u_{i,1}^{n+1}) - g(u_i^n) = \sqrt{f'(\zeta)}\,(u_{i,1}^{n+1} - u_i^n),$$

thus,

$$v_{i,1}^{n+1} - g(u_{i,1}^{n+1}) = v_i^n - g(u_i^n) + 2q\left(\sqrt{f'(\xi)} - \sqrt{f'(\zeta)}\right)(v_{i+1/2}^n - v_i^n). \tag{42}$$

By the Cauchy-Schwarz inequality,

$$g(u_{i+1/2}^n) - g(u_i^n) \leqslant \left\{\left[f(u_{i+1/2}^n) - f(u_i^n)\right]\left[u_{i+1/2}^n - u_i^n\right]\right\}^{1/2},$$

which gives to the power two

$$\sqrt{f'(\xi)} = \frac{f(u_{i+1/2}^n) - f(u_i^n)}{g(u_{i+1/2}^n) - g(u_i^n)} \geqslant \frac{g(u_{i+1/2}^n) - g(u_i^n)}{u_{i+1/2}^n - u_i^n} = \sqrt{f'(\eta)}\,.$$

Since g is a convex increasing function on $[u_i^n, u_{i+1/2}^n]$, we have

$$\forall\, k \in \left[u_i^n, u_{i+1/2}^n\right] \frac{g(k) - g(u_i^n)}{k - u_i^n} \leqslant \sqrt{f'(\eta)}\,,$$

thus, for $k = u_{i,1}^{n+1}$, we obtain $\sqrt{f'(\zeta)} \leqslant \sqrt{f'(\eta)} \leqslant \sqrt{f'(\xi)}$.

The last term in (42) is now nonnegative, and we get

$$v_{i,1}^{n+1} - g(u_{i,1}^{n+1}) \geqslant v_i^n - g(u_i^n). \tag{43}$$

We deduce from (40), (41), (43) that $(u_{i,1}^{n+1}, v_{i,1}^{n+1})$ belongs to K_0.

Case 2. $u_i^n \leqslant u_{i+1/2}^n \leqslant 0$. The function f is concave on $[u_i^n, u_{i+1/2}^n]$ and

$$G(u_i^n, u_{i+1/2}^n) = g(u_i^n) - g(u_{i+1/2}^n).$$

Then we solve a particular Riemann problem, and (36) gives its L^2-projection. We obtain that $(u_{i,1}^{n+1}, v_{i,1}^{n+1})$ belongs to K_0.

Case 3. $u_i^n \leqslant 0 \leqslant u_{i+1/2}^n$. We get, from (38), (39),

$$u_i^n \leqslant u_{i,1}^{n+1} \leqslant u_{i+1/2}^n, \quad v_i^n \leqslant v_{i,1}^{n+1} \leqslant v_{i+1/2}^n.$$

Thus we get (40). We suppose now that $u_{i,1}^{n+1}$ is negative and have, obviously,

$$v_{i,1}^{n+1} - g(u_{i,1}^{n+1}) \geqslant v_i^n + g(u_i^n). \tag{44}$$

We also have

$$v_{i,1}^{n+1} - g(u_{i,1}^{n+1}) \leqslant v_{i+1/2}^n \quad (\leqslant v_{i+1/2}^n + g(u_{i+1/2}^n)). \tag{45}$$

To prove (45), we state the following formula for $a\ (= u_i^n) < 0 < x\ (= u_{i+1/2}^n)$,

$$F(x, a) = 2q\left\{ f(x) - f(a) - \frac{1}{2q}\int_a^{a+2q(g(x)-g(a))} \sqrt{f'(y)}\, dy \right\} - g(x) \leqslant 0. \tag{46}$$

This is true for $x = 0$, as it was shown in Case 2. We have

$$\frac{\partial F}{\partial x}(x, a) = \left[2q\left\{\sqrt{f'(x)} - \sqrt{f'(a + 2q(g(x) - g(a)))}\ \right\} - 1\right]\sqrt{f'(x)}\ ,$$

which is negative by (35). We now integrate it with respect to x on $[0, x]$ and add $F(0, a) \leqslant 0$, to get (46).

The inequality (45) follows from

$$v_{i,1}^{n+1} - g(u_{i,1}^{n+1}) = v_{i+1/2}^n + F(u_{i+1/2}^n, u_i^n) \leqslant v_{i+1/2}^n,$$

and we obtain from (40), (44) and (45) that $(u_{i,1}^{n+1}, v_{i,1}^{n+1})$ belongs to K_0.

For $u_{i,1}^{n+1} > 0$, we use similar arguments. We get, obviously,

$$v_{i,1}^{n+1} - g(u_{i,1}^{n+1}) \leqslant v_{i+1/2}^n + g(u_{i+1/2}^n), \tag{47}$$

and we prove the formula

$$H(x, a) = 2q\left\{ f(x) - f(a) - \frac{1}{2q}\int_0^{a+2q(g(x)-g(a))} \sqrt{f'(y)}\, dy \right\} \geqslant 0. \tag{48}$$

We have

$$\frac{\partial H}{\partial x}(x, a) = 2q\sqrt{f'(x)}\ \left\{\sqrt{f'(x)} - \sqrt{f'(a + 2q(g(x) - g(a)))}\ \right\},$$

which is nonnegative from the convexity of f, integrate on $[x_0, x]$ with x_0 such that

$$a + 2q(g(x_0) - g(a)) = 0,$$

and add $H(x_0, a)$ which is obviously positive. We derive from (48) that

$$v_{i,1}^{n+1} - g(u_{i,1}^{n+1}) \geqslant v_i^n > v_i^n + g(u_i^n),$$

and then $(u_{i,1}^{n+1}, v_{i,1}^{n+1})$ belongs to K_0.

All other cases, that is when $u_{i+1/2}^n < u_i^n$, are similar to one of the three cases above, and an analogue of this proof would show that $(u_{i,2}^{n+1}, v_{i,2}^{n+1})$ belongs to K_0.

We now consider a boundary point, for example at $x = 0$. For the Lax-Friedrichs scheme (31), this occurs only for n even, and (u_0^{n+1}, v_0^{n+1}) is given by (32). The straight line

$$v = v_1^n + \frac{1}{q}(u_1^n - u)$$

cuts the axis $v = 0$ inside K_0, by (35), thus (u_0^{n+1}, v_0^{n+1}) belongs to K_0.

For the scheme (24), (27), we get that $(u_{1/2}^n, v_{1/2}^n)$, defined by (25), is in K_0 by writing that the part of the generalized Riemann invariant with a positive speed, from (u_1^n, v_1^n) to its intersection with the axis $v = 0$, lies in K_0. This uses similar

arguments as those for Theorem 1. Now (27) is the L^2-projection of a solution with values in the convex set K_0, thus (u_1^{n+1}, v_1^{n+1}) is in K_0.

For the scheme (29), (27), $(u_{1/2}^n, v_{1/2}^n)$ obviously belongs to K_0 and then (u_1^{n+1}, v_1^{n+1}) is in K_0, from the first part of this proof.

Similar arguments may be used to state the stability at $x = 1$, and then Theorem 2 is proved.

VI. Conclusion. The technique used to build the schemes (24), (27) and (29), (27) is suitable for other first order hyperbolic systems such as the shallow water model or some equations of gas dynamics, as stated in the introduction; see [5], [6], [7].

The stability of the Glimm scheme (see [1]) may be deduced from Theorem 1. Indeed, this scheme uses the same mesh as the Lax-Friedrichs scheme, with an interpolation at a random point instead of a projection, to build (u_i^{n+1}, v_i^{n+1}) at each step. Note that the convexity is not used explicitly.

The numerical tests show a good treatment of the reflection of the waves on the boundaries, by (25), (26), (30) or (32). From (33), it is easy to come back to the wave equation (1) since the boundary conditions (3) are satisfied. The schemes (24), (27) and (29), (27) give two approximate solutions whose difference is very small. These two schemes do not depend on the value of q, provided that the stability condition (35) is verified; moreover, the stability seems to remain true even for $\alpha = 1$.

The Lax-Friedrichs scheme gives results which are subordinate to the value of q, the best of them are found with the greatest value of q satisfying (35). However, the speeds are weakened, which implies that the displacement is deadened. This phenomenon does not appear with significance for the two other schemes, for which the amount of viscosity is smaller; they may be corrected up to order two but on a few points; see [4].

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