

Multiple Grid Methods for the Solution of Fredholm Integral Equations of the Second Kind

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Abstract. In this paper multiple grid methods are applied for the fast solution of the large nonsparse systems of equations that arise from the discretization of Fredholm integral equations of the second kind. Various multiple grid schemes, both with Nyström and with direct interpolation, are considered. For these iterative methods, the rates of convergence are derived using the collectively compact operator theory by Anselone and Atkinson. Estimates for the asymptotic computational complexity are given, which show that the multiple grid schemes result in $\mathcal{O}(N^2)$ arithmetic operations.

1. Introduction. Multiple grid methods have been advocated by Brandt [5], [6] for solving sparse systems of equations that arise from discretization of partial differential equations. Convergence and computational complexity of such multiple grid techniques have been studied since by Hackbusch [7], [8] and Wesseling [12], [13]. We intend to show that multiple grid methods can also be used advantageously for the nonsparse systems that occur in numerical methods for integral equations.

In a recent paper [10], the second author applied the multiple grid technique to the solution of Fredholm integral equations of the second kind

$$(1.1) \quad f(x) - \int_0^1 k(x, y)f(y) dy = g(x), \quad x \in [0, 1],$$

where g belongs to a Banach space X . At the same time, Hackbusch [7] also used a multiple grid technique for these problems. Moreover, he gave a proof of convergence. In this proof he assumed the operator K , associated with the kernel $k(x, y)$, to map from X into a "smooth" subspace $\tilde{X} \subset X$, which has a stronger topology. In the present paper, for Hackbusch's method we give another proof, which fits into the theoretical framework developed by Anselone [1] and Atkinson [2], [3] for Fredholm equations. We assume that K is compact from X into X . In contrast to Hackbusch's analysis, this approach enables us to consider also Nyström interpolation as a permissible interpolation method. In addition, we introduce a new multiple grid method for Fredholm integral equations, which can deal with a larger class of problems than the method proposed by Hackbusch.

In 1978 Stetter [11] introduced the Defect Correction Principle for the formulation of various iterative methods. We shall apply this principle because it also appears to be an expedient tool to formulate multiple grid techniques.

Received January 17, 1980.

1980 *Mathematics Subject Classification.* Primary 65B99, 65F10, 65R20.

Key words and phrases. Fredholm integral equations of the second kind, multiple grid methods.

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0025-5718/81/0000-0017/\$05.50

In Section 2 we collect some results from papers by Atkinson [2], [3] and Prenter [9]. In Section 3 we cast the iterative schemes of Brakhage [4] and Atkinson [2], [3] into the context of the Defect Correction Principle and multiple grid iteration. Furthermore, we give the proof of convergence of the multiple grid schemes with Nyström interpolation. In Section 4 we treat other interpolation methods and we extend the iterative schemes of Section 3 for subspaces X_p of X of finite dimension N_p . These schemes are used as a basis for the construction of a general algorithm for the solution of Fredholm equations of the second kind. This algorithm is more efficient than the algorithms by Brakhage [4] and Atkinson [2], [3] because these schemes take $\mathcal{O}(N_p^3)$ and $\mathcal{O}(N_p^2 \log N_p)$ operations, respectively, whereas the multiple grid schemes result in $\mathcal{O}(N_p^2)$ operations. In Section 5 we illustrate the theoretical results of the previous sections by some numerical examples and we comment on the computational complexity.

2. Basic Assumptions. Equation (1.1) can be written symbolically as

$$(2.1) \quad Af = g, \quad g \in X,$$

where X is a Banach space and $A = I - K$, with I the identity operator on X and K the linear operator associated with the kernel $k(x, y)$. A is assumed to have a bounded inverse on X . We shall discuss the convergence of a sequence of approximations to the unique solution of (2.1).

Let X_p , $p = 0, 1, 2, \dots$, be finite-dimensional subspaces of X and let T_p , $p = 0, 1, 2, \dots$, be a bounded projection operator from X onto X_p , i.e. $T_p f = f$ for all $f \in X_p$. We need the following assumptions for $\{X_p\}$ and $\{T_p\}$

- (A1) $X_0 \subset X_1 \subset \dots \subset X_p \subset \dots \subset X$,
- (A2) $\lim_{p \rightarrow \infty} \|f - T_p f\| = 0$ for all $f \in X$.

LEMMA 2.1.

$$C_1 = \sup_{p > 0} \|T_p\| < \infty.$$

Proof. The lemma follows from the principle of uniform boundedness; see Atkinson [3, p. 18]. \square

The sequence $\{X_p\}$ is thought to be associated with a sequence of decreasing meshsizes $\{h_p\}$ with $\lim_{p \rightarrow \infty} h_p = 0$. Corresponding with this sequence $\{h_p\}$, we approximate K by a sequence of operators $\{K_p\}$, $K_p: X \rightarrow X$. Analogous to $A = I - K$, we also write $A_p = I - K_p$. In the context of multiple grid iteration, the subscript p is called "level".

We use the following assumptions on K_p , $p = 0, 1, 2, \dots$,

- (A3) K_p is a linear operator $X \rightarrow X$.
- (A4) $\{K_p\}$ is a collectively compact family of operators.
- (A5) $\lim_{p \rightarrow \infty} \|K_p f - Kf\| = 0$ for all $f \in X$.
- (A6) $K_p = K_p T_p$.

LEMMA 2.2. *From the assumptions (A3)–(A5) follow:*

- (i) K should be compact.
- (ii) The sequence $\{K_p\}$ is uniformly bounded, i.e.

$$C_2 = \sup_{p > 0} \|K_p\| < \infty.$$

(iii) $\lim_{p \rightarrow \infty} \|(K - K_p)M\| = 0$ for any compact operator $M: X \rightarrow X$.

(iv) Let

$$a_p = \sup_{q > p} \sup_{l > 0} \|(K - K_q)K_l\|.$$

Then $\lim_{p \rightarrow \infty} a_p = 0$.

Proof. See Atkinson [3, pp. 96 and 138]. \square

LEMMA 2.3. Let the finite-dimensional subspace $X_0 \subset X$ be sufficiently large (i.e. the meshwidth of the coarsest discretization is sufficiently small). From the existence of a bounded inverse of $A = I - K$ and the assumptions (A3)–(A5) follow:

(i) $(I - K_p)^{-1}$ exists on X for $p \geq 0$ and

$$C_3 = \sup_{p > 0} \|(I - K_p)^{-1}\| < \infty.$$

(ii) $\|f - f_p\| \leq C_3 \|Kf - K_p f\|$, where f is the solution of (2.1) and f_p of

$$(2.2) \quad (I - K_p)f_p = g.$$

Proof. See Atkinson [2, p. 18]. \square

The following lemma is a summary of results given by Prenter [9].

LEMMA 2.4. From the assumptions (A2)–(A6) follow:

(i) For any compact operator M on X into X ,

$$\lim_{p \rightarrow \infty} \|(I - T_p)M\| = 0.$$

(ii) If X_0 is sufficiently large, then $(I - T_p K_p)^{-1}$ exists on X for $p \geq 0$ and

$$C_4 = \sup_{p > 0} \|(I - T_p K_p)^{-1}\| < \infty.$$

Let \tilde{f}_p be a solution of

$$(2.3) \quad (I - T_p K_p)\tilde{f}_p = T_p g.$$

According to Lemma 2.4(ii), \tilde{f}_p exists and is unique; it follows from (2.3) that $\tilde{f}_p \in X_p$.

LEMMA 2.5. Let

$$b_p = \sup_{q > p} \sup_{l > 0} \|(I - T_q)K_l\|.$$

Then $\lim_{p \rightarrow \infty} b_p = 0$.

Proof. Let $\Psi = \{K_p f \mid p \geq 0 \text{ and } \|f\| < 1\}$. By assumption (A4), Ψ has compact closure in the Banach space X . Then

$$b_p = \sup_{q > p} \sup_{z \in \Psi} \|(I - T_q)z\|$$

and the proof follows by assumption (A2). \square

LEMMA 2.6. Let the subspace $X_0 \subset X$ be sufficiently large. Then

$$\|\tilde{f}_p - T_p f\| \leq C_1 C_4 \|Kf - K_p f\|,$$

where f is the solution of (2.1).

Proof. See Prenter [9, Theorem 6.3]. \square

As a consequence of assumption (A1), the following lemma is trivial.

LEMMA 2.7. *Let $q \leq p$, i.e. $\dim(X_q) \leq \dim(X_p)$. Then*

$$T_p T_q = T_q T_p = T_q.$$

3. Iteration Schemes With Nyström Interpolation. In this section we use the Defect Correction Principle (cf. Stetter [11]) to formulate a class of iterative methods for the solution of (2.2). This equation is written as

$$(3.1) \quad A_p f_p = g, \quad g \in X,$$

with $A_p = I - K_p$. The defect correction principle defines the following iterative process:

$$(3.2) \quad \begin{cases} f_{p,0} = 0, \\ f_{p,i+1} = B_p g + (I - B_p A_p) f_{p,i}. \end{cases}$$

Here B_p denotes some approximate inverse of A_p , which is bijective and continuous in X . The solution f_p of (3.1) is a fixed point of (3.2), and (3.2) will converge to f_p if the rate of convergence $\|I - B_p A_p\| < 1$.

Several well-known iterative schemes for solving Fredholm integral equations of the second kind can be formulated within this framework. The iterative scheme of Brakhage [4] is obtained by taking the following approximate inverse

$$(3.3) \quad B_p^{(1)} = I + (I - K_{p-1})^{-1} K_p.$$

Here we notice that the operator $(I - K_q)^{-1}$, $q \geq 0$, as a mapping on X into X describes the process of discretization, solution of the discrete problem (i.e. the solution of a square linear system), and subsequent Nyström interpolation; see e.g. [10]. Other kinds of interpolation are treated in the next section.

The second iterative scheme of Atkinson [2, p. 19], arises from

$$(3.4) \quad B_p^{(2)} = I + (I - K_0)^{-1} K_p.$$

The rates of convergence of the corresponding iterative processes are estimated in the following theorem.

- THEOREM 3.1. (i) $\|I - B_p^{(1)} A_p\| \rightarrow 0$ as $p \rightarrow \infty$,
(ii) $\|I - B_p^{(2)} A_p\| \leq C(X_0)$ as $p \rightarrow \infty$, $C(X_0) < 1$ for X_0 sufficiently large.

Proof. (i) Substitution of the explicit expressions for A_p and $B_p^{(1)}$ yields

$$\begin{aligned} I - B_p^{(1)} A_p &= I - \{I + (I - K_{p-1})^{-1} K_p\} (I - K_p) \\ &= K_p - (I - K_{p-1})^{-1} K_p (I - K_p) = (I - K_{p-1})^{-1} (K_p - K_{p-1}) K_p. \end{aligned}$$

From Lemmas 2.2 and 2.3, we get the following bound for the norm

$$\|(I - K_{p-1})^{-1} (K_p - K_{p-1}) K_p\| \leq C_3 (a_p + a_{p-1}).$$

(ii) Analogously, we get, for $B_p^{(2)}$ with X_0 sufficiently large,

$$\|I - B_p^{(2)} A_p\| \leq C_3 (a_p + a_0) \equiv C(X_0).$$

From Lemma 2.2 it follows that $C(X_0) < 1$ for all sufficiently large X_0 . \square

We remark that the approximate inverses $B_p^{(1)}$ and $B_p^{(2)}$ use only two levels: $B_p^{(1)}$ uses the levels $p - 1$ and p , whereas $B_p^{(2)}$ uses the levels 0 and p . We now introduce approximate inverses $B_p^{(3)}$ and $B_p^{(4)}$, which use $p + 1$ levels. They are defined recursively as follows:

$$(3.5) \quad \begin{cases} B_0^{(3)} = (I - K_0)^{-1}, \\ B_p^{(3)} = I + Q_{p-1}^{(3)}K_p, \quad p = 1, 2, \dots, \end{cases}$$

and

$$(3.6) \quad \begin{cases} B_0^{(4)} = (I - K_0)^{-1}, \\ B_p^{(4)} = Q_{p-1}^{(4)}(I - K_{p-1} + K_p), \quad p = 1, 2, \dots, \end{cases}$$

with $Q_p^{(j)}, j = 3, 4, p = 0, 1, 2, \dots$, given by

$$Q_p^{(j)} = \sum_{m=0}^{\gamma-1} (I - B_p^{(j)}A_p)^m B_p^{(j)}$$

for some positive integer γ .

From the fact that $Q_p^{(j)}$ satisfies the equality

$$(I - Q_p^{(j)}A_p) = (I - B_p^{(j)}A_p)^\gamma,$$

we see that $Q_p^{(j)}$ is an approximate inverse of A_p and its application is equivalent with the application of γ iteration steps of (3.2) with the use of the approximate inverse $B_p^{(j)}$. In fact, this is the motivation for this definition of $Q_p^{(j)}$ and it is the basis for the actual (recursive) implementation of the method.

In the following definition we give a short notation for the rates of convergence for the various iterative processes.

Definition.

$$\zeta_p^{(j)} = \|I - B_p^{(j)}A_p\|, \quad j = 1, 2, 3, 4.$$

THEOREM 3.2. (i) $\zeta_p^{(3)} \leq \zeta_p^{(1)} + \zeta_{p-1}^{(3)\gamma}(\zeta_p^{(1)} + \|K_p\|)$,
 (ii) $\zeta_p^{(4)} \leq \zeta_p^{(1)} + \zeta_{p-1}^{(4)\gamma}(\zeta_p^{(1)} + 1)$.

Proof. (i) By definition

$$I - B_p^{(1)}A_p = I - \{I + A_{p-1}^{-1}(I - A_p)\}A_p$$

and

$$\begin{aligned} I - B_p^{(3)}A_p &= I - \{I + [I - (I - B_{p-1}^{(3)}A_{p-1})^\gamma]A_{p-1}^{-1}(I - A_p)\}A_p \\ &= I - B_p^{(1)}A_p + (I - B_{p-1}^{(3)}A_{p-1})^\gamma A_{p-1}^{-1}(I - A_p)A_p \\ &= I - B_p^{(1)}A_p + (I - B_{p-1}^{(3)}A_{p-1})^\gamma (B_p^{(1)} - I)A_p. \end{aligned}$$

Hence,

$$\|I - B_p^{(3)}A_p\| \leq \|I - B_p^{(1)}A_p\| + \|I - B_{p-1}^{(3)}A_{p-1}\|^\gamma \{\|I - B_p^{(1)}A_p\| + \|I - A_p\|\},$$

i.e.

$$\zeta_p^{(3)} \leq \zeta_p^{(1)} + \zeta_{p-1}^{(3)\gamma}(\zeta_p^{(1)} + \|K_p\|).$$

(ii) Similar to the case (i), now we have

$$\begin{aligned} I - B_p^{(4)}A_p &= I - \left[I - (I - B_{p-1}^{(4)}A_{p-1})^\gamma \right] A_{p-1}^{-1}(A_{p-1} - A_p + I)A_p \\ &= I - B_p^{(1)}A_p + (I - B_{p-1}^{(4)}A_{p-1})^\gamma B_p^{(1)}A_p. \end{aligned}$$

Hence,

$$\|I - B_p^{(4)}A_p\| \leq \|I - B_p^{(1)}A_p\| + \|I - B_{p-1}^{(4)}A_{p-1}\|^\gamma \|I - B_p^{(1)}A_p - I\|,$$

i.e.

$$\zeta_p^{(4)} \leq \zeta_p^{(1)} + \zeta_{p-1}^{(4)\gamma}(\zeta_p^{(1)} + 1). \quad \square$$

By Theorem 3.1 we know that $\zeta_p^{(1)} \rightarrow 0$ as $p \rightarrow \infty$; conditions for $\zeta_p^{(3)}$ to vanish depend on γ , $\|K_p\|$, and $\zeta_p^{(1)}$, whereas the conditions for $\zeta_p^{(4)}$ depend on γ and $\zeta_p^{(1)}$ only. In order to study this dependence further, we prove the following lemma.

LEMMA 3.3. *Let $k \in \mathbf{R}$ and $\gamma \in \mathbf{N}$ be given, let $\{v_p \mid v_p > 0, p = 0, 1, 2, \dots\}$ be a nonincreasing sequence with $d = \inf_p v_p/v_{p-1}$ and let $\{w_p\}$ be defined by*

$$\begin{cases} w_0 = v_0 \\ w_p = v_p + w_{p-1}^\gamma(v_p + k). \end{cases}$$

If either

$$(i) \quad \begin{cases} \gamma \geq 1, & 0 < k < d < 1, \text{ and} \\ v_0 < \frac{1}{2}(1 - k/d), \end{cases}$$

or

$$(ii) \quad \begin{cases} \gamma \geq 2, \text{ and} \\ v_0 < \frac{1}{2} \left\{ \sqrt{1 + (k/d)^2} - k/d \right\}, \end{cases}$$

then a $C > 0$ exists such that $v_p \leq w_p \leq Cv_p$.

Proof. (i) We define $c = (d + k)/(d - k)$. Then $c > 1$ and the conditions on $\{v_p\}$ are written as

$$(3.7) \quad k \frac{c + 1}{c - 1} < \frac{v_p}{v_{p-1}} \leq 1$$

and

$$(3.8) \quad (1 + c)v_0 < 1.$$

We show that the lemma is true for $C = 1 + c$. From (3.8) we see $w_0 = v_0 < (1 + c)v_0 < 1$. Now we show by induction that $w_p < (1 + c)v_p < 1$ assuming that $w_{p-1} < (1 + c)v_{p-1} < 1$.

From (3.7) follows

$$\frac{v_{p-1}}{v_p} k < \frac{c - 1}{c + 1},$$

$$\left(\frac{1}{c + 1} + \frac{v_{p-1}}{v_p} k \right) < \frac{c}{c + 1},$$

$$(1 + c)^{\gamma-1} v_{p-1}^{\gamma-1} \left(v_{p-1} + \frac{v_{p-1}}{v_p} k \right) < \frac{c}{c + 1},$$

$$w_{p-1}^\gamma (v_p + k) < (1 + c)^\gamma v_{p-1}^\gamma (v_p + k) < cv_p,$$

$$w_p = v_p + w_{p-1}^\gamma (v_p + k) < (1 + c)v_p.$$

(ii) We assume $v_0 < \sqrt{(k/2d)^2 + c/(1+c)^2} - k/2d$ for some $0 < c < 1$ and we show $w_p \leq (1+c)v_p$. Then the lemma is proven by taking $c = 1$.

For any $v \in [0, v_0]$, we have

$$v^2 + \frac{k}{d}v - \frac{c}{(1+c)^2} < 0.$$

Hence, $(1+c)v(v+k/d) < c/(1+c)$. By assumption, we know

$$w_0 = v_0 < (1+c)v_0 < \sqrt{\left\{ \frac{k(1+c)}{2d} \right\}^2 + c} - \frac{k(1+c)}{2d} < \sqrt{c} < 1.$$

Now we show by induction that $w_p < (1+c)v_p < 1$, assuming that $w_{p-1} < (1+c)v_{p-1} < 1$,

$$\begin{aligned} (1+c)v_{p-1}\left(v_{p-1} + \frac{k}{d}\right) &< \frac{c}{1+c}, \\ (1+c)v_{p-1}\left(v_{p-1} + \frac{v_{p-1}}{v_p}k\right) &< \frac{c}{1+c}, \\ (1+c)^2v_{p-1}^2\left(1 + \frac{k}{v_p}\right) &< c, \\ (1+c)^\gamma v_{p-1}^\gamma\left(1 + \frac{k}{v_p}\right) &< c, \quad \gamma = 2, 3, \dots, \\ w_{p-1}^\gamma\left(1 + \frac{k}{v_p}\right) &< c, \\ w_p = v_p + w_{p-1}^\gamma(v_p + k) &< (1+c)v_p. \quad \text{Q.E.D.} \end{aligned}$$

THEOREM 3.4. Let $\gamma \geq 2$ and let $\zeta_p^{(1)}$ satisfy

$$\zeta_p^{(1)} \leq v_p = d^p v_0 \quad \text{for some } 0 < d < 1.$$

Then,

- (i) if $v_0 \leq \{\sqrt{d^2 + C_2^2} - C_2\}/2d$, it follows that $\zeta_p^{(3)} \leq 2d^p v_0$, and
- (ii) if $v_0 \leq \{\sqrt{1 + d^2} - 1\}/2d$, it follows that $\zeta_p^{(4)} \leq 2d^p v_0$.

Proof. (i) Let $\{w_p\}$ be defined as in Lemma 3.3 with $k = C_2 = \sup_{p>0} \|K_p\|$. Then it follows from the proof of Lemma 3.3 that $w_p \leq 2v_p$. Therefore we show $\zeta_p^{(3)} \leq w_p$ by induction: from the definition of $\zeta_p^{(3)}$ we derive

$$\zeta_0^{(3)} = \zeta_0^{(1)} \leq v_0 = w_0$$

and, by Theorem 3.2,

$$\zeta_p^{(3)} \leq \zeta_p^{(1)} + \zeta_{p-1}^{(3)\gamma}(\zeta_p^{(1)} + \|K_p\|) \leq v_p + w_{p-1}^\gamma(v_p + C_2) = w_p.$$

(ii) Similarly, with $\{w_p\}$ defined as in Lemma 3.3 with $k = 1$, we prove $\zeta_p^{(4)} < w_p$ and, hence,

$$\zeta_p^{(4)} < w_p < 2v_p = 2d^p v_0. \quad \text{Q.E.D.}$$

Remark. If $B_p^{(3)}$ is defined with $\gamma = 1$, then a similar proof yields that, for any decreasing sequence $\{v_p\}$ with

$$\sup_{p > 0} \|K_p\| = C_2 < d = \inf_p \frac{v_p}{v_{p-1}} < 1,$$

for which $\zeta_p^{(1)} \leq v_p$, we have $\zeta_p^{(3)} < (2d/(d - C_2))v_p$. \square

By $f_{p,\sigma}^{(j)}$ we denote the result of σ applications of the Defect Correction Process on level p with approximate inverse $B_p^{(j)}$, $j = 1, 3, 4$, when we take zero as the initial approximant.

With the aid of the previous theorem and Lemma 2.3, now the following theorem is immediate.

THEOREM 3.5 [APPROXIMATION THEOREM]. *Under the hypotheses of Theorem 3.4, the Multi Grid process yields approximate solutions for which the following error estimates hold*

$$\|f - f_{p,\sigma}^{(j)}\| \leq C_3 \|K_p f - Kf\| + (2d^p v_0)^\sigma \|f_p\|, \quad j = 3, 4,$$

where f and f_p are the solutions of (2.1) and (2.2) respectively.

Proof. The proof follows immediately from

$$\|f - f_{p,\sigma}^{(j)}\| \leq \|f - f_p\| + \|f_p - f_{p,\sigma}^{(j)}\|. \quad \square$$

4. Iteration Schemes With Projection into Finite-Dimensional Subspaces. In this section we expand the technique used in Section 3 to find the solution in X_p of the equation (2.3):

$$(4.1) \quad \tilde{A}_p \tilde{f}_p = g_p, \quad g_p \in X_p,$$

where $\tilde{A}_p = I - T_p K_p$ is a mapping on X into X . We assume that X_0 is sufficiently large such that $(I - T_p K_p)^{-1}$ exists for all $p > 0$.

Analogous to the approximate inverses of A_p in the previous section, we now introduce

$$\begin{aligned} \tilde{B}_p^{(1)} &= T_p + T_{p-1}(I - T_{p-1}K_{p-1})^{-1}T_pK_p, \\ \tilde{B}_p^{(2)} &= T_p + T_0(I - T_0K_0)^{-1}T_pK_p, \\ \begin{cases} \tilde{B}_0^{(3)} &= T_0(I - T_0K_0)^{-1}, \\ \tilde{B}_p^{(3)} &= T_p + T_{p-1}\tilde{Q}_{p-1}^{(3)}T_pK_p, \end{cases} \\ \begin{cases} \tilde{B}_0^{(4)} &= T_0(I - T_0K_0)^{-1}, \\ \tilde{B}_p^{(4)} &= T_p - T_{p-1} + T_{p-1}\tilde{Q}_{p-1}^{(4)}(T_{p-1} - T_{p-1}K_{p-1} + T_pK_p), \end{cases} \end{aligned}$$

with

$$\tilde{Q}_p^{(j)} = \sum_{m=0}^{\gamma-1} (T_p - \tilde{B}_p^{(j)}\tilde{A}_p)^m \tilde{B}_p^{(j)}, \quad j = 3, 4,$$

for some positive integer γ .

The operators $\tilde{B}_p^{(j)}$, $j = 1, 2, 3, 4$, are all mappings on X into X_p . The solution $\tilde{f}_p \in X_p$ of (4.1) is approximated by a defect correction process of the form

$$(4.2) \quad \begin{cases} \tilde{f}_{p,0} = 0, \\ \tilde{f}_{p,i+1} = \tilde{B}_p g_p + (I - \tilde{B}_p \tilde{A}_p) \tilde{f}_{p,i}. \end{cases}$$

We notice that $\tilde{B}_p^{(1)}$ and $\tilde{B}_p^{(3)}$ yield iterative processes that are equivalent, respectively, with the "One Step Method" and the "Multi Grid Method" discussed in Hackbusch [7]. $\tilde{B}_p^{(2)}$ yields an iterative process analogous to Atkinson's method, whereas $\tilde{B}_p^{(4)}$ yields a new multiple grid method with better convergence properties than $\tilde{B}_p^{(3)}$.

Analogously to Section 3, but restricting the domain of the operators to X_p , we see that here $\tilde{Q}_p^{(j)}$: $X_p \rightarrow X_p$ is an approximate inverse of \tilde{A}_p : $X_p \rightarrow X_p$, and the amplification operator on X_p into X_p of a defect correction step with $\tilde{Q}_p^{(j)}$ is

$$T_p - \tilde{Q}_p^{(j)} \tilde{A}_p = (T_p - \tilde{B}_p^{(j)} \tilde{A}_p)^\gamma.$$

Thus, one application of $\tilde{Q}_p^{(j)}$ is equivalent to the γ times application of $\tilde{B}_p^{(j)}$, and we may write

$$\tilde{Q}_p^{(j)} = [T_p - (T_p - \tilde{B}_p^{(j)} \tilde{A}_p)^\gamma] \tilde{A}_p^{-1}.$$

The convergence of the process (4.2) depends on the Lipschitz constant of the operator $I - \tilde{B}_p \tilde{A}_p$ as a mapping $X_p \rightarrow X_p$. Therefore its rate of convergence is given by $\|T_p(I - \tilde{B}_p \tilde{A}_p)\|$. This rate of convergence is studied in the remainder of this section.

- THEOREM 4.1.** (i) $\|T_p(I - \tilde{B}_p^{(1)} \tilde{A}_p)\| \rightarrow 0$ as $p \rightarrow \infty$,
 (ii) $\|T_p(I - \tilde{B}_p^{(2)} \tilde{A}_p)\| \leq \tilde{C}(X_0)$ as $p \rightarrow \infty$, $\tilde{C}(X_0) < 1$ for X_0 sufficiently large.

Proof. (i) Substitution of the explicit expressions for $\tilde{B}_p^{(1)}$ and \tilde{A}_p yields

$$\begin{aligned} T_p(I - \tilde{B}_p^{(1)} \tilde{A}_p) &= (T_p - T_{p-1})K_p \\ &\quad + T_{p-1}(I - T_{p-1}K_{p-1})^{-1}T_p(K_p - T_{p-1}K_{p-1})K_p. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|T_p(I - \tilde{B}_p^{(1)} \tilde{A}_p)\| &\leq \|(I - T_p)K_p\| + \|(I - T_{p-1})K_p\| \\ &\quad + \|T_{p-1}\| \|(I - T_{p-1}K_{p-1})^{-1}\| \|T_p\| \\ &\quad \cdot \{ \|(K_p - K)K_p\| + \|(K - K_{p-1})K_p\| \\ &\quad \quad + \|(K_{p-1} - T_{p-1}K_{p-1})K_p\| \}. \end{aligned}$$

Using Lemmas 2.1 to 2.5 we obtain the proof of (i) by the same arguments as used for the proof of Theorem 3.1.

(ii) Replacing the subscript $p - 1$ by 0 in the first part of the proof, we get

$$\begin{aligned} \|T_p(I - \tilde{B}_p^{(2)} \tilde{A}_p)\| &\leq \|(I - T_p)K_p\| + \|(I - T_0)K_p\| + \|T_0\| \|(I - T_0K_0)^{-1}\| \|T_p\| \\ &\quad \cdot \{ \|(K_p - K)K_p\| + \|(K - K_0)K_p\| + \|(I - T_0)K_0\| \|K_p\| \}. \end{aligned}$$

For $p \rightarrow \infty$ $\|(I - T_p)K_p\|$ and $\|(K_p - K)K_p\|$ vanish, whereas the other terms tend to a constant value depending on X_0 . \square

Definition.

$$\eta_p^{(j)} = \|T_p(I - \tilde{B}_p^{(j)} \tilde{A}_p)\|, \quad j = 1, 2, 3, 4.$$

THEOREM 4.2.

$$\begin{aligned} \eta_p^{(3)} &< \eta_p^{(1)} + \eta_{p-1}^{(3)\prime} [\eta_p^{(1)} + \|T_p\| \|K_p\|], \\ \eta_p^{(4)} &< \eta_p^{(1)} + \eta_{p-1}^{(4)\prime} [\eta_p^{(1)} + \|T_p\|]. \end{aligned}$$

Proof. We use the notation $M_p^{(j)} = T_p(I - \tilde{B}_p^{(j)}\tilde{A}_p)$, $j = 1, 3, 4$. From assumption (A6) and the definitions of A_p , \tilde{A}_p , and $\tilde{B}_p^{(j)}$, $j = 1, 3, 4$, it is clear that

$$T_p A_p = T_p \tilde{A}_p = \tilde{A}_p T_p$$

and

$$\tilde{B}_p^{(j)} = T_p \tilde{B}_p^{(j)} = \tilde{B}_p^{(j)} T_p, \quad j = 1, 3, 4.$$

Hence,

$$M_p^{(j)} = T_p M_p^{(j)} = M_p^{(j)} T_p, \quad j = 1, 3, 4,$$

and also

$$\tilde{Q}_p^{(j)} = [T_p - M_p^{(j)\prime}] \tilde{A}_p^{-1}.$$

(i) From the definition of $\tilde{B}_p^{(1)}$, we get

$$M_p^{(1)} = T_p - T_p \tilde{A}_p - T_{p-1} \tilde{A}_{p-1}^{-1} T_p K_p \tilde{A}_p = T_p K_p - T_{p-1} \tilde{A}_{p-1}^{-1} T_p K_p \tilde{A}_p.$$

These relations are used to prove that

$$\begin{aligned} M_p^{(3)} &= T_p [I - \{T_p + T_{p-1}(T_{p-1} - M_p^{(3)\prime})\} \tilde{A}_{p-1}^{-1} T_p K_p] \tilde{A}_p \\ &= T_p - T_p \tilde{A}_p - T_{p-1} (T_{p-1} - M_p^{(3)\prime}) \tilde{A}_{p-1}^{-1} T_p K_p \tilde{A}_p \\ &= M_p^{(1)} + M_p^{(3)\prime} T_{p-1} \tilde{A}_{p-1}^{-1} T_p K_p \tilde{A}_p \\ &= M_p^{(1)} + M_p^{(3)\prime} (T_p K_p - M_p^{(1)}). \end{aligned}$$

Hence,

$$\eta_p^{(3)} \leq \eta_p^{(1)} + \eta_{p-1}^{(3)\prime} (\eta_p^{(1)} + \|T_p\| \|K_p\|).$$

(ii)

$$\begin{aligned} M_p^{(4)} &= T_p - T_p \tilde{A}_p + T_{p-1} \tilde{A}_p - (T_{p-1} - M_p^{(4)\prime}) \tilde{A}_{p-1}^{-1} (T_{p-1} \tilde{A}_{p-1} + T_p K_p) \tilde{A}_p \\ &= M_p^{(1)} + M_p^{(4)\prime} \tilde{A}_{p-1}^{-1} (\tilde{A}_{p-1} T_{p-1} + T_p K_p) \tilde{A}_p \\ &= M_p^{(1)} + M_p^{(4)\prime} T_{p-1} T_p (\tilde{A}_p + \tilde{A}_{p-1}^{-1} T_p K_p \tilde{A}_p) \\ &= M_p^{(1)} + M_p^{(4)\prime} (T_p - M_p^{(1)}). \end{aligned}$$

Hence,

$$\eta_p^{(4)} \leq \eta_p^{(1)} + \eta_{p-1}^{(4)\prime} (\eta_p^{(1)} + \|T_p\|). \quad \text{Q.E.D.}$$

THEOREM 4.3. Let $\gamma \geq 2$ and let $\eta_p^{(1)}$ satisfy

$$\eta_p^{(1)} \leq v_p = d^p v_0 \quad \text{for some } 0 < d < 1.$$

Then,

- (i) if $v_0 \leq \{\sqrt{d^2 + C_1^2 C_2^2} - C_1 C_2\} / 2d$, it follows that $\eta_p^{(3)} \leq 2v_0 d^p$, and
- (ii) if $v_0 \leq \{\sqrt{d^2 + C_1^2} - C_1\} / 2d$, it follows that $\eta_p^{(4)} \leq 2v_0 d^p$.

Proof. (i) Use Lemma 3.3 with $k = C_1 C_2$ and Theorem 4.2.

(ii) Analogously with $k = C_1$. \square

By $\tilde{f}_{p,\sigma}^{(j)}$ we denote the result of σ applications of the Defect Correction Process on level p with approximate inverse $\tilde{B}_p^{(j)}$, $j = 1, 3, 4$, when we take zero as the initial approximate.

THEOREM 4.4 [APPROXIMATION THEOREM]. *Under the hypotheses of Theorem 4.3 the Multi Grid process yields $\tilde{f}_{p,\sigma}^{(j)}$, for which the following error estimates hold*

$$\|f - \tilde{f}_{p,\sigma}^{(j)}\| \leq \|f - T_p f\| + C_1 C_4 \|Kf - K_p f\| + (2d^p v_0)^\sigma \|\tilde{f}_p\|,$$

where f and \tilde{f}_p are the solutions of (2.1) and (2.3) respectively.

Proof. For $j = 1, 3, 4$ we have

$$\|f - \tilde{f}_{p,\sigma}^{(j)}\| \leq \|f - T_p f\| + \|T_p f - \tilde{f}_p\| + \|\tilde{f}_p - \tilde{f}_{p,\sigma}^{(j)}\|$$

and the proof follows from Lemma 2.6 and Theorem 4.3. \square

We notice that the usual discretization methods easily satisfy the first condition of Theorem 4.3 as is illustrated in Section 5. The other condition of Theorem 4.3, which requires an upper bound on v_0 , essentially is a requirement on the coarsest discretization used in the multiple grid algorithm. This condition is also discussed in the next section.

5. Numerical Results. In this section we illustrate the theoretical convergence results from the previous sections, and we make some remarks about the computational complexity of the various methods. We shall only show numerical results obtained with the methods that appear to be the most efficient. These methods are defined by the approximate inverses $B_p^{(2)}$ (Atkinson's method), $\tilde{B}_p^{(3)}$ (Hackbusch's method) and $\tilde{B}_p^{(4)}$ (a new method with better convergence properties).

As an example, the integral equation

$$(5.1) \quad f(x) - \lambda \int_0^1 \cos(\pi xy) f(y) = g(x)$$

is solved for various values of the parameter λ (cf. Hackbusch [7] who gives results for the same equation); $g(x)$ is chosen such that $f(x) = e^x \cos(7x)$.

The operators K_p are defined by means of the repeated trapezoidal rule:

$$K_p f(x) = \sum_{j=0}^{N_p} w_j k(x, x_j) f(x_j),$$

where the nodal points $\{x_j\}$ are uniformly distributed ($x_0 = 0, x_{N_p} = 1$) and the weights $\{w_j\}$ are given by $\{\frac{1}{2}h_p, h_p, h_p, \dots, h_p, \frac{1}{2}h_p\}$ with $h_p = (N_p)^{-1}$. The projection operators are defined by piecewise linear interpolation at the nodal points $\{x_j\}$. The different grid-levels are related by $N_p = 2N_{p-1}$.

For the operators $\{K_p\}$ and $\{T_p\}$ we know (cf. Atkinson [3] and Prenter [9])

$$(5.2) \quad \|K_p f - Kf\| = \mathcal{O}(h_p^2),$$

$$(5.3) \quad \|T_p f - f\| = \mathcal{O}(h_p^2),$$

$$(5.4) \quad a_p = \mathcal{O}(h_p^2), \quad \text{and}$$

$$(5.5) \quad b_p = \mathcal{O}(h_p^2) \quad \text{for } p \rightarrow \infty.$$

Using these estimates, we easily derive (see the proof of Theorem 4.1)

$$\eta_p^{(1)} \leq v_p = Ch_p^2.$$

Because the successive meshsizes are related by $h_p = h_0 2^{-p}$, we have

$$(5.6) \quad \eta_p^{(1)} \leq v_p = 4Ch_0^2 4^{-p}.$$

Comparing this expression with the assumption on $\eta_p^{(1)}$ in Theorem 4.3, we see that $d = 1/4$. In the same theorem, conditions on $\eta_0^{(1)}$ are formulated for the multiple grid methods to converge. Comparing these conditions, we conclude that the condition on $\eta_0^{(1)}$ in the process defined by $\tilde{B}_p^{(4)}$ is independent of $C_2 = \sup_{p>0} \|K_p\|$, whereas in the process defined by $\tilde{B}_p^{(3)}$ the condition on $\eta_0^{(1)}$ becomes stronger as $\sup_{p>0} \|K_p\|$ increases. In Figure 1 we sketch the regions of convergence induced by $\tilde{B}_p^{(3)}$ and $\tilde{B}_p^{(4)}$, as derived from Theorem 4.3 with $d = 1/4$ and $\gamma = 2$.

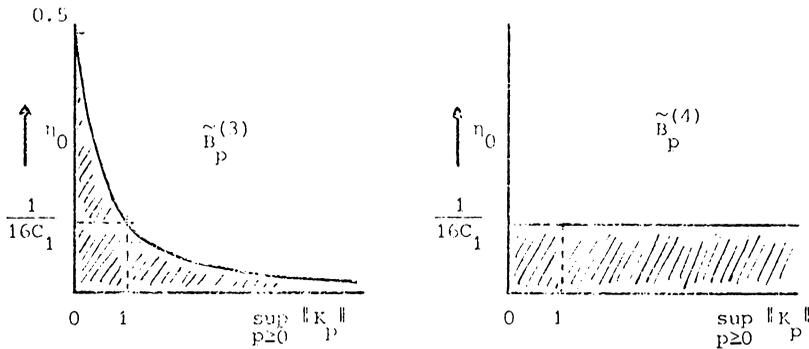


FIGURE 1

The multiple grid convergence regions. The coarsest grid convergence factor η_0 versus $C_2 = \sup_{p>0} \|K_p\|$.

Hence, from Theorem 4.3 one may expect that both multiple grid methods yield similar results as $\|K\| \approx 1$, whereas they differ for $\|K\| \gg 1$. For the integral equation (5.1), $\|K\| \gg 1$ holds for $\lambda \gg 1$.

In Tables 5.1–5.3 we give the observed rates of convergence

$$\eta(N_p; N_0) = \left[\frac{\|\tilde{f}_{p,i+1} - \tilde{f}_{p,i}\|}{\|\tilde{f}_{p,1} - \tilde{f}_{p,0}\|} \right]^{1/i},$$

for the iterative methods defined by $B_p^{(2)}$, $\tilde{B}_p^{(3)}$ and $\tilde{B}_p^{(4)}$, respectively, with $\gamma = 2$. The dependence of $\eta(N_p; N_0)$ on N_p , the number of mesh intervals in the finest grid, and on N_0 , the number in the coarsest grid, is shown. The value of i is suitably chosen and $\|\cdot\|$ denotes the maximum norm. From Table 5.1 we see that the rates of convergence of Atkinson's method tend to a constant value as $N_p \rightarrow \infty$. As was expected, it decreases as N_0 increases.

In the case of convergence, the Tables 5.2 and 5.3 asymptotically show similar results. However, for larger values of λ , the new multiple grid method needs fewer subintervals in the coarsest grid. The quotients $\eta(N_p; N_0)/\eta(N_{p-1}; N_0)$ approximate the value $d = 1/4$, which is in agreement with the theory.

TABLE 5.1

Rates of convergence for the two-grid method defined by $B_p^{(2)}$ (Atkinson's method).

λ	$N_p \backslash N_0$	2	4	8	16	32	64
	1	4	$.23 \cdot 10^{-1}$				
8		$.28 \cdot 10^{-1}$	$.58 \cdot 10^{-2}$				
16		$.30 \cdot 10^{-1}$	$.72 \cdot 10^{-2}$	$.15 \cdot 10^{-2}$			
32		$.30 \cdot 10^{-1}$	$.76 \cdot 10^{-2}$	$.18 \cdot 10^{-2}$	$.38 \cdot 10^{-3}$		
64		$.30 \cdot 10^{-1}$	$.78 \cdot 10^{-2}$	$.19 \cdot 10^{-2}$	$.47 \cdot 10^{-3}$	$.83 \cdot 10^{-4}$	
128		$.30 \cdot 10^{-1}$	$.78 \cdot 10^{-2}$	$.19 \cdot 10^{-2}$	$.52 \cdot 10^{-3}$	$.10 \cdot 10^{-3}$	$.24 \cdot 10^{-4}$
10	4	$.11 \cdot 10^{+1}$					
	8	$.16 \cdot 10^{+1}$	$.18 \cdot 10^0$				
	16	$.17 \cdot 10^{+1}$	$.22 \cdot 10^0$	$.36 \cdot 10^{-1}$			
	32	$.17 \cdot 10^{+1}$	$.23 \cdot 10^0$	$.45 \cdot 10^{-1}$	$.86 \cdot 10^{-2}$		
	64	$.17 \cdot 10^{+1}$	$.24 \cdot 10^0$	$.48 \cdot 10^{-1}$	$.11 \cdot 10^{-1}$	$.21 \cdot 10^{-2}$	
	128	$.17 \cdot 10^{+1}$	$.24 \cdot 10^0$	$.48 \cdot 10^{-1}$	$.11 \cdot 10^{-1}$	$.27 \cdot 10^{-2}$	$.38 \cdot 10^{-3}$
100	4	$.64 \cdot 10^{+1}$					
	8	$.11 \cdot 10^{+2}$	$.14 \cdot 10^{+1}$				
	16	$.14 \cdot 10^{+2}$	$.16 \cdot 10^{+1}$	$.40 \cdot 10^0$			
	32	$.15 \cdot 10^{+2}$	$.16 \cdot 10^{+1}$	$.42 \cdot 10^0$	$.99 \cdot 10^{-1}$		
	64	$.15 \cdot 10^{+2}$	$.16 \cdot 10^{+1}$	$.45 \cdot 10^0$	$.15 \cdot 10^0$	$.33 \cdot 10^{-1}$	
	128	$.15 \cdot 10^{+2}$	$.16 \cdot 10^{+1}$	$.49 \cdot 10^0$	$.16 \cdot 10^0$	$.41 \cdot 10^{-1}$	$.68 \cdot 10^{-2}$

Using (5.2), (5.3), and (5.6) for the approximation errors, we conclude, from Theorem 4.4, that for the multiple grid methods $\sigma = 2$ iteration steps are sufficient to get an iteration error which is of the same order of magnitude as the approximation errors $\|f - T_p f\|$ and $\|Kf - K_p f\|$. Of course, this is not the case with Atkinson's method for which one has to perform $\mathcal{O}(\log N_p)$ iteration steps. That these asymptotic arguments hold already for relatively small N_p is shown in the Tables 5.4–5.5, where we compare the approximation errors with the iteration error after $\sigma = 2$ iteration steps.

We conclude this section with some remarks about the asymptotic computational complexity. We only count multiplications, ignoring the multiplications with the weights w_j and the computations involved in the evaluation of $k(s, t)$. Then, asymptotically for $N_p \rightarrow \infty$, the operation counts per iteration sweep for the various approximate inverses are:

$$\begin{aligned}
 B_p^{(1)} &: 2.75N_p^2, & \tilde{B}_p^{(1)} &: 1.75N_p^2, \\
 B_p^{(2)} &: 2N_p^2, & \tilde{B}_p^{(2)} &: 1N_p^2, \\
 B_p^{(3)} &: 2N_p^2 \log N_p, & \tilde{B}_p^{(3)} &: 3N_p^2, \\
 B_p^{(4)} &: 2.5N_p^2 \log N_p, & \tilde{B}_p^{(4)} &: 3.5N_p^2.
 \end{aligned}$$

TABLE 5.2

Rates of convergence for the multiple grid method
defined by $\tilde{B}_p^{(3)}$ (Hackbusch's method).

λ	N_p	N_0					
		2	4	8	16	32	64
1	4	$.31 \cdot 10^{-1}$					
	8	$.98 \cdot 10^{-2}$	$.94 \cdot 10^{-2}$				
	16	$.24 \cdot 10^{-2}$	$.24 \cdot 10^{-2}$	$.23 \cdot 10^{-2}$			
	32	$.62 \cdot 10^{-3}$	$.62 \cdot 10^{-3}$	$.62 \cdot 10^{-3}$	$.62 \cdot 10^{-3}$		
	64	$.14 \cdot 10^{-3}$	$.14 \cdot 10^{-3}$	$.14 \cdot 10^{-3}$	$.14 \cdot 10^{-3}$	$.14 \cdot 10^{-3}$	
	128	$.35 \cdot 10^{-4}$	$.35 \cdot 10^{-4}$	$.35 \cdot 10^{-4}$	$.35 \cdot 10^{-4}$	$.34 \cdot 10^{-4}$	$.35 \cdot 10^{-4}$
10	4	$.32 \cdot 10^0$					
	8	$.12 \cdot 10^{+1}$	$.10 \cdot 10^0$				
	16	$.42 \cdot 10^{+1}$	$.12 \cdot 10^0$	$.25 \cdot 10^{-1}$			
	32	$.20 \cdot 10^{+3}$	$.18 \cdot 10^{-1}$	$.13 \cdot 10^{-1}$	$.62 \cdot 10^{-2}$		
	64	$.23 \cdot 10^{+6}$	$.91 \cdot 10^{-2}$	$.24 \cdot 10^{-2}$	$.23 \cdot 10^{-2}$	$.19 \cdot 10^{-2}$	
	128	$.40 \cdot 10^{+12}$	$.46 \cdot 10^{-3}$	$.57 \cdot 10^{-3}$	$.52 \cdot 10^{-3}$	$.53 \cdot 10^{-3}$	$.51 \cdot 10^{-3}$
100	4	$.43 \cdot 10^{+1}$					
	8	$.11 \cdot 10^{+4}$	$.11 \cdot 10^{+1}$				
	16	$.66 \cdot 10^{+7}$	$.77 \cdot 10^{+2}$	$.29 \cdot 10^0$			
	32	$.17 \cdot 10^{+17}$	$.79 \cdot 10^{+4}$	$.51 \cdot 10^{+1}$	$.10 \cdot 10^0$		
	64	$.82 \cdot 10^{+34}$	$.46 \cdot 10^{+11}$	$.15 \cdot 10^{+3}$	$.33 \cdot 10^0$	$.29 \cdot 10^{-1}$	
	128	$.80 \cdot 10^{+70}$	$.86 \cdot 10^{+23}$	$.96 \cdot 10^{+7}$	$.43 \cdot 10^0$	$.24 \cdot 10^{-1}$	$.85 \cdot 10^{-2}$

Here we ignored the direct solution on the coarsest grid and we applied the multiple grid methods with $\sigma = \gamma = 2$ on all levels.

Note. The number of kernel-function evaluations is N_p^2 in the linear case when they are computed once and stored. In the nonlinear case or in the case when kernel-functions are reevaluated whenever they are used, the number of kernel-function evaluations is of the same order as the number of arithmetic operations.

Asymptotically, all methods need only 2 iterations to obtain a result of the order of the truncation error, except the methods with $B_p^{(2)}$ and $\tilde{B}_p^{(2)}$ which need $\mathcal{O}(\log N_p)$ sweeps. For the methods with $B_p^{(1)}$ and $\tilde{B}_p^{(1)}$ the coarsest grid still has $N_p/2$ mesh intervals; on this grid the problem is solved by a direct method (e.g. Gauss-elimination) and therefore we have to add $\frac{1}{12} N_p^3$ to the total computational complexity. Thus, for the total amount of asymptotic computational work we get the following table:

$$\begin{aligned}
 B_p^{(1)} &: \frac{1}{12} N_p^3 + 5.5 N_p^2, & \tilde{B}_p^{(1)} &: \frac{1}{12} N_p^3 + 3.5 N_p^2, \\
 B_p^{(2)} &: \frac{2}{3} N_0^3 + \mathcal{O}(N_p^2 \log N_p), & \tilde{B}_p^{(2)} &: \frac{2}{3} N_0^3 + \mathcal{O}(N_p^2 \log N_p),
 \end{aligned}$$

$$B_p^{(3)} : \frac{2}{3} N_0^3 + 4N_p^2 \log N_p, \quad \tilde{B}_p^{(3)} : \frac{2}{3} N_0^3 + 6N_p^2,$$

$$B_p^{(4)} : \frac{2}{3} N_0^3 + 5N_p^2 \log N_p, \quad \tilde{B}_p^{(4)} : \frac{2}{3} N_0^3 + 7N_p^2.$$

TABLE 5.3

Rates of convergence for the multiple grid method defined by $\tilde{B}_p^{(4)}$.

λ	N_0								
	N	p	2	4	8	16	32	64	
1	4		.31 10 ⁻¹						
	8		.95 10 ⁻²	.94 10 ⁻²					
	16		.23 10 ⁻²	.23 10 ⁻²	.23 10 ⁻²				
	32		.62 10 ⁻³	.62 10 ⁻³	.62 10 ⁻³	.62 10 ⁻³			
	64		.14 10 ⁻³	.14 10 ⁻³	.14 10 ⁻³	.14 10 ⁻³	.14 10 ⁻³		
	128		.35 10 ⁻⁴	.34 10 ⁻⁴	.34 10 ⁻⁴	.35 10 ⁻⁴	.35 10 ⁻⁴	.35 10 ⁻⁴	.35 10 ⁻⁴
10	4		.32 10 ⁰						
	8		.18 10 ⁰	.10 10 ⁰					
	16		.40 10 ⁻¹	.12 10 ⁻¹	.25 10 ⁻¹				
	32		.70 10 ⁻²	.69 10 ⁻²	.60 10 ⁻²	.62 10 ⁻²			
	64		.19 10 ⁻²	.19 10 ⁻²	.19 10 ⁻²	.19 10 ⁻²	.19 10 ⁻²		
	128		.50 10 ⁻³	.50 10 ⁻³	.50 10 ⁻³	.50 10 ⁻³	.50 10 ⁻³	.51 10 ⁻³	
100	4		.43 10 ⁺¹						
	8		.72 10 ⁺¹	.11 10 ⁺¹					
	16		.30 10 ⁺²	.11 10 ⁺¹	.29 10 ⁰				
	32		.13 10 ⁺⁴	.13 10 ⁺¹	.20 10 ⁰	.10 10 ⁰			
	64		.17 10 ⁺⁷	.16 10 ⁺¹	.41 10 ⁻¹	.36 10 ⁻¹	.29 10 ⁻¹		
	128		.29 10 ⁺¹³	.26 10 ⁺¹	.75 10 ⁻²	.94 10 ⁻²	.10 10 ⁻¹	.85 10 ⁻²	

From these tables we see that the multiple grid methods become cheaper than Atkinson's method, whenever the latter needs more than three iterations.

In order to get an impression of the qualities of the various methods, we suggest to measure by experiments the following ratio (which shows the amount of computational work per digit accuracy obtained):

$$\kappa_\sigma = - \frac{\text{Number of multiplications to obtain } \tilde{f}_{p,\sigma}}{N_p^2 * 10 \log \|f - \tilde{f}_{p,\sigma}\|}.$$

Because of their rates of convergence, for the multiple grid methods we choose $\sigma = 2$, whereas for Atkinson's method we determine σ such that κ_σ is minimal. Better methods are now characterized by a smaller κ_σ .

TABLE 5.4

The ratio: iteration error after 2 sweeps / approximation errors, i.e.

$$\|\tilde{f}_{p,2} - \tilde{f}_{p,\infty}\| / \|f - \tilde{f}_{p,\infty}\|,$$

Numbers of subintervals: $N_0 = 2$, (a divergent iteration process is denoted by -).

λ	N_p	$B_p^{(2)}$	$\tilde{B}_p^{(3)}$	$\tilde{B}_p^{(4)}$
1	4	.0046	.0018	.0018
	8	.0267	.0003	.0003
	16	.1162	.0001	.0001
	32	.4743	.0000	.0000
	64	.0930	.0000	.0000
	128	5.6378	.0000	.0000
10	4	-	3.3089	3.3089
	8	-	-	.0568
	16	-	-	.3899
	32	-	-	.0694
	64	-	-	.0194
	128	-	-	.0050

TABLE 5.5

The ratio: iteration error after 2 sweeps / approximation errors.

As Table 5.4, but $N_0 = 8$.

λ	N_p	$B_p^{(2)}$	$\tilde{B}_p^{(3)}$	$\tilde{B}_p^{(4)}$
1	16	.0003	.0001	.0001
	32	.0017	.0000	.0000
	64	.0075	.0000	.0000
	128	.0306	.0000	.0000
10	16	.0936	.3202	.3202
	32	.6088	.2000	.0692
	64	2.7111	.0341	.0194
	128	11.1310	.0056	.0050
100	16	91.0760	34.0160	34.0160
	32	563.4392	-	34.7089
	64	2480.5082	-	24.3138
	128	$> 10^4$	-	0.1220

TABLE 5.6

For problem (5.1) with $\lambda = 100$ and $N_p = 128$ the experimental ratios κ_σ , where σ is given between parentheses; for this problem ${}^{10}\log\|f - \tilde{f}_{p,\infty}\| = -3.5$.

N_0	8	16	32	64
$B_p^{(2)}$	7.55 (11)	3.46 (5)	2.73 (3)	6.20 (2)
$\tilde{B}_p^{(3)}$	-	-8.42 (2)	2.97 (2)	5.95 (2)
$\tilde{B}_p^{(4)}$	2.00 (2)	1.86 (2)	2.20 (2)	6.10 (2)

Table 5.6 shows, for the multiple grid methods, that small values of N_0 are more efficient as long as the process converges. However, within a reasonable range of small N_0 , it seems not worthwhile to determine an optimal N_0 .

The asymptotic work estimates and the convergence property discussed in Section 4 lead us to prefer $\tilde{B}_p^{(3)}$ for $\|K\| \approx 1$ and $\tilde{B}_p^{(4)}$ for $\|K\| \gg 1$. Finally, we remark that the same multiple grid techniques can be applied to nonlinear problems as well. Moreover, the structure of multiple grid algorithms yields estimates for the approximation and truncation errors in a natural way. All these features together can be used to construct an automatic program for solving Fredholm integral equations of the second kind. In fact, such a program has been constructed and some of the results have been reported in [10].

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