The Finite Element Method With Nonuniform Mesh Sizes for Unbounded Domains*

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Abstract. The finite element method with nonuniform mesh sizes is employed to approximately solve elliptic boundary value problems in unbounded domains. Consider the following model problem:

$$-\Delta u = f \quad \text{in } \Omega^C, \quad u = g \quad \text{on } \partial\Omega, \quad \frac{\partial u}{\partial r} + \frac{1}{r}u = o\left(\frac{1}{r}\right) \quad \text{as } r = |x| \to \infty,$$

where Ω^C is the complement in R^3 (three-dimensional Euclidean space) of a bounded set Ω with smooth boundary $\partial\Omega$, f and g are smooth functions, and f has bounded support. This problem is approximately solved by introducing an artificial boundary Γ_R near infinity, e.g. a sphere of sufficiently large radius R. The intersection of this sphere with Ω^C is denoted by Ω_R^C and the given problem is replaced by

$$-\Delta u_R = f \quad \text{in } \Omega_R^C, \quad u_R = g \quad \text{on } \partial\Omega, \quad \frac{\partial u_R}{\partial r} + \frac{1}{r}u_R = 0 \quad \text{on } \Gamma_R.$$

This problem is then solved approximately by the finite element method, resulting in an approximate solution u_R^h for each $h > 0$. In order to obtain a reasonably small error for $u - u_R^h = (u - u_R) + (u_R - u_R^h)$, it is necessary to make R large. This necessitates the solution of a large number of linear equations, so that this method is often not very good when a uniform mesh size h is employed. It is shown that a nonuniform mesh may be introduced in such a way that optimal error estimates hold and the number of equations is bounded by Ch^{-3} with C independent of h and R.

1. Introduction. The finite element method has been studied extensively in connection with elliptic boundary value problems on bounded domains; see, e.g., [1]–[3] and the references cited there. In particular, it has been shown that optimal error estimates hold under suitable assumptions on the differential operator and the finite element subspace. It is the purpose of this paper to show that, by employing the finite element method with an appropriately graded mesh near infinity, analogous results hold for unbounded domains.

We consider the following model problem:

$$(1.1) \quad -\Delta u = f \quad \text{in } \Omega^C, \quad u = g \quad \text{on } \partial\Omega \quad \text{and} \quad \frac{\partial u}{\partial r} + \frac{1}{r}u = o\left(\frac{1}{r}\right) \quad \text{as } r = |x| \to \infty,$$

where Ω^C is the complement in R^3 (three-dimensional Euclidean space) of a bounded domain Ω with smooth boundary $\partial\Omega$, f and g are smooth functions, and f has bounded support. We shall approximate this problem by introducing a sphere

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Γ_R of sufficiently large radius R, as well as an approximate boundary condition on this sphere. We denote the region bounded by $\partial\Omega$ and Γ_R by Ω_R and replace problem (1.1) by

$$(1.2) \quad -\Delta u_R = f \quad \text{in } \Omega_R, \quad u_R = g \quad \text{on } \partial\Omega \quad \text{and} \quad \frac{\partial u_R}{\partial r} + \frac{1}{r}u_R = 0 \quad \text{on } \Gamma_R.$$

Problem (1.2) may be approximately solved using the finite element method, resulting in an approximate solution u_R^h for each $h > 0$. In order to obtain a suitable estimate for $u - u_R^h = (u - u_R) + (u_R - u_R^h)$, where u and u_R satisfy (1.1) and (1.2), respectively, it will be necessary to make R large. In the usual finite element formulation with a uniform mesh size h, this results in an excessively large number of linear equations. We shall overcome this difficulty by grading the mesh systematically in such a way that the element mesh sizes become larger as the distance of the element from the origin increases. It will be proved that optimal error estimates for $u - u_R^h$ hold (in the sense to be described in Section 3), while the number of equations is bounded by Ch^{-3} with C independent of h and R. The results will follow by combining certain rates of decay estimates for the derivatives of u with approximation theory.

We now outline the remainder of the paper. In Section 2, we introduce our notation and describe the boundary value problems under consideration, as well as a variational formulation of problem (1.2). The main result of Section 2 is an estimate for $u - u_R$ in both the energy norm and the L^2 norm. In Section 3, we describe the finite element method for solving problem (1.2) and obtain an estimate for $u_R - u_R^h$ in the energy norm when a uniform mesh size is employed. We shall see that this yields a suboptimal error estimate for $u - u_R^h$. In Section 4, we describe the mesh grading technique and obtain an optimal error estimate for $u - u_R^h$ in the energy norm. In Section 5, we obtain an optimal error estimate in the L^2 norm. The main results of the paper are embodied in Theorems 4.1 and 5.1.

For other references, treating problems in unbounded domains using the finite element method, see [2, pp. 276–280] and [4]–[6], as well as additional references cited there. The mesh grading technique has been employed in [7]–[10] to treat domains with corners.

2. The Boundary Value Problems. In this section, we shall investigate problems (1.1) and (1.2). We begin by defining some notation and formulating the given problem on an unbounded domain in R^3. We shall then formulate an approximate problem on a bounded domain and establish estimates for the difference between the solutions of these two problems.

We shall employ the usual notation for Sobolev spaces. In particular, suppose that M is a nonnegative integer and B is a subset of R^3. (We shall obtain our results in R^3 although the arguments carry over to R^2 in a straightforward manner.) Set

$$|u|_{H^M(B)} = \left(\sum_{|\alpha| = M} \|D_u^\alpha\|_{L^2(B)}^2 \right)^{1/2}$$

and

$$\|u\|_{H^M(B)} = \left(\sum_{s=0}^{M} |u|^2_{H^s(B)} \right)^{1/2},$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with each integer $\alpha_j \geqslant 0$, $|\alpha| = \sum_{j=1}^{3} \alpha_j$, and D^α denotes the weak derivative of the real-valued function u. Thus, $|\ |_{H^M(B)}$ defines a seminorm, whereas $\|\ \|_{H^M(B)}$ defines a norm. Set $H^M(B) = \{u: \|u\|_{H^M(B)} < \infty\}$. An arbitrary point in R^3 will be denoted by $x = (x_1, x_2, x_3)$ in Cartesian coordinates. We shall also employ spherical polar coordinates, (r, ω, θ), defined by $x_1 = r \sin\theta \cos\omega$, $x_2 = r \sin\theta \sin\omega$, and $x_3 = r \cos\theta$.

We next define our boundary value problem as follows. Let Ω denote a bounded domain in R^3 with C^∞ boundary $\partial\Omega$. Denote the complement of $\overline{\Omega} = \Omega \cup \partial\Omega$ by Ω^C. We shall also require the sets S_R and Ω_R given by $S_R = \{x: |x| < R\}$ and $\Omega_R = \Omega^C \cap S_R$. Suppose that $g \in C^\infty(\partial\Omega)$, $f \in C^\infty(\Omega^C)$, and f has bounded support. (For the sake of simplicity, we are assuming more smoothness of f, g and, $\partial\Omega$ than necessary.) Throughout the paper, we shall assume that R is sufficiently large that $\Omega \subset S_R$ and $\text{supp}(f) \subset \overline{\Omega}_R$. Our aim is to solve the boundary value problem given by (1.1). (The results and arguments of this paper go through without essential change if the Dirichlet boundary condition on $\partial\Omega$ is replaced by the Neumann condition.)

It suffices to consider the following boundary value problem:

$$(2.1) \quad -\Delta u = f \quad \text{in } \Omega^C, \quad u = 0 \quad \text{on } \partial\Omega \quad \text{and} \quad \frac{\partial u}{\partial r} + \frac{1}{r} u = o\left(\frac{1}{r}\right) \quad \text{as } r \to \infty.$$

To obtain (2.1) from (1.1), we observe that g may be extended to a smooth function G with bounded support. If v satisfies (1.1), we set $u = v - G$ and observe that u satisfies (2.1) with f replaced by $f + \Delta G$. We now establish some important properties of the solution u of (2.1) that will be useful throughout this paper.

LEMMA 2.1. *Suppose that $u \in C^\infty(\Omega^C)$ satisfies (2.1). Then for $r = |x|$ sufficiently large and each multi-index α, we have*

(a)
$$D_x^\alpha u(x) = \sum_{n=1}^{\infty} \frac{a_n^\alpha(\omega, \theta)}{r^{n+|\alpha|}},$$

(b)
$$|D_x^\alpha u(x)|^2 \leqslant \frac{C}{r^{2(|\alpha|+1)}} \left(\oint_{\partial\Omega} \left| \frac{\partial u}{\partial n} \right|^2 ds_x + \oint_{\Omega^C} |f|^2 \, dx \right),$$

and

(c)
$$\left| \frac{\partial u(x)}{\partial r} + \frac{1}{r} u(x) \right|^2 \leqslant \frac{C}{r^6} \left(\oint_{\partial\Omega} \left| \frac{\partial u}{\partial n} \right|^2 ds_x + \int_{\Omega^C} |f|^2 \, dx \right).$$

The constant C is independent of r and u.

Note that we shall often use the same letter C to denote different constants when there is no danger of confusion.

Proof. (a) We may apply (2.1) and integration by parts to see that

$$(2.2) \quad u(x) = \frac{1}{4\pi} \left(\oint_{\partial\Omega} \frac{1}{\rho} \frac{\partial u(x')}{\partial n'} \, ds_{x'} - \int_{\Omega^C} \frac{1}{\rho} f(x') \, dx' \right) \quad \forall x \in \Omega^C,$$

where $\rho = |x - x'|$, $x' = (r', \omega', \theta')$, and n' is the outward directed normal to $\partial\Omega$ at the point x'. It may be seen (as, e.g., in [11]) that

$$(2.3) \qquad \frac{1}{\rho} = \frac{1}{r\sqrt{1 - \dfrac{2r'}{r}\cos\psi + \left(\dfrac{r'}{r}\right)^2}}.$$

Here ψ is the angle between the rays joining x and x' to the origin, so that $\cos\psi = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\omega - \omega')$. Since f has bounded support, we readily obtain (a) using (2.2) and (2.3).

(b) It is clear that we may differentiate under the integral sign in (2.2). Hence, (b) follows from (2.2), (2.3), and the Schwarz inequality.

(c) We again apply (2.2), (2.3), and the Schwarz inequality to see that (c) holds. Q.E.D.

We next establish the well-posedness of problem (2.1).

LEMMA 2.2. *There exists a unique solution $u \in C^\infty(\Omega^C)$ of problem* (2.1).

Proof. The existence of a solution u of (2.1) is well known and may be established using the method of integral equations; see, e.g., [12] and the references cited there. The smoothness of u follows from the smoothness of f and $\partial\Omega$ using standard regularity theory for elliptic differential equations, [13]. The uniqueness of u may be established as follows. Suppose that $f = 0$ in (2.1) and apply integration by parts on Ω_R with R large to obtain:

$$O = \int_{\Omega_R} - u\Delta u\, dx = \int_{\Omega_R} |\nabla u|^2\, dx - \oint_{\partial S_R} u \frac{\partial u}{\partial r}\, ds_x.$$

It follows from Lemma 2.1(a) that

$$u(x) = O\left(\frac{1}{r}\right) \quad \text{and} \quad \frac{\partial u(x)}{\partial r} = O\left(\frac{1}{r^2}\right) \quad \text{as } r \to \infty.$$

Hence, we may let $R \to \infty$ to see that $\int_{\Omega^C} |\nabla u|^2\, dx = 0$. Thus, u is a constant in Ω^C and $u = O(1/r)$ as $r \to \infty$, so that $u = 0$ in Ω^C. We have thus proved uniqueness. Q.E.D.

Note. Even if $f \notin C^\infty(\Omega^C)$, $u(x)$ is still infinitely differentiable outside of the support of f.

As we indicated earlier, we shall not approximate the solution of (2.1) directly but instead shall introduce an intermediate problem on a bounded domain. We thus consider the following problem:

$$(2.4) \qquad -\Delta_R = f \quad \text{in } \Omega_R, \quad u_r = 0 \quad \text{on } \partial\Omega, \quad \frac{\partial u_R}{\partial r} + \frac{1}{r} u_R = 0 \quad \text{on } \partial S_R.$$

In view of the last equation in (2.4), it follows from Lemma 2.1(c) that

$$(2.5) \qquad \max_{x \in \partial S_R} \left| \frac{\partial}{\partial r}(u(x) - u_R(x)) + \frac{1}{r}(u(x) - u_R(x)) \right| = \max_{x \in \partial S_R} \left| \frac{\partial u(x)}{\partial r} + \frac{1}{r} u(x) \right|$$

$$< \frac{C}{R^3} \left(\oint_{\partial\Omega} \left| \frac{\partial u}{\partial n} \right|^2 ds_x + \int_{\Omega^C} |f|^2\, dx \right)^{1/2}.$$

Our main goal in this section is to establish mean-square estimates for $u - u_R$ and $\nabla(u - u_R)$ over Ω_R. First, however, we prove that problem (2.4) is well posed.

LEMMA 2.3. *There exists a unique solution $u_R \in C^\infty(\Omega_R)$ of (2.4).*

Proof. First we prove uniqueness. Suppose that u_R satisfies (2.4) with $f = 0$ and note that

$$0 = \int_{\Omega_R} - u_R \Delta u_R \, dx = \int_{\Omega_R} |\nabla u_R|^2 \, dx - \oint_{\partial \Omega_R} u_R \frac{\partial u_R}{\partial r} \, ds_x$$

$$= \int_{\Omega_R} |\nabla u_R|^2 \, dx + \frac{1}{R} \oint_{\partial S_R} |u_R|^2 \, ds_x.$$

Hence, u_R is a constant in Ω_R and $u_R = 0$ on ∂S_R. Thus, $u_R = 0$ in Ω_R and we have proved uniqueness. The remainder of the theorem now follows from [13, Theorem 2.1], in view of the smoothness of f and $\partial \Omega$. Q.E.D.

In order to apply the finite element method to problem (2.4), we first express this problem in a variational formulation. Set

$$H_R^E = \left\{ u : u \in H^1(\Omega_R), u = 0 \text{ on } \partial \Omega \right\}$$

and define

(2.6)
$$\begin{cases} a_R(u, v) = \int_{\Omega_R} \nabla u \cdot \nabla v \, dx + \frac{1}{R} \oint_{\partial S_R} uv \, ds_x \quad \forall u, v \in H_R^E, \\ \text{and} \\ \|u\|_{H_R^E} = (a_R(u, u))^{1/2} \quad \forall u \in H_R^E. \end{cases}$$

LEMMA 2.4. (a) *There exists a constant $C(R)$ such that*

$$\|v\|_{H^1(\Omega_R)} \leqslant C(R) \|v\|_{H_R^E} \quad \forall v \in H_R^E,$$

where $C(R)$ is independent of v (but depends on R in general),

(b) *there exists a unique function $u^1 \in H_R^E$ satisfying the equation*

$$a_R(u^1, v) = (f, v) \quad \forall v \in H_R^E,$$

and

(c) *$u^1 = u_R$, where u_R is the solution of (2.4).*

Proof. (a) This follows from Rellich's theorem as in [2, Theorem 1.2.1].

(b) This follows readily from (a); see, e.g., [2] or [3].

(c) This follows easily from (2.4), (2.6), and (b) using integration by parts. Q.E.D.

We are now ready to obtain estimates for $e_R(x) = u(x) - u_R(x)$, where u and u_R satisfy (2.1) and (2.4), respectively. For convenience, we set

(2.7)
$$w_R(x) = \left(\frac{\partial}{\partial r} + \frac{1}{R} \right) u(x) \quad \text{for each } x \text{ in } \partial S_R.$$

LEMMA 2.5. *Suppose that u satisfies (2.1), u_R satisfies (2.4), and $e_R = u - u_R$. Then there exists a constant C, independent of R, such that*

(a)
$$\|e_R\|_{H_R^E}^2 \leqslant \frac{C}{R^3} \left(\oint_{\partial \Omega} \left| \frac{\partial u}{\partial n} \right|^2 \, ds_x + \int_{\Omega_R} |f|^2 \, dx \right),$$

and

(b)
$$\int_{\Omega_R} |e_R|^2 \, dx \leqslant \frac{C}{R} \left(\oint_{\partial \Omega} \left| \frac{\partial u}{\partial n} \right|^2 \, ds_x + \int_{\Omega_R} |f|^2 \, dx \right).$$

Proof. (a) It follows from (2.1), (2.4), and (2.7) that

$$(2.8) \qquad \frac{1}{R} \oint_{\partial S_R} e_R^2 \, ds_x = -\oint_{\partial S_R} e_R \frac{\partial e_R}{\partial r} \, ds_x + \oint_{\partial S_R} e_R w_R \, ds_x.$$

We apply the Schwarz inequality and Lemma 2.1(c) to the last term in (2.8) to deduce

$$\left| \oint_{\partial S_R} e_R w_R \, ds_x \right| \leqslant \left(\oint_{\partial S_R} e_R^2 \, ds_x \right)^{1/2} \left(\oint_{\partial S_R} w_R^2 \, ds_x \right)^{1/2}$$

$$(2.9) \qquad \leqslant \frac{C}{R^3} \left(\oint_{\partial S_R} e_R^2 \, ds_x \right)^{1/2} \left(\oint_{\partial \Omega} \left| \frac{\partial u}{\partial n} \right|^2 \, ds_x + \int_{\Omega_R} f^2 \, dx \right)^{1/2}$$

$$\leqslant \frac{C_\varepsilon}{R^3} \left(\oint_{\partial \Omega} \left| \frac{\partial u}{\partial n} \right|^2 \, ds_x + \int_{\Omega_R} f^2 \, dx \right) + \frac{\varepsilon}{R} \oint_{\partial S_R} e_R^2 \, ds_x,$$

where $\varepsilon > 0$ is arbitrarily small and C_ε depends on ε but not on R. Applying (2.1), (2.4), and integration by parts, we see that

$$(2.10) \quad \int_{\Omega_R} |\nabla e_R|^2 \, dx - \oint_{\partial S_R} e_R \frac{\partial e_R}{\partial r} \, ds_x = -\int_{\Omega_R} e_R \Delta e_R \, dx - \oint_{\partial \Omega} e_R \frac{\partial e_R}{\partial n} \, ds_x = 0.$$

Finally, we combine (2.8)–(2.10) to conclude that

$$\int_{\Omega_R} |\nabla e_R|^2 \, dx + \frac{1}{R} \oint_{\partial S_R} e_R^2 \, ds_x = \int_{\Omega_R} |\nabla e_R|^2 \, dx - \oint_{\partial S_R} e_R \frac{\partial e_R}{\partial r} \, ds_x + \oint_{\partial S_R} e_R w_R \, ds_x$$

$$\leqslant \frac{C_\varepsilon}{R^3} \left(\oint_{\partial \Omega} \left| \frac{\partial u}{\partial n} \right|^2 \, ds_x + \int_{\Omega_R} f^2 \, dx \right) + \frac{\varepsilon}{R} \oint_{\partial S_R} e_R^2 \, ds_x.$$

We may now choose $\varepsilon = \frac{1}{2}$ in the last estimate to obtain (a).

(b) To begin with, we observe that

$$\int_{\Omega_R} |\nabla(r e_R)|^2 \, dx = \int_{\Omega_R} |r \nabla e_R + e_R \nabla r|^2 \, dx$$

$$= \int_{\Omega_R} \left(r^2 |\nabla e_R|^2 + e_R^2 |\nabla r|^2 + 2 r e_R \nabla r \cdot \nabla e_R \right) \, dx.$$

Since $|\nabla r| = 1$, we deduce

$$\int_{\Omega_R} e_R^2 \, dx = \int_{\Omega_R} |\nabla(r e_R)|^2 \, dx - \int_{\Omega_R} r^2 |\nabla e_R|^2 \, dx - \int_{\Omega_R} 2 r e_R \nabla r \cdot \nabla e_R \, dx$$

$$(2.11) \qquad \leqslant \int_{\Omega_R} |\nabla(r e_R)|^2 \, dx + \frac{1}{2} \int_{\Omega_R} e_R^2 \, dx + C \int_{\Omega_R} r^2 |\nabla e_R|^2 \, dx.$$

Using (a), we obtain

$$(2.12) \qquad \int_{\Omega_R} r^2 |\nabla e_R|^2 \, dx \leqslant R^2 \int_{\Omega_R} |\nabla e_R|^2 \, dx \leqslant \frac{C}{R} \left(\oint_{\partial \Omega} \left| \frac{\partial u}{\partial n} \right|^2 \, ds_x + \int_{\Omega_R} f^2 \, dx \right).$$

Combining (2.11) and (2.12), we see that

$$(2.13) \qquad \int_{\Omega_R} e_R^2 \, dx \leqslant \frac{C}{R} \left(\oint_{\partial \Omega} \left| \frac{\partial u}{\partial n} \right|^2 \, ds_x + \int_{\Omega_R} f^2 \, dx \right) + 2 \int_{\Omega_R} |\nabla(r e_R)|^2 \, dx.$$

We next estimate the last term in (2.13). We apply integration by parts to obtain

$$(2.14) \qquad \int_{\Omega_R} |\nabla(re_R)|^2 \, dx = \int_{\Omega_R} re_R(-\Delta(re_R)) \, dx + \oint_{\partial S_R} re_R \frac{\partial}{\partial r}(re_R) \, ds_x.$$

To estimate the last term on the right, we observe that $r = R$ on ∂S_R and we employ (2.8), (2.9), and (a) to conclude that

$$\left| \oint_{\partial S_R} re_R \frac{\partial}{\partial r}(re_R) \, ds_x \right| = \left| \oint_{\partial S_R} \left(re_R^2 + r^2 e_R \frac{\partial e_R}{\partial r} \right) ds_x \right| = \left| \oint_{\partial S_R} R^2 e_R w_R \, ds_x \right|$$

$$(2.15) \qquad\qquad \leq R^2 \left(\frac{C}{R^3} \left(\oint_{\partial \Omega} \left| \frac{\partial u}{\partial n} \right|^2 ds_x + \oint_{\Omega_R} f^2 \, dx \right) + \frac{1}{R} \oint_{\partial S_R} e_R^2 \, ds_x \right)$$

$$\leq \frac{C}{R} \left(\oint_{\partial \Omega} \left| \frac{\partial u}{\partial n} \right|^2 ds_x + \int_{\Omega_R} f^2 \, dx \right).$$

Finally, we estimate

$$\int_{\Omega_R} re_R(-\Delta(re_R)) \, dx = \int_{\Omega_R} re_R(e_R(-\Delta r) + r(-\Delta e_R) - 2\nabla r \cdot \nabla e_R) \, dx.$$

Noting that $\Delta e_R = 0$ and $\Delta r = 2/r$, we deduce

$$(2.16) \qquad \int_{\Omega_R} re_R(-\Delta(re_R)) \, dx = -2 \int_{\Omega_R} \left(e_R^2 + re_R \nabla r \cdot \nabla e_R \right) dx.$$

Using (a), it follows readily that

$$\left| \int_{\Omega_R} 2re_R \nabla r \cdot \nabla e_R \, dx \right| \leq \frac{1}{2} \int_{\Omega_R} e_R^2 \, dx + C \int_{\Omega_R} r^2 |\nabla e_R|^2 \, dx$$

$$(2.17) \qquad\qquad \leq \frac{1}{2} \int_{\Omega_R} e_R^2 \, dx + \frac{C}{R} \left(\oint_{\partial \Omega} \left| \frac{\partial u}{\partial n} \right|^2 ds_x + \int_{\Omega_R} f^2 \, dx \right).$$

Combining (2.14)–(2.17) with (2.13), we now conclude that

$$\int_{\Omega_R} e_R^2 \, dx \leq \frac{C}{R} \left(\oint_{\partial \Omega} \left| \frac{\partial u}{\partial n} \right|^2 ds_x + \int_{\Omega_R} f^2 \, dx \right).$$

This completes the proof of the lemma. Q.E.D.

The following result enables us to simplify the estimates in Lemma 2.5.

LEMMA 2.6. *Suppose that u satisfies (2.1). Then there exists a constant C, independent of u, such that*

$$\oint_{\partial \Omega} \left| \frac{\partial u}{\partial n} \right|^2 ds_x \leq C \int_{\Omega^C} f^2 \, dx.$$

Proof. Without loss of generality, we may assume that $\Omega \subset S_1$ and $\text{supp}(f) \subset S_1$. Let χ denote a cutoff function satisfying the following conditions: $\chi \in C^\infty(\Omega^C)$, $\chi \equiv 1$ in a neighborhood of $\partial \Omega$ and $\chi \equiv 0$ in a neighborhood of ∂S_1. Set $\Omega_1 = \Omega^C \cap S_1$. Since $\chi u = 0$ on $\partial \Omega$ and on ∂S_1, we may employ (2.1), the theory of elliptic differential equations and the triangle inequality, to deduce

$$\oint_{\partial \Omega} \left| \frac{\partial u}{\partial n} \right|^2 ds_x = \oint_{\partial \Omega} \left| \frac{\partial (\chi u)}{\partial n} \right|^2 ds_x \leq C \|\chi u\|_{H^2(\Omega_1)}^2$$

$$\leq C \|-\Delta(\chi u)\|_{L^2(\Omega_1)}^2 \leq C \left(\|f\|_{L^2(\Omega^C)}^2 + \|u\|_{H^1(\Omega_1)}^2 \right).$$

Since $u = 0$ on $\partial\Omega$, it follows that

$$\int_{\Omega_1} |u|^2 \, dx \leqslant C \int_{\Omega_1} |\nabla u|^2 \, dx$$

(see, e.g., [2]). We thus obtain

$$(2.18) \qquad \oint_{\partial\Omega} \left| \frac{\partial u}{\partial n} \right|^2 ds_x \leqslant C \left(\int_{\Omega_1} |\nabla u|^2 \, dx + \int_{\Omega^C} f^2 \, dx \right).$$

Now suppose that R_0 is a fixed large positive number and u_{R_0} satisfies (2.4) with R replaced by R_0. Choosing R_0 sufficiently large, we may apply Lemma 2.5(a) to obtain

$$(2.19) \qquad \int_{\Omega_{R_0}} |\nabla(u - u_{R_0})|^2 \, dx \leqslant \frac{C}{R_0^3} \left(\oint_{\partial\Omega} \left| \frac{\partial u}{\partial n} \right|^2 ds_x + \int_{\Omega^C} f^2 \, dx \right),$$

with C independent of R_0. Again, employing elliptic regularity theory, we see that

$$(2.20) \qquad \|u_{R_0}\|_{H^1(\Omega_{R_0})} \leqslant C_{R_0} \|f\|_{L^2(\Omega_{R_0})}.$$

Combining (2.18)–(2.20), we conclude that

$$(2.21) \qquad \oint_{\partial\Omega} \left| \frac{\partial u}{\partial n} \right|^2 ds_x \leqslant K_{R_0} \int_{\Omega^C} f^2 \, dx + \frac{C}{R_0^3} \oint_{\partial\Omega} \left| \frac{\partial u}{\partial n} \right|^2 ds_x,$$

where K_{R_0} depends on R_0 but C is independent of R_0. We now choose $R_0^3 = 2C$ to obtain the lemma.   Q.E.D.

We may now combine Lemmas 2.5 and 2.6 to immediately deduce the main result of this section.

THEOREM 2.1. *Suppose that R is sufficiently large and that u and u_R satisfy (2.1) and (2.4), respectively. Then there exists a constant C, independent of R and u, such that*

$$(a) \qquad \int_{\Omega_R} |\nabla(u - u_R)|^2 \, dx + \frac{1}{R} \oint_{\partial S_R} (u - u_R)^2 \, ds_x \leqslant \frac{C}{R^3} \int_{\Omega_R} f^2 \, dx,$$

and

$$(b) \qquad \int_{\Omega_R} (u - u_R)^2 \, dx \leqslant \frac{C}{R} \int_{\Omega_R} f^2 \, dx.$$

Remark 2.1. There has recently been developed, [14], a hierarchy of approximate boundary conditions at infinity for calculating the solution of problems such as (2.1). The construction of these boundary conditions is based on the series expansion for u given in Lemma 2.1(a). The first boundary condition in this hierarchy is given by $\partial u/\partial r + u/r = 0$ on ∂S_R and corresponds to the outer boundary condition in problem (2.4). The arguments of this paper appear to be applicable to higher order boundary conditions as well, although this will not be considered here.

We also observe that the following error estimate was proved in [14]:

$$(2.22) \qquad \|u - u_R\|_{L^\infty(\Omega_R)} \leqslant \frac{C}{R^2},$$

where C is independent of R. The proof of (2.22) is based on a version of the maximum principle dealing with impedence boundary conditions. Since we are analyzing the finite element method, which is based on a variational principle, the mean-square and energy estimates of Theorem 2.1 are more appropriate to the purposes of this paper.

3. The Finite Element Method. In this section we shall construct an approximate solution u_R^h of (2.4) using the finite element method with element mesh size of order $O(h)$ on Ω_R. We begin by considering a discretized version of the variational problem in Section 2, obtained by replacing H_R^E by a finite-dimensional subspace. We shall then describe the finite element method and estimate the error $\|u - u_R^h\|_{H_R^E}$, where u satisfies the original problem (2.1). We shall see that this error estimate is suboptimal, thus motivating the last two sections of the paper.

We begin with the following result.

LEMMA 3.1. *Suppose that R is sufficiently large, u_R satisfies (2.4), $a_R(,)$ and H_R^E are defined by (2.6), and M is a finite-dimensional subspace of H_R^E. Then,*

(a) *there exists a unique solution $\hat{u}_R \in M$ of the equation*

$$(3.1) \qquad a_R(\hat{u}_R, v) = (f, v) \quad \forall v \in M,$$

and

(b) $\|u_R - \hat{u}_R\|_{H_R^E} = \inf_{\chi \in M}\|u_R - \chi\|_{H_R^E}.$

The proof of Lemma 3.1 follows easily from Lemma 2.4(a) using the fact that $\hat{u}_R$ is the projection of u_R onto M with respect to the inner product defined by $a_R(,)$; see, e.g., [2] or [3]. We now describe how the finite element method may be employed to approximately solve problem (2.4). We begin by replacing H_R^E by a one parameter family of finite-dimensional subspaces S^h, defined for each $h \in (0, \infty)$. The subspaces S^h are typically obtained by subdividing Ω_R into simple subsets (elements), denoted by t^h, with diameter of order $O(h)$. S^h may then be defined as the subspace of H_R^E consisting of all continuous functions v^h such that the restriction of v^h to each element t^h belongs to some appropriate class of functions, denoted by F. For example, F may consist of all polynomials of degree less than K in a convenient coordinate system for a fixed integer K not less than 2.

Using Lemma 3.1, we may now define the finite element approximation $u_R^h \in S^h$ to the solution of problem (2.4) as the unique solution of

$$(3.2) \qquad a_R(u_R^h, v^h) = (f, v^h) \quad \forall v^h \in S^h.$$

We shall require certain approximation properties of our finite element spaces. Specifically, we assume that

$$\begin{cases} \inf_{\chi \in S^h} \int_{\Omega'} |\nabla(v - \chi)|^2 \, dx \leqslant Ch^{2s-2}|v|^2_{H^s(\Omega')} \\ \text{and} \\ \inf_{\chi \in S^h} \oint_{\partial S_R} |v - \chi|^2 \, ds_x \leqslant Ch^{2s-1}|v|^2_{H^s(\Omega_R)} \end{cases}$$

for each $v \in H^K(\Omega_R)$ and each integer $s \in [2, K]$, where K is an integer not less than 2 and Ω' is an arbitrary subset of Ω_R. The constant C is independent of Ω', h,

and v. It is important to note that only derivatives of order s appear on the right-hand side of these estimates. The estimates in (3.3) hold for typical finite element spaces; see, e.g., [1]–[3].

There is another condition, typically satisfied by finite element subspaces, that will be useful throughout this paper. Let Ω' denote an arbitrary subset of Ω_R. Set $S_{\Omega'}^{h\perp} = \{\chi \in S^h: \chi(x) = 0 \text{ for } x \in \Omega'\}$ and denote the orthogonal complement of $S_{\Omega'}^{h\perp}$ by $S_{\Omega'}^{h} = S^h - S_{\Omega'}^{h\perp}$. We assume that there exist constants, C_1 and C_2, independent of $h \in (0, 1]$ and Ω', such that

$$(3.4) \qquad C_1 h^{-3} \operatorname{meas}(\Omega') \leqslant \dim(S_{\Omega'}^h) \leqslant C_2 h^{-3} \operatorname{meas}(\Omega').$$

Here $\operatorname{meas}(\Omega')$ denotes the measure of Ω' and $\dim(S_{\Omega'}^h)$ denotes the dimension of the finite-dimensional subspace $S_{\Omega'}^h$.

Remark 3.1. We observe that since $S^h \subset H_R^E$, functions in S^h must vanish on $\partial\Omega$. This will cause problems, in general, when $\partial\Omega$ does not have a simple shape. (The boundary condition on ∂S_R is natural. Hence, no boundary condition needs to be imposed on ∂S_R.) Methods have been developed for overcoming the difficulties associated with essential boundary conditions; see [1] or [2]. For example, $\partial\Omega$ may be approximated by surfaces with simple shapes or the finite element method may be generalized. These techniques for treating essential boundary conditions on $\partial\Omega$ will not affect the arguments of this paper. For the sake of simplicity, we shall consider the usual finite element formulation described above. See [1]–[3] for more detailed descriptions of the finite element method.

We next estimate the error $u - u_R^h$ in the energy norm.

THEOREM 3.1. *Suppose that the hypotheses of Lemma 3.1 hold with M replaced by S_R^h for $h \in (0, 1]$, condition (3.3) holds with $K \geqslant 2$, and u satisfies (2.1). Then there exists a unique solution $u_R^h \in S^h$ of (3.2) and a constant C, independent of u, h, and R, such that*

$$(3.5) \qquad \|u - u_R^h\|_{H_R^E} \leqslant C\big(R^{-3/2}\|f\|_{L^2(\Omega^c)} + h^{K-1}|u|_{H^K(\Omega_R)}\big).$$

Proof. We may apply Lemma 3.1 and Theorem 2.1(a) to see that there exists a unique solution $u_R^h \in S^h$ of (3.2) such that

$$\|u - u_R^h\|_{H_R^E} \leqslant \|u - u_R\|_{H_R^E} + \|u_R - u_R^h\|_{H_R^E}$$

$$\leqslant CR^{-3/2}\|f\|_{L^2(\Omega^c)} + \inf_{\chi \in S^h}\|u_R - \chi\|_{H_R^E}$$

$$\leqslant 2CR^{-3/2}\|f\|_{L^2(\Omega^c)} + \inf_{\chi \in S^h}\|u - \chi\|_{H_R^E}.$$

Combining this estimate with (3.3), we have proved the theorem. Q.E.D.

We may obtain an L^2 estimate for $u - u_R^h$ in an analogous fashion using Theorem 2.1 and a duality argument. We postpone the details of this argument until Section 5 since both the energy and L^2 estimates are suboptimal when a uniform mesh size is employed, as in this section. We shall demonstrate the suboptimality of estimate (3.5) as follows. First observe from (3.5) that in order to obtain the estimate

$$(3.6) \qquad \|u - u_R^h\|_{H_R^E} \leqslant Ch^{K-1},$$

it suffices to make R sufficiently large that

$$(3.7) \qquad R^{-3/2} = h^{K-1}.$$

Estimate (3.6) does not tell the whole story since the number of equations to be solved to obtain u_R^h is expressed in terms of $\dim(S^h)$. We define a new parameter H by

$$(3.8) \qquad H = \left(\dim(S^h)\right)^{-1/3}.$$

H is often taken as the relevent parameter to measure convergence when using the finite element method; see [1]. In view of assumption (3.4), we see that

$$(3.9) \qquad C_1 h^{-3} R^3 < \dim(S^h) < C_2 h^{-3} R^3.$$

It follows from (3.7) that $R = h^{-2(K-1)/3}$. Hence we see, using (3.8) and (3.9), that

$$(3.10) \qquad C_2^{-1/3} h^{(2K+1)/3} < H < C_1^{-1/3} h^{(2K+1)/3}.$$

We may thus employ Theorem 3.1 to deduce

COROLLARY 3.1. *Suppose that* (3.4) *and the hypotheses of Theorem* 3.1 *hold. Then there exists a constant* C, *independent of* h, *such that, for* $R = h^{-2(K-1)/3}$, *we have*

$$\|u - u_R^h\|_{H_R^E} \leq C H^{3(K-1)/(2K+1)}.$$

Remark 3.2. Suppose that we replace (2.1) by the following problem

$$(3.11) \quad (-\Delta + \lambda)u = f \quad \text{in } \Omega^C, \quad u = 0 \quad \text{on } \partial\Omega, \quad u \in L^2(\Omega^C), \lambda > 0, f \in C_0^\infty(\Omega^C).$$

It can then be shown that $D^\alpha u(x) = O(e^{-\sigma|x|})$ for some $\sigma > 0$ and all $|x|$ sufficiently large. From this, it may be seen, using the arguments and assumptions of this section, that

$$\|u - u_R^h\|_{H_R^E} \leq C_\epsilon H^{K-1\epsilon} \quad \text{for arbitrary } \epsilon > 0,$$

with u_R^h defined as before. An analogous estimate was obtained by Babuška in [1] and [4] for a problem similar to (3.11).

We shall refer to the following estimates as optimal for problems in unbounded domains:

$$(3.12) \qquad \|u - u_R^h\|_{H_R^E} \leq C H^{K-1},$$

and

$$(3.13) \qquad \|u - u_R^h\|_{L^2(B)} \leq C H^K,$$

where B is a fixed bounded subset of Ω^C. (For problems on bounded domains, optimality is generally defined by estimates such as (3.12) and (3.13).) We see from Corollary 3.1 that the energy estimate is considerably weaker than optimal for problem (2.1). It may also be readily seen that L^2 estimates are suboptimal when uniform mesh sizes are employed. We shall show in Sections 4 and 5 that the optimal estimates (3.12) and (3.13), respectively, hold when the mesh is graded systematically in such a way that the mesh sizes of elements "near infinity" become large.

4. Energy Estimates. It is our purpose, in this section, to prove an optimal energy estimate of the form (3.12). We shall accomplish this by replacing the family of finite element spaces S^h, described in Section 3, by a new family of spaces $\tilde{S}^h$

obtained by a mesh grading process to be described below. We begin by construct-
ing the family of spaces $\tilde{S}^h$ in such a way that the following approximation
estimate holds:

$$(4.1) \qquad\qquad \inf_{\chi \in \tilde{S}^h} \|u - \chi\|_{H_R^E} \leqslant CH^{K-1},$$

where H is defined by (3.8) and u satisfies (2.1). We shall then combine this with
the results of Sections 2 and 3 to obtain our optimal error estimate. At the end of
the section we shall illustrate the mesh grading process with respect to a specific
example.

To begin with, suppose that we have a family of finite element spaces $S^h \subset H_R^E$
satisfying conditions (3.3) and (3.4) for each $h \in (0, \infty)$. Hence, in particular, we
assume that Ω_R is partitioned into a union of elements t^h with diameter of order
$O(h)$. Functions in S^h are continuous and are such that their restriction to each t^h
belongs to a simple class of functions, denoted by F, as described in Section 3. We
now define a new subspace $\tilde{S}^h \subset H_R^E$ by systematically increasing the mesh sizes of
elements as their distance from the origin increases in such a way that estimate
(4.1) will hold.

To be precise, we define

$$S_j = \{x : 2^{j-1} < |x| < 2^j\}, \qquad j = 1, 2, \ldots, J_R,$$

with J_R to be specified below. Without loss of generality, we may assume that Ω
and the support of f are contained in the unit sphere. Set $\Omega_j^R = S_j \cap \Omega_R$, $j = 1, 2, \ldots, J_R$, and $\Omega_0^R = \overline{\Omega}_R - \cup_{j=1}^{J_R} \Omega_j^R$. We may obtain a new partition of Ω_R by
assuming that elements t^h in each annular region Ω_j^R have diameter of order $O(h_j)$,
with h_j to be determined shortly, $j = 0, 1, \ldots, J_R$. We now define $\tilde{S}^h$ to consist of
those continuous functions $\chi \in H_R^E$ such that χ restricted to each t^{h_j} belongs to F.

In order to satisfy (4.1), we determine the parameters h_j as follows. Using (3.3),
we may choose a function $\chi_j \in S^{h_j}$ such that

$$(4.2) \qquad \begin{cases} \displaystyle\int_{\Omega_j} R|\nabla(u - \chi_j)|^2 \, dx \leqslant C h_j^{2K-2} |u|_{H^K(\Omega_j^R)}^2, \qquad j = 0, 1, \ldots, J_R, \\[2mm] \text{and} \\[2mm] \displaystyle\frac{1}{R} \oint_{\partial S_R} |u - \chi_{J_R}|^2 \, ds_x \leqslant C \frac{h_{J_R}^{2K-1}}{R} |u|_{H^K(\Omega_{J_R}^R)}^2, \end{cases}$$

where C is independent of h_j, u, and R. We shall next apply Lemma 2.1. For the
sake of simplicity, we assume in the remainder of this paper that Lemma 2.1 holds
for $r = |x| \geqslant 1$. We employ Lemma 2.1(b) to see that

$$(4.3) \qquad\qquad |u|_{H^K(\Omega_j^R)}^2 \leqslant C \int_{\Omega_j^R} r^{-2(K+1)} \, dx \leqslant C 2^{-(2K-1)j}.$$

In view of (4.2) and (4.3), we wish to choose h_j such that

$$(4.4) \qquad\qquad h_j^{2K-2} 2^{-(2K-1)j} \leqslant h^{2K-2}.$$

Hence, we set

$$(4.5) \qquad h_0 = h \quad \text{and} \quad h_j = 2^{K'j} h \quad \text{with } 1 < K' < \frac{K - 1/2}{K - 1}, j = 1, 2, \ldots, J_R.$$

We next choose J_R' so that $2^{J_R'} = R$. For the reasons given in Section 3, we define R by (3.7). We thus have

$$(4.6) \qquad\qquad R = h^{-2(K-1)/3},$$

so that $J_R' = \log_2(1/h)^{2(K-1)/3}$. Since J_R' is not necessarily an integer, set

$$(4.7) \qquad\qquad J_R = [J_R'] + 1 = \left[\log_2 \left(\frac{1}{h}\right)^{2(K-1)/3} \right] + 1,$$

where $[J_R']$ denotes the greatest integer less than J_R'. Set $\sigma = J_R - J_R'$ and note that

$$(4.8) \qquad\qquad 2^{J_R} = 2^{\sigma} R, \qquad 0 < \sigma < 1.$$

Combining (4.5), (4.6), and (4.8), we readily deduce

$$(4.9) \qquad\qquad h_{J_R} < R$$

for h sufficiently small.

We now let χ denote the function in $\tilde{S}^h$ such that χ, restricted to Ω_j^R, is equal to χ_j, where χ_j satisfies (4.2), $j = 0, 1, \ldots, J_R$. Hence, we may combine (4.2), (4.3), (4.5), and (4.9) to obtain

$$(4.10) \qquad \begin{aligned} \|u - \chi\|_{H_R^E} &< \sum_{j=0}^{J_R} C h_j^{K-1} 2^{-(K-1/2)j} \\ &< C \sum_{j=0}^{J_R} h^{K-1} 2^{K'(K-1)j} 2^{-(K-1/2)j} < C h^{K-1}, \end{aligned}$$

where h is sufficiently small and the constant C is independent of h. Furthermore, we may apply (3.4) and (4.5) to deduce

$$(4.11) \qquad \begin{aligned} \dim(\tilde{S}^h) &< \sum_{j=0}^{J_R} \dim\left(\tilde{S}_{\Omega_j^R}^h\right) < C \sum_{j=0}^{J_R} h_j^{-3} \operatorname{meas}\left(\Omega_j^R\right) \\ &< C \sum_{j=0}^{J_R} h_j^{-3} (2^j)^3 < C \left(h^{-3} + \sum_{j=1}^{J_R} h^{-3} (2^{-K'j})^3 (2^j)^3 \right) \\ &< C h^{-3} \sum_{j=0}^{J_R} (2^{-3(K'-1)})^j < C_2 h^{-3}, \end{aligned}$$

where h is sufficiently small and the constant C_2 is independent of h. In view of (3.4) and (4.5), we also readily obtain

$$(4.12) \qquad\qquad \dim(\tilde{S}^h) > C_1 h^{-3},$$

with h sufficiently small and C_1 independent of h. Combining (3.8), (4.11), and (4.12), we deduce

$$(4.13) \qquad\qquad K_1 h < H < K_2 h,$$

for h sufficiently small and suitable constants K_1 and K_2 independent of h. Finally, we combine (4.10) and (4.13) to conclude that (4.1) holds.

We next apply Lemma 3.1 to obtain a unique function $\tilde{u}_R^h \in \tilde{S}^h$ satisfying

$$(4.14) \qquad\qquad a_R(\tilde{u}_R^h, v^h) = (f, v^h) \quad \forall v^h \in \tilde{S}^h.$$

Applying the triangle inequality, Theorem 2.1(a), and Lemma 3.1, we see that

$$
(4.15) \quad
\begin{aligned}
\|u - \tilde{u}_R^h\|_{H_R^E} &\leqslant \|u - u_R\|_{H_R^E} + \|u_R - \tilde{u}_R^h\|_{H_R^E} \\
&\leqslant CR^{-3/2} + \inf_{\chi \in \tilde{S}^h} \|u_R - \chi\|_{H_R^E} \leqslant CR^{-3/2} + \inf_{\chi \in \tilde{S}^h} \|u - \chi\|_{H_R^E}.
\end{aligned}
$$

Finally, we combine (4.1), (4.6), and (4.13) with (4.15) to obtain the main result of this section.

THEOREM 4.1. *Suppose that $\tilde{S}^h$ is the subspace of H_R^E defined above with $R = h^{-2(K-1)/3}$, u satisfies (2.1), and $\tilde{u}_R^h$ satisfies (4.14). Then there exists an $h_0 \in (0, 1]$ such that, for $h \in (0, h_0]$, we have*

$$
\|u - \tilde{u}_R^h\|_{H_R^E} \leqslant CH^{K-1},
$$

where C is independent of h and H is defined by (3.8).

Remark 4.1. Observe that we assumed R and h to be related by (4.6) in order to obtain an optimal error estimate for $u - \tilde{u}_R^h$. However, the mesh grading process was defined independently of R. The main feature of this process is embodied in the following estimate:

$$
\operatorname{diam}(\tau) < hd^{(K-1/2)/(K-1)} \quad \text{on each annulus } \Omega_j^R,
$$

where d is the distance of the element τ from the origin. Note that if the mesh size is roughly doubled on successive annuli, so that $h_j = 2^j h$, then $\dim(\tilde{S}^h) = O(h^{-3}|\log h|)$ instead of $O(h^{-3})$.

Remark 4.2. The results of this paper may be extended in various directions. For example, these methods may be applied to problems in scattering theory by replacing the Laplace operator by the Helmholtz operator, given by $-\Delta - K^2$. In this case, the condition at infinity is replaced by a suitable radiation condition. There are, however, certain technical difficulties associated with the indefiniteness of the relevent bilinear form. We may also treat variable coefficient perturbations of the Laplace and Helmholtz operators by the present methods, provided the perturbations have bounded support. In a subsequent publication, we shall deal with these and other extensions of the present results (such as L^∞ error estimates). Finally, we mention that numerical computations demonstrating the theoretical results of this paper will appear elsewhere.

Example 4.1. In order to clarify the finite element method described above, we consider an axially symmetric problem in the exterior of the sphere $\Omega = \{x: |x| < \frac{1}{2}\}$. Using spherical polar coordinates and the axial symmetry, we may replace $\Omega_R = \{x: \frac{1}{2} < |x| < R\}$ by the following two-dimensional domain:

$$
\hat{\Omega}_R = \left\{(r, \theta): \tfrac{1}{2} < r < R, 0 < \theta < \pi\right\}.
$$

The solution u_R of problem (2.4) now satisfies the following boundary conditions on $\partial\hat{\Omega}_R$:

$$
u_R = 0 \quad \text{on } r = \frac{1}{2}, \quad \frac{\partial u_R}{\partial r} + \frac{1}{R} u_R = 0 \quad \text{on } r = R, \quad \text{and} \quad \frac{\partial u_R}{\partial \theta} = 0 \quad \text{at } \theta = 0, \pi.
$$

The last condition is due to the axial symmetry. The differential operator $-\Delta$ as well as the bilinear form $a_R(\ ,\ )$ are readily expressed in polar coordinates.

We now construct a finite element space S^h consisting of piecewise linear functions defined on $\hat{\Omega}_R$. We first partition $\hat{\Omega}_R$ into triangles of maximum diameter h, as shown in Figure 4.1, taking a uniform mesh spacing in the r and θ directions. Next, let S^h consist of all functions $\chi(r, \theta)$, defined on $\hat{\Omega}_R$, satisfying the following conditions:

(i) χ is continuous on $\hat{\Omega}_R$,

(ii) χ is linear on each triangle,

and

(iii) $\chi = 0$ on $r = \frac{1}{2}$.

The space S^h may be constructed using a nodal basis of Lagrange type in the usual way; see, e.g., [2] or [3]. It may be readily seen that S^h satisfies (3.3) (with $K = 2$) and (3.4). In Figure 4.1, we set $h_r = h_0 = 2^{-3}$ and $h_\theta = \pi/10$.
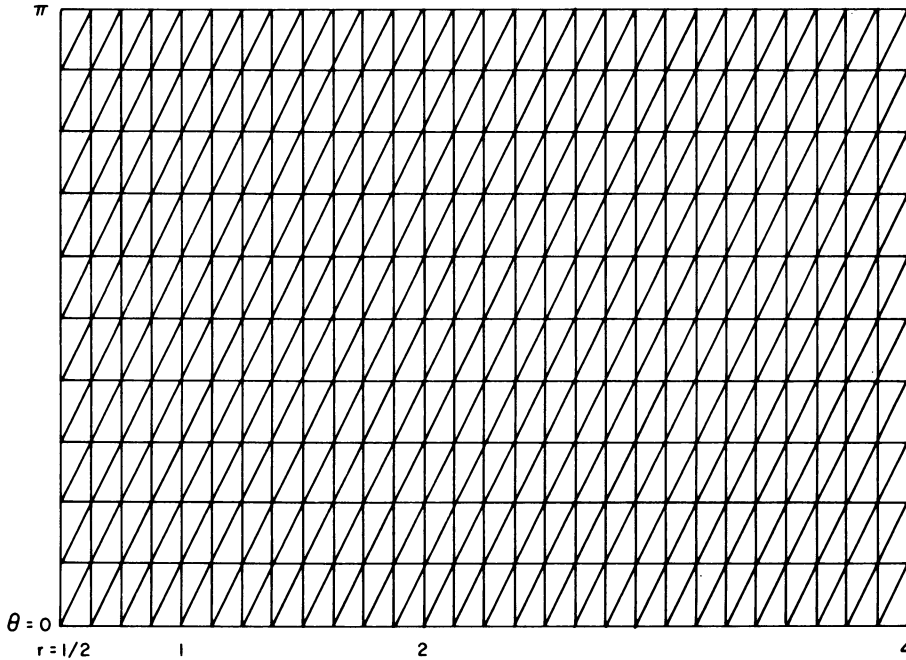

FIGURE 4.1 (*Uniform Mesh*)

We next define a new finite element space $\tilde{S}^h$ in accordance with the mesh grading procedure described above; see Figure 4.2. For simplicity, suppose that $\text{supp}(f) \subset \{x: |x| < 1\}$. We set $\Omega_0^R = \{x: \frac{1}{2} < |x| < 1\}$, $\Omega_j^R = \{x: 2^{j-1} < |x| < 2^j\}$, $j = 1, 2, \ldots, J_R$, $h_\theta = \pi/10$, and $h_0 = 2^{-3}$. We increase the mesh size in the r direction as described above. Hence, we set $h_{jr} = 2^{3j/2}h_0$, $j = 0, 1, \ldots, J_R$. Using (4.6) and (4.7), respectively, we set $R = 2^2$ and $J_R = 2$. Functions in $\tilde{S}^h$ are now defined with respect to this partition in the same manner as functions in S^h.
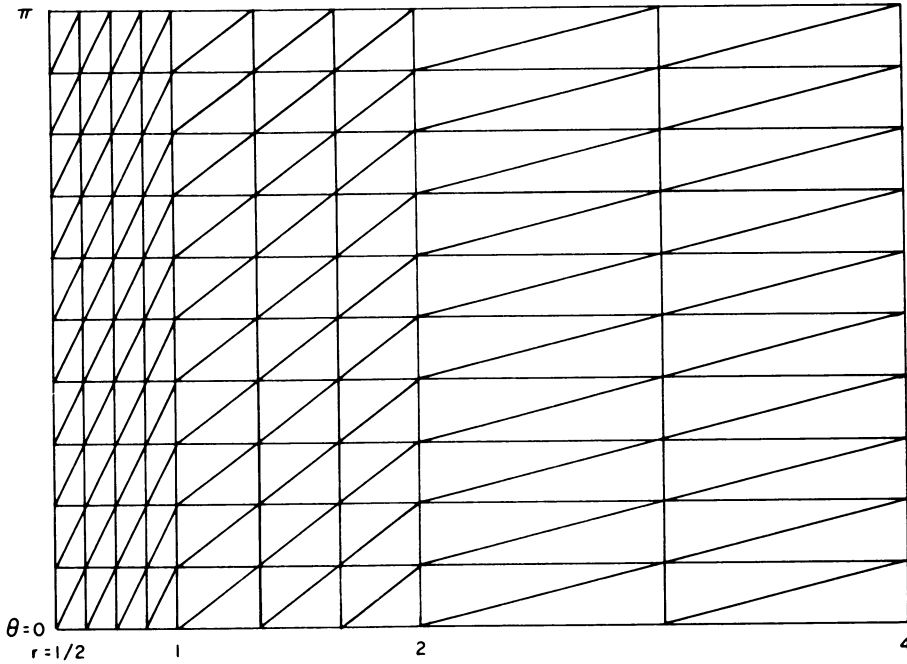
FIGURE 4.2 (*Nonuniform Mesh*)

5. L^2 Estimates. In this section, we shall establish the following optimal mean-square error estimate:

$$(5.1) \qquad \|u - \tilde{u}_R^h\|_{L^2(B)} \leqslant CH^K,$$

where B is a bounded subset of Ω^C and $\tilde{u}_R^h$ is defined by (4.14). Note that u, H, and K are defined as before. We first show that, by combining the results of Sections 2 and 3 with a duality argument, we may reduce our mean-square estimate to an approximation estimate. We shall then show that the mesh grading process described in Section 4 yields our optimal mean-square error estimate.

We may assume without loss of generality that Ω and the support of f are contained in the unit sphere and Lemma 2.1 holds for $r > 1$. In order to obtain estimate (5.1) we shall require the following lemma.

LEMMA 5.1. *Suppose that B is a bounded subset of Ω^C, R is sufficiently large, u_R satisfies (2.4), $a_R(\ ,\ )$ and H_R^E are defined by (2.6), M is a finite-dimensional subspace of H_R^E, and $\hat{u}_R \in M$ satisfies the equation*

$$(5.2) \qquad a_R(\hat{u}_R, v) = (f, v) \quad \forall v \in M.$$

Then

$$\|u_R - \hat{u}_R\|_{L^2(B)} \leqslant \inf_{\chi \in M} \|u_R - \chi\|_{H_R^E} \left(\sup_{\phi \in C_0^\infty(B)} \frac{1}{\|\phi\|_{L^2(B)}} \inf_{\psi \in M} \|\Phi_R - \psi\|_{H_R^E} \right),$$

where Φ_R and ϕ are related by

$$(5.3) \quad -\Delta\Phi_R = \phi \quad in\ \Omega_R, \quad \Phi_R = 0 \quad on\ \partial\Omega \quad and \quad \frac{\partial\Phi_R}{\partial r} + \frac{1}{R}\Phi_R = 0 \quad on\ \partial S_R.$$

Proof. It follows from the definition of the L^2 norm that

$$(5.4) \qquad \|u_R - \hat{u}_R\|_{L^2(B)} = \sup_{\phi \in C_0^\infty(B)} \frac{|(u_R - \hat{u}_R, \phi)|}{\|\phi\|_{L^2(B)}}.$$

Combining (2.4), (2.6), (5.2), (5.3), and integration by parts, we see that

$$(u_R - \hat{u}_R, \phi) = a_R(u_R - \hat{u}_R, \Phi_R) = a_R(u_R - \hat{u}_R, \Phi_R - \psi) \quad \forall \psi \in M.$$

Hence, we may apply the Schwarz inequality to obtain

$$(5.5) \qquad |(u_R - \hat{u}_R, \phi)| \leqslant \|u_R - \hat{u}_R\|_{H_R^E} \inf_{\psi \in M} \|\Phi_R - \psi\|_{H_R^E}.$$

Finally, we may employ Lemma 3.1(b), (5.4), and (5.5) to obtain the lemma. Q.E.D.

We next apply Lemma 2.5(b) and Lemma 5.1 to the family of finite element spaces $\tilde{S}^h$ defined in Section 4. Hence, we assume that $h_j \leqslant 2^{K'j}h, j = 0, 1, \ldots, J_R$, with K' defined in (4.5). Now we make the additional assumption that each $h_j \leqslant R$. Set

$$(5.6) \qquad R^{-1/2} = h^K.$$

The integer J_R may be calculated as in Section 4 using (5.6). We thus obtain $J_R = [\log_2(1/h)^{2K}] + 1$. It also follows, as in Section 4, that $\dim(\tilde{S}^h) = O(h^{-3})$ for h sufficiently small.

We recall that $\tilde{u}_R^h$ is defined as the unique solution of Eq. (4.14). Using the triangle inequality, we obtain

$$\|u - \tilde{u}_R^h\|_{L^2(B)} \leqslant \|u - u_R\|_{L^2(B)} + \|u_R - \tilde{u}_R^h\|_{L^2(B)}.$$

We thus see, from Lemma 2.5(b) and (5.6), that

$$(5.7) \qquad \|u - \tilde{u}_R^h\|_{L^2(B)} \leqslant CR^{-1/2} + \|u_R - \tilde{u}_R^h\|_{L^2(B)} \leqslant Ch^K + \|u_R - \tilde{u}_R^h\|_{L^2(B)}.$$

We next apply Lemma 5.1, with M replaced by $\tilde{S}^h$, to the last term in (5.7). In view of this, our goal is to prove the following two estimates:

$$\inf_{\chi \in \tilde{S}^h} \|u_R - \chi\|_{H_R^E} \leqslant Ch^{K-1},$$

and

$$\inf_{\psi \in \tilde{S}^h} \|\Phi_R - \psi\|_{H_R^E} \leqslant Ch\|\phi\|_{L^2(B)}.$$

Employing Theorem 2.1(a) and the triangle inequality, we readily see that it suffices to prove

$$(5.8) \qquad \inf_{\chi \in \tilde{S}^h} \|u - \chi\|_{H_R^E} \leqslant Ch^{K-1},$$

and

$$(5.9) \qquad \inf_{\psi \in \tilde{S}^h} \|\Phi - \chi\|_{H_R^E} \leqslant Ch\|\phi\|_{L^2(B)},$$

where Φ satisfies

$$(5.10) \quad -\Delta\Phi = \phi \quad \text{in } \Omega^C, \quad \Phi = 0 \quad \text{on } \partial\Omega \quad \text{and} \quad \frac{\partial\Phi}{\partial r} + \frac{1}{r}\Phi = o\left(\frac{1}{r}\right) \quad \text{as } r \to \infty.$$

The proof of estimate (5.8) follows as in Section 4. To prove (5.9), we may apply Lemma 2.1(b), Lemma 2.6, and the arguments of Section 4 (with K replaced by 2).

We thus obtain the following sufficient condition for (5.9) to hold:

$$(5.11) \qquad h_j \leqslant 2^{K''j}h, \qquad j = 1, 2, \ldots, J_R, \text{ with } K' < K'' < \frac{3}{2}.$$

Since $2^{K'j} \leqslant 2^{K''j}$, it is readily seen that (5.11) holds. Combining (5.7) with Lemma 5.1, (5.8), and (5.9), we obtain our main result as in Section 4.

THEOREM 5.1. *Suppose that B is a bounded subset of Ω^C, $\tilde{S}^h$ is the subspace of H_R^E defined in Section 4 with $R = h^{-2K}$, u satisfies (2.1), and $\tilde{u}_R^h$ satisfies (4.14). Then there exists an $h_0 \in (0, 1]$ such that, for $h \in (0, h_0]$, we have*

$$\|u - u_R^h\|_{L^2(B)} \leqslant CH^K,$$

where C is independent of h and H is defined by (3.8).

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