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A Method of Calculation of Lifting Flows Around Two-Dimensional Corner-Shaped Bodies

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Abstract. We study the flow problem of a perfect fluid around a two-dimensional corner-shaped body. By adding a singular function to the basis, we obtain better accuracy for the velocities, and we determine the lift with precision.

Introduction. In this paper, we study the flow problem of an incompressible inviscible fluid around a two-dimensional corner-shaped body.

Generally, one uses the potential function of such a flow, which is the solution of an exterior Neumann problem. Considering this solution as a single layer potential, one then gets a Fredholm integral equation of the second kind on the boundary which can be solved by collocation methods.

We shall use another approach. Let Ω represent the interior region of the body, Ω' the exterior one, and Γ its boundary. Then, the stream function of the perturbation flow, Ψ, is the solution of the exterior Dirichlet problem

$$\begin{cases} \Delta\Psi = 0 & \text{in } \Omega', \\ \Psi = \Psi_0 & \text{on } \Gamma. \end{cases}$$

Ψ_0 is $-\Psi_\infty$, Ψ_∞ being the stream function of the onset flow. This flow is usually uniform.

We shall write the solution Ψ as a single layer potential with some additional terms. We shall obtain the single layer potential by solving a variational problem on the boundary, as Nedelec-Planchard did for the three-dimensional case and M. N. Le Roux for the two-dimensional one.

The singularity of the solution near the corner and the Kutta-Joukowsky condition give the other terms and the value of the lift of such a potential.

1. Integral Equations for an Exterior Dirichlet Problem. We are given the following problem:

$$\text{(1.1)} \qquad \begin{cases} \Delta\psi = 0 & \text{in } \Omega', \\ \psi = \psi_0 & \text{on } \Gamma, \end{cases}$$

where Ω' is the exterior domain of a corner-shaped body and Γ its boundary.

Grisvard's results [1] allow us to define the Sobolev spaces on the boundary in quite the same way as Lions and Magenes [2] for $\mathcal{C}^\infty$ boundaries. Furthermore, the spaces thus defined have properties which are quite similar to those of classical Sobolev spaces [3].

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Given ψ_0 in the space $H^{1/2}(\Gamma)$, let us consider a function u of $H^1(\Omega')$, with compact support, such that $u = \psi_0$ on Γ. Problem (1.1) can then be written

$$\begin{cases} \Delta(\psi - u) = -\Delta u & \text{in } \Omega', \\ (\psi - u) = 0 & \text{on } \Gamma. \end{cases}$$

The variational formulation of this problem is

$$\int_{\Omega'} \operatorname{grad}(\psi - u) \cdot \operatorname{grad} \phi = -\int_{\Omega'} \operatorname{grad} u \cdot \operatorname{grad} \phi, \tag{1.2}$$

for all ϕ in some space of functions which vanish at the boundary.

The bilinear form in (1.2) is unfortunately not coercive in the space $H_0^1(\Omega')$:

$$H_0^1(\Omega') = \left\{ u \in L^2(\Omega');\ \frac{\partial u}{\partial x_i} \in L^2(\Omega');\ u|_\Gamma = 0 \right\}.$$

We then have to choose a smaller space by regularizing the behavior of the functions at infinity.

Then we consider the weighted Sobolev space, introduced by M. N. Le Roux [4],

$$W^1(\Omega') = \left\{ \psi;\ \frac{\psi}{\rho(1 + \operatorname{Log} \rho)} \in L^2(\Omega');\ \frac{\partial \psi}{\partial x_i} \in L^2(\Omega'), i = 1, 2 \right\}, \tag{1.3}$$

where the weight ρ is given by $\rho = (1 + r^2)^{1/2}$. Since the functions of $W^1(\Omega')$ and $H^1(\Omega')$ coincide locally, it makes sense to define

$$W_0^1(\Omega') = \{\psi \in W^1(\Omega');\ \psi = 0 \text{ on } \Gamma\}.$$

PROPOSITION 1.1. *Problem* (1.1) *has a unique solution in the space* $W^1(\Omega')$.

Proof. Using a Hardy inequality, one can prove [5] that the expression

$$\|\psi\| = \left(\int_{\Omega'} |\operatorname{grad} \psi|^2 \right)^{1/2} \tag{1.4}$$

is a norm on $W_0^1(\Omega')$, equivalent to the "natural norm" of $W^1(\Omega')$, which is

$$\|\psi\|_{W^1(\Omega')} = \left(\left| \frac{\psi}{\rho(1 + \operatorname{Log} \rho)} \right|^2_{L^2(\Omega')} + \sum_{i=1}^{2} \left| \frac{\partial \psi}{\partial x_i} \right|^2_{L^2(\Omega')} \right)^{1/2}.$$

The bilinear form (1.2) is then coercive on $W_0^1(\Omega')$, and by the Lax-Milgram theorem, we obtain the existence and uniqueness of the solution. □

We shall now consider the same problem for the interior domain:

$$\begin{cases} \Delta\psi = 0 & \text{in } \Omega, \\ \psi = \psi_0 & \text{on } \Gamma. \end{cases} \tag{1.5}$$

It is well known that this problem has a unique solution in the space $H^1(\Omega)$.

Joining the solutions of Problems (1.1) and (1.5), we obtain a function ψ which, since it is continuous at the boundary, belongs to the space

$$W^1(\mathbf{R}^2) = \left\{ \psi;\ \frac{\psi}{\rho(1 + \operatorname{Log} \rho)} \in L^2(\mathbf{R}^2);\ \frac{\partial \psi}{\partial x_i} \in L^2(\mathbf{R}^2), i = 1, 2 \right\}. \tag{1.6}$$

Any function ϕ being given in the space $\mathfrak{D}(\mathbf{R}^2)$ of smooth functions with compact support, we have

$$\langle \Delta\psi, \phi \rangle = -\int_{\mathbf{R}^2} \operatorname{grad} \psi \cdot \operatorname{grad} \phi,$$

where $\langle \cdot , \cdot \rangle$ represents the duality pairing between $\mathcal{D}'(\mathbf{R}^2)$ and $\mathcal{D}(\mathbf{R}^2)$. By using Green's formulas in Ω and Ω' [3], we obtain

$$\int_{\mathbf{R}^2} \operatorname{grad} \psi \cdot \operatorname{grad} \phi = \left\langle \frac{\partial \psi}{\partial n}\bigg|_{\text{int}} - \frac{\partial \psi}{\partial n}\bigg|_{\text{ext}}, \phi|_\Gamma \right\rangle.$$

We denote by $[\partial \psi / \partial n]$ the jump of the normal derivative of the solution across the boundary. Since $\mathcal{D}(\mathbf{R}^2)$ is dense in $W^1(\mathbf{R}^2)$ [4], the equation is exact for all ϕ in $W^1(\mathbf{R}^2)$. It is then valid for the constants which are elements of $W^1(\mathbf{R}^2)$. Thus $[\partial \psi / \partial n]$ has the following property:

$$\left\langle \left[\frac{\partial \psi}{\partial n} \right], 1|_\Gamma \right\rangle = 0,$$

where $\langle \, , \, \rangle$ represents the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. Finally, $[\partial \psi / \partial n]$ is in the space

$$H_0^{-1/2}(\Gamma) = \{ v \in H^{-1/2}(\Gamma); \langle v, 1|_\Gamma \rangle = 0 \}. \tag{1.7}$$

LEMMA 1.1. *For any given v in $H_0^{-1/2}(\Gamma)$, the problem*

$$\begin{cases} \textit{Find } \psi \textit{ such that} \\ \displaystyle\int_{\mathbf{R}^2} \operatorname{grad} \psi \cdot \operatorname{grad} \phi = \langle v, \phi|_\Gamma \rangle, \ \forall \phi \in W^1(\mathbf{R}^2) \end{cases} \tag{1.8}$$

has a unique solution in $W^1(\mathbf{R}^2)/\mathbf{R}$.

Proof. It is a very easy consequence of the coercivity of the bilinear form on the space $W^1(\mathbf{R}^2)/\mathbf{R}$ [4]. □

Moreover, if v is very regular and if it satisfies $\langle v, 1|_\Gamma \rangle = 0$, the solution of (1.8) is given by

$$\psi(x) = -\frac{1}{2\pi} \int_\Gamma v(y) \operatorname{Log}|x - y| \, d\gamma_y + C. \tag{1.9}$$

In order to solve our initial problem, it is then sufficient to know the value of v, which determines ϕ as the sum of a single layer potential and a constant.

The symmetry of the left-hand side of (1.8) leads to the following theorem (proof in [4]):

THEOREM 1.1. *The expression* (1.9) *is an isomorphism of the space $H_0^{-1/2}(\Gamma)$ onto $K/\mathbf{R}$, where $K = \{\psi \in W^1(\mathbf{R}^2); \Delta\psi = 0 \text{ in } \Omega \text{ and } \Omega'\}$. This isomorphism is associated with the following variational problem which is coercive in $H_0^{-1/2}(\Gamma)$:*

$$\begin{cases} \textit{Find } v \in H_0^{-1/2}(\Gamma) \textit{ such that} \\ a(v, v') = -\dfrac{1}{2\pi} \displaystyle\int_\Gamma \int_\Gamma v(x) v'(y) \operatorname{Log}|x - y| \, d\gamma_x d\gamma_y = \langle v', \psi_0 \rangle \\ \textit{for all } v' \in H_0^{-1/2}(\Gamma). \end{cases} \tag{1.10}$$

The left-hand side of (1.10) *is a bilinear symmetric form on $H_0^{-1/2}(\Gamma) \times H_0^{-1/2}(\Gamma)$, and the expression*

$$\|v\| = \left(-\frac{1}{2\pi} \int_\Gamma \int_\Gamma v(x) v(y) \operatorname{Log}|x - y| \, d\gamma_x d\gamma_y \right)^{1/2}$$

is a norm on $H_0^{-1/2}(\Gamma)$, equivalent to its definition norm. □

Remark 1.1. The solution ψ found in this paragraph is the stream function of a nonlifting flow, since $\langle v, 1|_\Gamma \rangle$, which is a multiple of the lift, is set to be zero. Nevertheless, it can also be interpreted as the electric potential in $\mathbf{R}^2$, in the presence of the conductor Ω; v then represents the electric charge on Γ. □

Remark 1.2. This approach to exterior problems, which leads to integral equations of the first kind and, by integration, to variational problems on the boundary, was first introduced by Nedelec-Planchard [5] for a three-dimensional Dirichlet problem. □

2. Approximation of the Nonlifting Flow. Theorem 1.1 shows that it is sufficient to calculate the solution of Problem (1.10) to obtain the stream function ψ by Formula (1.9).

Since we want to use a finite element method to approximate the variational and coercive problem (1.10), we are interested in the regularity of the solution of this problem, or, which is the same, of the initial problem.

2.1. *Some Regularity Results.* Regularity properties of the solution are local. Suppose ψ_0 is given in a "more regular" space–say $H^{3/2}(\Gamma)$. Then the solution ψ of Problem (1.1) will locally behave as a $H^2(\Omega')$ function, except in the vicinity of the singular parts of the boundary.

It is then sufficient to consider the following interior problem:

$$\begin{cases} \Delta\psi = 0 & \text{in } \Omega, \\ \psi = \psi_0 & \text{on } \Gamma, \end{cases} \tag{2.1}$$

where Ω represents a corner-shaped bounded domain in $\mathbf{R}^2$. The corner is assumed to be linear, and the angle is set to be ω, where ω is a real number, greater than π and less than 2π (if $\omega < \pi$, which means Ω is convex, and if $\psi_0 \in H^{3/2}(\Gamma)$, then ψ is in $H^2(\Omega)$).

We have then Grisvard's fundamental theorem [1]:

THEOREM 2.1. *Assume $\psi_0 \in H^{3/2}(\Gamma)$. Then, the solution of Problem* (2.1) *has the form*

$$\psi = ar^{\pi/\omega}\sin\left(\frac{\pi\theta}{\omega}\right) + \phi,$$

where (r, θ) are polar coordinates with origin at the vertex of the corner, and $\phi \in H^2(\Omega)$.

Remark 2.1. If ψ_0 is more regular–say in $H^{s+3/2}(\Gamma)$, $s > 0$, the solution ψ of Problem (2.1) can be written as the sum of a regular function ϕ (in $H^{s+2}(\Omega)$) and some singular functions, whose number and form are perfectly known [1].

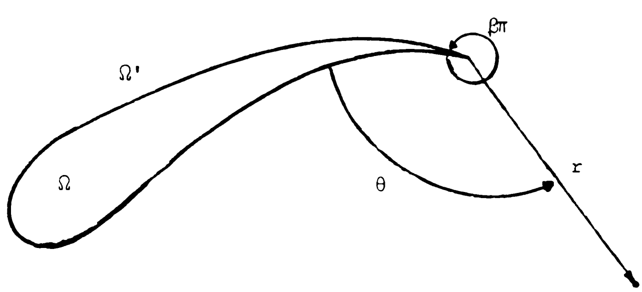


FIGURE 1

We know that the integral formulation is obtained by combining both exterior and interior problems

$$\text{(2.2)} \qquad \begin{cases} \Delta\psi = 0 & \text{in } \Omega \text{ and } \Omega', \\ \psi = \psi_0 & \text{on } \Gamma. \end{cases}$$

Assume that the interior angle is acute and ψ_0 is in $H^{3/2}(\Gamma)$. The solution of Problem (2.2) will be such that

$$\psi|_\Omega \in H^2(\Omega), \qquad \psi|_{\Omega'} = ar^{1/\beta}\sin\left(\frac{\theta}{\beta}\right) + \phi \text{ near the corner},$$

where ϕ is a function of $H^2(\Omega')$ and $\beta = 2 - \alpha$.

The solution v of Problem (1.10), which is the jump of the normal derivative of ψ, will then be written in the following way near the corner:

$$v = \frac{\partial\psi}{\partial n}\bigg|_{\text{int}} - \frac{\partial\psi}{\partial n}\bigg|_{\text{ext}} = -a\frac{\partial}{\partial n}\left(r^{1/\beta}\sin\frac{\theta}{\beta}\right) + \frac{\partial\psi}{\partial n}\bigg|_{\text{int}} - \frac{\partial\psi}{\partial n}\bigg|_{\text{ext}}.$$

Since the corner is supposed to be linear, we have

$$\frac{\partial}{\partial n}\left(r^{1/\beta}\sin\frac{\theta}{\beta}\right) = r^{(1/\beta)-1}.$$

Let v_0 be a distribution in $H_0^{-1/2}(\Gamma)$ such that $v_0 = r^{(1/\beta)-1}$ in the vicinity of the corner. Thus, $v = A \cdot v_0 + V$, where $V \in H^{1/2}(\Gamma) \cap H_0^{-1/2}(\Gamma)$.

2.2. *The Discrete Problem*. Let n be an integer and consider $(n+1)$ points on the boundary Γ, $(A_i)_{i=1,\dots,n+1}$, such that

$$A_1 = A_{n+1} = \text{vertex of the corner}.$$

The points $(A_i)_{i=1,\dots,n}$ are assumed to be distinct.

We denote by Γ_i the arc A_iA_{i+1} and by Γ_{ih} the associated chord with length h_i. Let h be the maximum of $(h_i)_{i=1,\dots,n}$ and h_0 their minimum. We assume that $h/h_0 < c$; c positive constant. Then we define the space

$$W_h = \left\{ w_h \colon w_h|_{\Gamma_{ih}} = c_i,\ i = 1, \dots, n;\ \int_{\Gamma_n} w_h d\gamma_h = 0 \right\}, \quad c_i \text{ constant},$$

where $\Gamma_h = \bigcup_{i=1}^n \Gamma_{ih}$, and

$$\text{(2.3)} \qquad V_h = W_h \oplus \{v_0\}.$$

The approximate problem set in the so defined space V_h then takes the form

$$\text{(2.4)} \qquad \begin{cases} \text{Find } v_h \in V_h \text{ such that} \\ a_h(v_h, v_h') = -\dfrac{1}{2\pi}\displaystyle\int_{\Gamma_h}\int_{\Gamma_h} v_h(x)v_h'(y)\operatorname{Log}|x-y|\,d\gamma_h(x)\,d\gamma_h(y) \\ \qquad = \displaystyle\int_{\Gamma_h} v_h' u_{0h}\, d\gamma_h, \end{cases}$$

where u_{0h} is an approximation of u_0, defined on Γ_h.

Since V_h is not included in $H_0^{-1/2}(\Gamma)$, but in $H_0^{-1/2}(\Gamma_h)$, we need two mappings

$$r_h \colon V_h \to H^{-1/2}(\Gamma), \qquad p_h \colon V_h \to H_0^{-1/2}(\Gamma).$$

The parametrization of Γ_i being given by

$$F_i(t) = \begin{cases} t \cdot h_i, \\ f_i(t), \end{cases} \qquad t \in [0, 1],$$

we have the following parametrization of the approximate arc Γ_{ih}, in local coordinates:

$$F_{ih}(t) = \begin{cases} t \cdot h_i, \\ 0, \end{cases} \qquad t \in [0, 1].$$

We then define

$$r_h w_h = w_h \circ F_{ih} \circ F_i^{-1} \qquad \text{for all } w_h \in W_h,$$

$$p_h w_h \left| \frac{dF_i}{dt} \right| = (w_h \circ F_{ih} \circ F_i^{-1}) \cdot h \quad \text{for all } w_h \in W_h.$$

We then extend these definitions to all $v_h \in V_h$ by setting, since v_0 has its support in the common part of the boundary to Γ and Γ_h,

$$r_h v_0 = p_h v_0 = v_0.$$

LEMMA 2.1. *For any v_h in V_h, we have the following inequalities*:

$$c_1 |r_h v_h|_{L^2(\Gamma)} \leq |v_h|_{L^2(\Gamma_h)} \leq c_2 |r_h v_h|_{L^2(\Gamma)}, \tag{2.5}$$

$$c_1' |p_h v_h|_{L^2(\Gamma)} \leq |v_h|_{L^2(\Gamma_h)} \leq c_2' |p_h v_h|_{L^2(\Gamma)}, \tag{2.6}$$

with $c_1, c_2, c_1', c_2' > 0$.

Proof. We have

$$|v_h|^2_{L^2(\Gamma_h)} = \sum_{i=1}^{n} \int_{\Gamma_{ih}} |v_h|^2 \, d\gamma_h = \sum_{i=1}^{n} \int_0^1 |v_h \circ F_{ih}(t)|^2 \left| \frac{dF_{ih}}{dt} \right| dt.$$

But

$$\left| \frac{dF_{ih}}{dt} \right| = \begin{cases} h_i, \\ 0, \end{cases} \quad \text{and} \quad \left| \frac{dF_i}{dt} \right| = \begin{cases} h_i, \\ f_i'(t). \end{cases}$$

Since 0 is the first order interpolate of f_i, there exist two constants α_1 and α_2 such that

$$\alpha_1 \left| \frac{dF_i}{dt} \right| \leq \left| \frac{dF_{ih}}{dt} \right| \leq \alpha_2 \left| \frac{dF_i}{dt} \right|,$$

$$|r_h v_h|^2_{L^2(\Gamma)} = \sum_{i=1}^{n} \int_{\Gamma_i} |r_h v_h|^2 \, d\gamma(x) = \sum_{i=1}^{n} \int_0^1 |v_h \circ F_{ih}|^2 \left| \frac{dF_i}{dt} \right| dt.$$

Then, with the previous inequalities, we obtain (2.5). In the same way

$$|p_h v_h|_{L^2(\Gamma)} = \sum_{i=1}^{n} \int_{\Gamma_i} |v_h \circ F_{ih} \circ F_i^{-1}(x)|^2 \frac{\left| \dfrac{dF_{ih}}{dt} \right|^2}{\left| \dfrac{dF_i}{dt} \right|^2} dx$$

$$= \sum_{i=1}^{n} \int_0^1 |v_h \circ F_{ih}(t)|^2 \frac{\left| \dfrac{dF_{ih}}{dt} \right|}{\left| \dfrac{dF_i}{dt} \right|} \left| \frac{dF_{ih}}{dt} \right| dt.$$

Since

$$\alpha_1\left|\frac{dF_i}{dt}\right| \leq \left|\frac{dF_{ih}}{dt}\right| \leq \alpha_2\left|\frac{dF_i}{dt}\right|,$$

we get

$$\alpha_1|v_h|^2_{L^2(\Gamma_h)} \leq |p_h v_h|_{L^2(\Gamma)} \leq \alpha_2|v_h|^2_{L^2(\Gamma_h)}.$$

Thus, we derive (2.6). □

LEMMA 2.2. *For all $v_h \in V_h$, the following inequalities, called inverse inequalities, hold*:

(2.7) $$\|v_h\|_{H^s(\Gamma_h)} \leq \frac{c}{h^{s-t}}\|v_h\|_{H^t(\Gamma_h)}, \qquad -1 \leq t \leq s \leq 0.$$

Proof (A. Bamberger, personal communication). Let us prove (2.7) for $t = -1$ and $s = 0$. Thus, we obtain all the other cases by interpolation. We have

(2.8) $$\|v_h\|_{H^{-1}(\Gamma_h)} = \max_{\Psi \in H^1(\Gamma_h)} \frac{\left|\int_{\Gamma_h} v_h \Psi\right|}{\|\Psi\|_{1,\Gamma_h}}.$$

Let us construct a particular function Ψ, which will of course minimize the right-hand side of (2.8). Let us set

$$\Psi|_{\Gamma_{ih}} = \Psi_i.$$

We shall then choose Ψ_i such that

- $\Psi_i \in H^1_0(\Gamma_{ih})$ (i.e.: $\Psi_i(A_i) = \Psi_i(A_{i+1}) = 0$),
- $\int_{\Gamma_{ih}} \Psi_i v_h = \int_{\Gamma_{ih}} (v_h)^2,$
- $\int_{\Gamma_{ih}} \frac{d\Psi_i}{dx} \cdot \frac{dv}{dx} = 0$ for any $v \in H^1_0(\Gamma_{ih})$ such that $\int_{\Gamma_{ih}} v \cdot v_h = 0.$

Such a choice is possible (and unique). Since $\Psi_i \in H^1_0(\Gamma_{ih})$, we have, for h small enough,

$$\|\Psi_i\|_{H^1(\Gamma_{ih})} \leq c|\Psi_i|_{1,\Gamma_{ih}} \quad (c \text{ constant}).$$

Thus,

$$\|v_h\|_{H^{-1}(\Gamma_h)} \geq c\frac{\sum_{i=1}^{N} \int_{\Gamma_{ih}} (v_h)^2}{\left(\sum_{i=1}^{N} \int_{\Gamma_{ih}} \left(\frac{d\Psi_i}{dx}\right)^2\right)^{1/2}},$$

and

$$\|v_h\|^2_{H^{-1}(\Gamma_h)} \geq c\frac{\left(\sum_{i=1}^{N} \int_{\Gamma_{ih}} (v_h)^2\right)^2}{\sum_{i=1}^{N} \int_{\Gamma_{ih}} \left(\frac{d\Psi_i}{dx}\right)^2}.$$

Let us now consider the function θ_i, the solution of the following problem:

$$(\mathrm{P}_i) \qquad \begin{cases} \theta_i \in H_0^1(\Gamma_{ih}) \\ -\dfrac{d^2\theta_i}{dx^2} = v_h \quad \text{on } \Gamma_{ih}. \end{cases}$$

One can easily see that θ_i verifies

$$\int_{\Gamma_{ih}} \frac{d\theta_i}{dx} \cdot \frac{dv}{dx} = 0 \quad \text{for any } v \in H_0^1(\Gamma_{ih}) \text{ such that } \int_{\Gamma_{ih}} v \cdot v_h = 0.$$

Thus, $\Psi_i = \lambda_i \theta_i$ with

$$\lambda_i = \int_{\Gamma_{ih}} v_h^2 \bigg/ \int_{\Gamma_{ih}} \left(\frac{d\theta_i}{dx}\right)^2.$$

We then have

$$\|v_h\|^2_{H^{-1}(\Gamma_h)} \geq c \frac{\left(\int_{\Gamma_h} v_h^2\right)^2}{\sum_{i=1}^N \lambda_i^2 \int_{\Gamma_{ih}} \left|\frac{d\theta_i}{dx}\right|^2}, \qquad \|v_h\|^2_{H^{-1}(\Gamma_h)} \geq c \frac{\left(\int_{\Gamma_h} v_h^2\right)^2}{\sum_{i=1}^N \lambda_i \left(\int_{\Gamma_{ih}} v_h^2\right)}.$$

Suppose now that we could prove

$$\lambda_i \leq \frac{c}{h^2}. \tag{2.9}$$

Thus,

$$\|v_h\|^2_{H^{-1}(\Gamma_h)} \geq ch^2 \frac{\left(\int v_h^2\right)^2}{\sum_{i=1}^N \left(\int_{\Gamma_{ih}} v_h^2\right)} \geq ch^2 \|v_h\|^2_{L^2(\Gamma_h)},$$

and we get (2.7).

Let us then prove (2.9).

One can first notice that

$$\lambda_i = \|v_h\|^2_{0,\Gamma_{ih}} / \|\theta_i\|^2_{1,\Gamma_{ih}}.$$

But, since θ_i is the solution of Problem (P_i),

$$|\theta_i|^2_{1,\Gamma_{ih}} = \int_{\Gamma_{ih}} v_h \cdot \theta_i,$$

$$|\theta_i|_{1,\Gamma_{ih}} = \frac{\int_{\Gamma_{ih}} v_h \cdot \theta_i}{|\theta_i|_{1,\Gamma_{ih}}} \leq \max_{\theta \in H_0^1(\Gamma_{ih})} \frac{\left|\int_{\Gamma_{ih}} v_h \cdot \theta\right|}{|\theta|_{1,\Gamma_{ih}}} \leq c \|v_h\|_{H^{-1}(\Gamma_{ih})}.$$

On the other hand,

$$\|v_h\|_{H^{-1}(\Gamma_{ih})} = \max_{\theta \in H_0^1(\Gamma_{ih})} \frac{\left|\int v_h \cdot \theta\right|}{|\theta|_{1,\Gamma_{ih}}} = \max_{\theta \in H_0^1(\Gamma_{ih})} \frac{\int \frac{d\theta_i}{dx} \cdot \frac{d\theta}{dx}}{|\theta|_{1,\Gamma_{ih}}} \leq |\theta_i|_{1,\Gamma_{ih}}.$$

Thus

$$c_1 \frac{\|v_h\|_{0,\Gamma_{ih}}}{\|v_h\|_{-1,\Gamma_{ih}}} \leqslant \lambda_i \leqslant c_2 \frac{\|v_h\|_{0,\Gamma_{ih}}}{\|v_h\|_{-1,\Gamma_{ih}}}.$$

Now, by using the mapping given by $\hat{x} = (x - x_i)/h_i$ and by defining the functions $\hat{\theta}_i$ and $\hat{v}_h$ by

$$\hat{\theta}_i(\hat{x}) = \theta_i(x), \qquad \hat{v}_h(\hat{x}) = v_h(x),$$

we are led to the following problem:

$$(\hat{\mathrm{P}}) \qquad \begin{cases} \dfrac{-d^2\hat{\theta}_i}{d\hat{x}^2} = h_i^2\hat{v}_h, & \hat{x} \in \left]0, 1\right[, \\ \hat{\theta}_i(0) = \hat{\theta}_i(1) = 0. \end{cases}$$

We can then easily derive that

$$\lambda_i = h_i^2 \frac{\|\hat{v}_h\|^2_{L^2(0,1)}}{|\hat{\theta}_i|^2_{H_0^1(0,1)}}.$$

Let us consider the general problem

$$(\mathrm{Q}) \qquad \begin{cases} -\dfrac{d^2u}{dx^2} = F(x), & x \in (0, 1), \\ u(0) = u(1) = 0, \end{cases}$$

and denote by $R(F)$ the quantity

$$R(F) = \frac{\|F\|^2_{L^2(0,1)}}{|u|^2_{H_0^1(0,1)}}.$$

We obtain

$$\lambda_i = \frac{1}{h_i^2} R(\hat{v}_h). \tag{2.10}$$

We suppose now that $h_i = h$ for any $i = 1, \dots, N$. The calculations proceed in a similar fashion in the case where this equality does not hold. Thus,

$$\hat{v}_h(x) = v_h(x_i + hx), \qquad x \in \left]0, 1\right[.$$

Since $v_h = w_i + v_0$ (we can suppose that $a_0 = 1$),

$$\hat{v}_h(x) = v_i + (x_i + hx)^{\nu - 1}, \qquad x \in \left]0, 1\right[.$$

Let us define the vector space

$$\mathcal{G}_i = \left\{ g_i;\, g_i = a + b(ih + hx)^{\nu-1},\, a, b \in \mathbf{R} \right\}.$$

We also have

$$\mathcal{G}_i = \left\{ g_i;\, g_i = a + b(i + x)^{\nu-1},\, a, b \in \mathbf{R} \right\},$$

which means that $\mathcal{G}_i$ is independent of h. Hence

$$R(\hat{v}_h) < R(\mathcal{G}_i) = \max_{g_i \in \mathcal{G}_i} R(g_i).$$

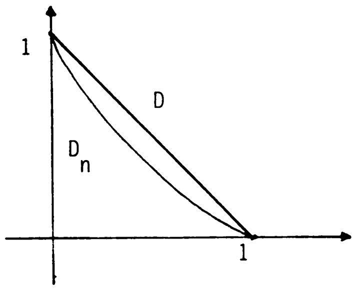
One can prove that $\mathcal{G}_n$, for $n \neq 0$, can be identified to the space

$$\mathcal{G}_n = \{ g_n;\, g_n = a + bD_n \},$$

where D_n is defined by

$$D_n(x) = \frac{(n+1)^{\nu-1} - (n+x)^{\nu-1}}{(n+1)^{\nu-1} - n^{\nu-1}}.$$

D_n is a convex function and the sequence $\{D_n\}$ is a convergent sequence in $C^0[0, 1]$, and $D_n \to 1 - x$ uniformly on [0, 1].



Let us define function D by $D(x) = 1 - x$. Thus, the subspace $\mathcal{G}_n$ "converges" and its limit is

$$\mathcal{G} = \{g; g = a + b \cdot D, a, b \in \mathbf{R}\}.$$

For any function G of $L^2(0, 1)$, we are now led to consider the quantity

$$R^*(G) = \max_{(a, b) \in \mathbf{R}^2} R(a + bG).$$

We can prove that, for a sequence $\{G_n\}$ in $L^2(0, 1)$ such that $G_n \to G$ in $L^2(0, 1)$, we have $R^*(G_n) \to R^*(G)$. Thus, for h small enough, we shall have $R^*(\mathcal{G}_i) \leqslant c_0$ for $i \neq 0$ and, for $i = 0$, $R^*(\mathcal{G}_0) = c_1$. We can deduce $R^*(\mathcal{G}_i) \leqslant c = \max(c_0, c_1)$. By (2.10), we then obtain $\lambda_i \leqslant c/h_i^2$, which was the desired result. □

Remark. We also have, by a similar proof,

$$\|p_h v_h\|_{H^s(\Gamma)} \leqslant \frac{c}{h^{s-t}} \|p_h v_h\|_{H^t(\Gamma)}, \qquad -1 \leqslant t \leqslant s \leqslant 0. \quad \square \tag{2.11}$$

LEMMA 2.3. *For all v_h in the space V_h, the following inequalities hold*:

$$|p_h v_h - r_h v_h|_{L^2(\Gamma)} \leqslant ch^2 |v_h|_{L^2(\Gamma)}, \tag{2.12}$$

$$\|p_h v_h - r_h v_h\|_{H^{-1/2}(\Gamma)} \leqslant ch^{3/2} |v_h|_{H^{-1/2}(\Gamma)}. \tag{2.13}$$

Proof. Inequality (2.19) is obtained as in [6], since v_h is an element of $L^2(\Gamma_h)$. To obtain (2.13), we note that

$$\|p_h v_h - r_h v_h\|_{H^{-1/2}(\Gamma)} \leqslant |p_h v_h - r_h v_h|_{L^2(\Gamma)} \leqslant ch^2 |v_h|_{L^2(\Gamma_h)}.$$

Using the inverse inequality (2.7) with $s = 0$ and $t = -\frac{1}{2}$, we get

$$\|p_h v_h - r_h v_h\|_{H^{-1/2}(\Gamma)} \leqslant ch^{2-1/2} \|v_h\|_{H^{-1/2}(\Gamma_h)}. \quad \square$$

PROPOSITION 2.1. *Assuming the boundary is smooth enough out of the singular point, and v_h, v_h' are two elements of V_h, we have*

$$|a(p_h v_h, p_h v_h') - a_h(v_h, v_h')| \leqslant ch^2 |v_h|_{L^2(\Gamma)} |v_h'|_{L^2(\Gamma)}. \tag{2.14}$$

Proof. Using the definitions of p_h and of the bilinear forms $a(\cdot, \cdot)$ and $a_h(\cdot, \cdot)$, we get

$$a(p_h v_h, p_h v_h') - a_h(v_h, v_h') = -\frac{1}{2\pi} \sum_{i,j=1}^{n} \int_0^1 \int_0^1 v_h(F_{ih}(t)) v_h'(F_{jh}(s)) \cdot \operatorname{Log} \frac{|F_i(t) - F_j(s)|}{|F_{ih}(t) - F_{jh}(s)|} \left| \frac{dF_{ih}}{dt} \right| \left| \frac{dF_{jh}}{ds} \right| dtds.$$

By M. N. Le Roux [6], we know that

$$\operatorname{Log} \frac{|F_i(t) - F_j(t)|}{|F_{ih}(t) - F_{jh}(s)|} \leqslant ch^2,$$

thus

$$|a(p_h v_h, p_h v_h') - a_h(v_h, v_h')| \leqslant ch^2 |r_h v_h|_{L^2(\Gamma)} |r_h v_h'|_{L^2(\Gamma)}.$$

By using inequality (2.5), we then obtain (2.14). □

THEOREM 2.2. ELLIPTICITY OF THE APPROXIMATE PROBLEM. *For h small enough, there exists a positive constant β such that*

$$a_h(v_h, v_h) \geqslant \beta \| p_h v_h \|^2_{H^{-1/2}(\Gamma)}. \tag{2.15}$$

This result leads to the existence and uniqueness (by the Lax-Milgram theorem) of the approximate solution, which we can "compare" with the exact solution by using the mappings defined above.

Proof of Theorem 2.2. By using (2.14), we have

$$a_h(v_h, v_h) \geqslant a(p_h v_h, p_h v_h) - ch^2 |v_h|^2_{L^2(\Gamma)}.$$

Since the bilinear form $a(\cdot, \cdot)$ is coercive, one gets

$$a(p_h v_h, p_h v_h) \geqslant \alpha_1 \| p_h v_h \|^2_{H^{-1/2}(\Gamma)}.$$

Thus, by (2.6),

$$a_h(v_h, v_h) \geqslant \alpha_1 \| p_h v_h \|^2_{H^{-1/2}(\Gamma)} - ch^2 | p_h v_h |^2_{H^{-1/2}(\Gamma)}.$$

Using now the inverse inequality (2.11), we get

$$a_h(v_h, v_h) \geqslant (\alpha_1 - ch^{3/2}) \| p_h v_h \|^2_{H^{-1/2}(\Gamma)},$$

so, for h small enough,

$$a_h(v_h, v_h) \geqslant \beta \| p_h v_h \|^2_{H^{-1/2}(\Gamma)}. \quad \square$$

2.3. *Error Estimates.*

THEOREM 2.3. *Let v be the solution of the exact problem and v_h the solution of the approximate problem. Then, the following estimate holds*:

$$\| v - p_h v_h \|_{H^{-1/2}(\Gamma)} \leqslant c \Big\{ \operatorname*{Inf}_{v_h' \in V_h} \big[\| v - p_h v_h' \|_{H^{-1/2}(\Gamma)} + h^{3/2} |v_h'|_{L^2(\Gamma_h)} \big] \times + \| \psi_0 - r_h \psi_{0h} \|_{H^{1/2}(\Gamma)} \Big\}. \tag{2.16}$$

Proof. Let $v_h' \in V_h$, and let us use (2.15). We obtain

$$\beta \| p_h(v_h - v_h') \|^2 \leq a_h(v_h - v_h', v_h - v_h'),$$

$$\begin{aligned} a_h(v_h - v_h', v_h - v_h') &= a_h(v_h, v_h - v_h') - a(v, p_h v_h - p_h v_h') + a(v - p_h v_h', p_h v_h - p_h v_h') \\ &\quad + a(p_h v_h', p_h v_h - p_h v_h') - a_h(v_h', v_h - v_h'). \end{aligned}$$

Using the Estimates (2.11) and (2.14), and since v and v_h are the solutions of the continuous problem and of the discrete problem, we derive

$$\begin{aligned} \beta \| p_h(v_h - v_h') \|^2_{H^{-1/2}(\Gamma)} &\leq |\langle \psi_{0h}, v_h - v_h' \rangle - \langle \psi_0, p_h v_h - p_h v_h' \rangle| \\ &\quad + M \| v - p_h v_h' \|_{H^{-1/2}(\Gamma)} \| p_h v_h - p_h v_h' \|_{H^{-1/2}(\Gamma)} \\ &\quad + c h^{3/2} |v_h'|_{L^2(\Gamma_h)} \| p_h(v_h - v_h') \|_{H^{-1/2}(\Gamma)}. \end{aligned}$$

Thus,

$$\| p_h v_h - p_h v_h' \|_{H^{-1/2}(\Gamma)} \leq c \left\{ \| v - p_h v_h' \|_{H^{-1/2}(\Gamma)} + h^{3/2} |v_h'|_{L^2(\Gamma_h)} + \frac{|\langle \psi_{0h}, v_h - v_h' \rangle - \langle \psi_0, p_h v_h - p_h v_h' \rangle|}{\| p_h v_h - p_h v_h' \|_{H^{-1/2}(\Gamma)}} \right\}.$$

The problem is then to estimate, for any $x_h \in V_h$, the expression

$$\frac{|\langle \psi_{0h}, x_h \rangle - \langle \psi_0, p_h x_h \rangle|}{\| p_h x_h \|_{H^{-1/2}(\Gamma)}}.$$

$$\begin{aligned} |\langle \psi_{0h}, x_h \rangle - \langle \psi_0, p_h x_h \rangle| &= \sum_{i=1}^{n} \int_0^1 (r_h \psi_{0h} - \psi_0)(F_i(t)) p_h x_h(F_i(t)) \left| \frac{dF_i}{dt} \right| dt \\ &= \sum_{i=1}^{n} \int_{\Gamma_i} (r_h \psi_{0h} - \psi_0) \cdot p_h x_h \, d\gamma; \end{aligned}$$

thus

$$|\langle \psi_{0h}, x_h \rangle - \langle \psi_0, p_h x_h \rangle| \leq c \| \psi_0 - r_h \psi_{0h} \|_{H^{1/2}(\Gamma)} \| p_h x_h \|_{H^{-1/2}(\Gamma)}.$$

It follows that

$$\begin{aligned} &\| p_h v_h - p_h v_h' \|_{H^{-1/2}(\Gamma)} \\ &\quad \leq c \left\{ \| v - p_h v_h' \|_{H^{-1/2}(\Gamma)} + h^{3/2} |v_h'|_{L^2(\Gamma_h)} + \| \psi_0 - r_h \psi_{0h} \|_{H^{1/2}(\Gamma)} \right\}, \end{aligned}$$

and

$$\| v - p_h v_h \| \leq c \left\{ \inf_{v_h' \in V_h} \left[\| v - p_h v_h' \|_{H^{-1/2}(\Gamma)} + h^{3/2} |v_h'|_{L^2(\Gamma_h)} \right] + \| \psi_0 - r_h \psi_{0h} \|_{H^{1/2}(\Gamma)} \right\},$$

which ends the proof. □

Remark 2.2. This result can easily be extended to the case of

$$V_h = W_h \bigoplus_{l=1}^{M} \{v_l\},$$

where $\{v_l\}_{l=1,\ldots,M}$ are the M first singularities, and w_h is a space of polynomials over Γ_h. If Γ_h is a union of p-polynomial arcs, we have the estimate

$$\|v - p_h v_h\|_{H^{-1/2}(\Gamma)} \leqslant c\left\{ \inf_{v'_h \in V_h} \left[\|v - p_h v'_h\| + h^{p+1/2} |v'_h|_{L^2(\Gamma_h)} \right] + \|\psi_0 - r_h \psi_{0h}\|_{H^{1/2}(\Gamma)} \right\}.$$

Thus, we gain precision by fitting the boundary better.

In order to bound each one of the quantities appearing in the right-hand side of (2.16), we shall use the following result of M. N. Le Roux [6].

LEMMA 2.4. *Let s_h denote the orthogonal projection from $L^2(\Gamma)$ on W_h. Then, we obtain the following inequality*

$$\|w - s_h w\|_{H^{-1/2}(\Gamma)} \leqslant c h^{s+1/2} \|w\|_{H^s(\Gamma)} \tag{2.17}$$

for $0 \leqslant s \leqslant 1$ *and* $w \in H^s(\Gamma)$.

THEOREM 2.4. *Let v be the solution of Problem* (1.10) *and v_h be the solution of Problem* (2.4); *then,*

$$\|v - p_h v_h\|_{H^{-1/2}(\Gamma)} \leqslant c\left\{ h^{s+1/2} \|w\|_{H^s(\Gamma)} + h^{3/2}\left(|a_0 v_0|_{L^2(\Gamma)} + |w|_{L^2(\Gamma)}\right) + \|\psi_0 - r_h \psi_{0h}\|_{H^{1/2}(\Gamma)} \right\} \tag{2.18}$$

for $-\frac{1}{2} \leqslant s \leqslant 1$ *and* $v = a_0 v_0 + w$.

Proof. By Theorem 2.3, we have the estimate

$$\|v - p_h v_h\|_{H^{-1/2}(\Gamma)} \leqslant c\left\{ \operatorname*{Inf}_{v'_h \in V_h} \left[\|v - p_h v'_h\|_{H^{-1/2}(\Gamma)} + h^{3/2} |v'_h|_{L^2(\Gamma_h)} \right] + \|\psi_0 - r_h \psi_{0h}\| \right\}.$$

Let us choose a particular v'_h in the following way: v can be written as $v = a_0 v_0 + w$. Thus, $s_h w_h \in r_h W_h$ and there exists some $w_h \in W_h$ such that $s_h w = r_h w_h$. We then consider

$$v'_h = a_0 v_0 + w_h,$$

$$\begin{aligned} \|v - p_h v'_h\|_{H^{-1/2}(\Gamma)} &\leqslant \|w - p_h w_h\|_{H^{-1/2}(\Gamma)} \\ &\leqslant \|w - s_h w\|_{H^{-1/2}(\Gamma)} + \|r_h w_h - p_h w_h\|_{H^{-1/2}(\Gamma)}. \end{aligned}$$

But

$$\|w - s_h w\|_{H^{-1/2}(\Gamma)} \leqslant c h^{s+1/2} \|w\|_{H^s(\Gamma)} \quad \text{(by Lemma 2.4)},$$

$$\|r_h w_h - p_h w_h\|_{H^{-1/2}(\Gamma)} \leqslant c h^{3/2} |w_h|_{L^2(\Gamma_h)} \leqslant c h^{3/2} |w|_{L^2(\Gamma)},$$

since $w_h \in W_h$ [6] and $r_h w_h = s_h w$. Thus

$$\|v - p_h v'_h\|_{H^{-1/2}(\Gamma)} \leqslant c h^{s+1/2} \|w\|_{H^s(\Gamma)} + h^{3/2} |w|_{L^2(\Gamma)}.$$

On the other hand,

$$|v'_h|_{L^2(\Gamma_h)} \leqslant |a_0 v_0|_{L^2(\Gamma_h)} + |w_h|_{L^2(\Gamma_h)} \leqslant |a_0 v_0|_{L^2(\Gamma)} + |w|_{L^2(\Gamma)},$$

since $r_h w_h = s_h w$. We can then obtain

$$\|v - p_h v_h\|_{H^{-1/2}(\Gamma)} \leqslant c\left\{ h^{s+1/2} \|w\|_{H^s(\Gamma)} + h^{3/2}\left(|a_0 v_0|_{L^2(\Gamma)} + |w|_{L^2(\Gamma)}\right) + \|\psi_0 - r_h \psi_{0h}\|_{H^{1/2}(\Gamma)} \right\}. \quad \square$$

THEOREM 2.5. *For all x such that $d(x, \Gamma) \geq \delta > 0$ and for h small enough, the following estimates hold, for* $0 \leq s \leq 1$,

$$|\psi(x) - \psi_h(x)| \leq \frac{c}{d(x, \Gamma)}\{h^2(|a_0 v_0|_{L^2(\Gamma)} + |w|_{L^2(\Gamma)}) + \|\psi_0 - r_h\psi_{0h}\|_{L^2(\Gamma)} + h^{s+1}|w|_{H^s(\Gamma)} + h^{1/2}\|\psi_0 - r_h\psi_{0h}\|_{H^{1/2}(\Gamma)}\}, \tag{2.19}$$

$$|D^\alpha\psi(x) - D^\alpha\psi_h(x)| \leq \frac{c}{d(x, \Gamma)^{|\alpha|}}\{h^2(|a_0 v_0|_{L^2(\Gamma)} + |w|_{L^2(\Gamma)}) + h^{s+1}\|w\|_{H^s(\Gamma)} + \|\psi_0 - r_h\psi_{0h}\|_{L^2(\Gamma)} + h^{1/2}\|\psi_0 - r_h\psi_{0h}\|_{H^{1/2}(\Gamma)}\}. \tag{2.20}$$

Proof. First, we shall bound $\|v - p_h v_h\|_{H^{-1}(\Gamma)}$:

$$\|v - p_h v_h\|_{H^{-1}(\Gamma)} = \sup_{\phi \in H^1(\Gamma)} \frac{\langle v - p_h v_h, \phi\rangle}{\|\phi\|_{H^1(\Gamma)}}.$$

Since the singular function $v_0 \in L^2(\Gamma)$, the mapping

$$A: v \to -\frac{1}{2\pi}\int_\Gamma v(y)\mathrm{Log}|x - y|\, d\gamma(y)$$

defines an isomorphism from $L_0^2(\Gamma) = L^2(\Gamma) \cap H_0^{-1/2}(\Gamma)$ onto $H^1(\Gamma)/\mathbf{R}$. Therefore,

$$\|v - p_h v_h\|_{H^{-1}(\Gamma)} \leq c \sup_{g \in L_0^2(\Gamma)} \frac{|\langle v - p_h v_h, Ag\rangle|}{\|g\|_{L^2(\Gamma)}} \leq c \sup_{g \in L_0^2(\Gamma)} \frac{|a(v - p_h v_h, g)|}{\|g\|_{L^2(\Gamma)}}.$$

Let $s_h g$ be the projection of g on $r_h W_h$. Then, $s_h g = r_h G_h$, $G_h \in W_h$; thus,

$$a(v - p_h v_h, g) = a(v - p_h v_h, g - p_h G_h) + a(v - p_h v_h, p_h G_h).$$

By using (2.13), we get for the first term

$$|a(v - p_h v_h, g - p_h G_h)| \leq ch^{1/2}\|v - p_h v_h\|_{H^{-1/2}(\Gamma)}\|g\|_{L^2(\Gamma)}.$$

The second term can be written

$$a(v - p_h v_h, p_h G_h) = \langle\psi_0, p_h G_h\rangle - \langle\psi_{0h}, G_h\rangle + a_h(v_h, G_h) - a(p_h v_h, p_h G_h)$$
$$= \langle\psi_0 - r_h\psi_{0h}, p_h G_h\rangle + a_h(v_h, G_h) - a(p_h v_h, p_h G_h).$$

Thus, by using (2.14), we obtain

$$|a(v - p_h v_h, p_h G_h)| \leq \|\psi_0 - r_h\psi_{0h}\|_{L^2(\Gamma)}\|G_h\|_{L^2(\Gamma_h)} + ch^2\|v_h\|_{L^2(\Gamma_h)}\|G_h\|_{L^2(\Gamma_h)}.$$

Then

$$\|v - p_h v_h\|_{H^{-1}(\Gamma)} \leq c\{h^{1/2}\|v - p_h v_h\|_{H^{-1/2}(\Gamma)} + \|\psi_0 - r_h\psi_{0h}\|_{L^2(\Gamma)} + h^2\|v_h\|_{L^2(\Gamma_h)}\}. \tag{2.21}$$

We can now write

$$v = a_0 v_0 + w, \qquad v_h = a_{0h} v_0 + w_h,$$

$$\psi(x) - \psi_h(x) = -\frac{1}{2\pi}\int_\Gamma w(y)\mathrm{Log}|x - y|\, d\gamma_y + \frac{1}{2\pi}\int_\Gamma w_h(y)\mathrm{Log}|x - y|\, d\gamma_h(y) - \frac{1}{2\pi}(a_0 - a_{0h})\int_\Gamma v_0(y)\mathrm{Log}|x - y|\, d\gamma(y).$$

By an easy computation using replacement of variables (defined by functions F_i and F_{ih}), we obtain

$$\psi(x) - \psi_h(x) = -\frac{1}{2\pi}\int_\Gamma (v - p_h v_h)(y)\operatorname{Log}|x - y|\, d\gamma(y)$$
$$= -\frac{1}{2\pi}\sum_{i=1}^{n}\int_0^1 w_h(F_{ih}(t))\operatorname{Log}\frac{|x - F_{ih}(t)|}{|x - F_i(t)|}\left|\frac{dF_{ih}}{dt}\right| dt.$$

Using the fact that $d(x, \Gamma) \geqslant \delta > 0$, we get

$$\left|\operatorname{Log}\frac{|x - F_{ih}(t)|}{|x - F_i(t)|}\right| \leqslant ch^2\frac{1}{d(x, \Gamma)}$$

and

$$\left|-\frac{1}{2\pi}\sum_{i=1}^{n}\int_0^1 w_h(F_{ih}(t))\operatorname{Log}\frac{|x - F_{ih}(t)|}{|x - F_i(t)|}\left|\frac{dF_{ih}}{dt}\right| dt\right| \leqslant \frac{ch^2}{d(x, \Gamma)}\|w\|_{L^2(\Gamma)}.$$

On the other hand,

$$\left|-\frac{1}{2\pi}\int_\Gamma (v - p_h v_h)(y)\operatorname{Log}|x - y| d\gamma(y)\right| \leqslant \|v - p_h v_h\|_{H^{-1}(\Gamma)}\|\operatorname{Log}|x - y|\|_{H^1(\Gamma)},$$

and

$$\|\operatorname{Log}|x - y|\|_{H^1(\Gamma)} \leqslant \frac{c}{d(x, \Gamma)},$$

by the Taylor formula.

Regrouping all these results and using (2.21), we obtain

$$|\psi(x) - \psi_h(x)| \leqslant \frac{c}{d(x, \Gamma)}\{h^{s+1}\|w\|_{H^s(\Gamma)} + h^{1/2}\|\psi_0 - r_h\psi_{0h}\|_{H^{1/2}(\Gamma)}$$
$$+ \|\psi_0 - r_h\psi_{0h}\|_{L^2(\Gamma)} + h^2(|a_0 v_0|_{L^2(\Gamma)} + |w|_{L^2(\Gamma)})\},$$

which is the Estimate (2.19).

(2.20) is obtained in the same way, by remarking that

$$\operatorname{grad}\phi(x) = \frac{1}{2\pi}\int_\Gamma v(y)\frac{x - y}{|x - y|^2}\, d\gamma(y),$$
$$\operatorname{grad}\phi_h(x) = \frac{1}{2\pi}\int_{\Gamma_h} v_h(y)\frac{x - y}{|x - y|^2}\, d\gamma_h(y). \quad \square$$

Remark 2.3. By a better fitting of the boundary (by piecewise p-polynomial functions), if we use the approximation space $V_h = W_h \oplus_{l=1}^{M}\{v_l\}$, with a sufficient number of singular functions and if W_h is the space of piecewise k-polynomial functions over Γ_h, we obtain an order of error in $h^{p+1} + h^{k+2}$. The optimal error is then obtained when $p = k + 1$ (in our case, $p = 1, k = 0$).

Remark 2.4. We have no error estimates in the vicinity of the boundary. That is because grad ψ is discontinuous at the points of Γ, while grad ψ_h is discontinuous at the points of Γ_h.

3. Approximation of the Lifting Flow. The stream function ψ of the flow is the solution of the following problem:

$$\begin{cases} \Delta\psi = 0 & \text{in } \Omega', \\ \psi = \psi_0 & \text{on } \Gamma. \end{cases}$$

Since the lift is nonzero, the behavior of the stream function is logarithmic at infinity. We can then try to solve this problem in a space allowing such a behavior.

First, we prolong ψ to Ω by the solution of the interior problem, and we try to solve simultaneously both problems

$$\begin{cases} \Delta\psi = 0 & \text{in } \Omega \text{ and } \Omega', \\ \psi = \psi_0 & \text{on } \Gamma. \end{cases} \tag{3.1}$$

The following space, introduced by Giroire [7], allows logarithmic behaviors

$$W^1_{-1}(\mathbf{R}^2) = \left\{ u \in \mathcal{D}'(\mathbf{R}^2);\ \frac{u}{1+r^2} \in L^2(\mathbf{R}^2),\ (1+r^2)^{-1/2}\frac{\partial u}{\partial x_i} \in L^2(\mathbf{R}^2),\ i = 1, 2 \right\}.$$

In Section 1, we solved Problem (3.1) in the space $W^1(\mathbf{R}^2)$ by integral equations. But in the space $W^1_{-1}(\mathbf{R}^2)$, Problem (3.1) has an infinity of solutions. One of them is the (unique) solution which is in $W^1(\mathbf{R}^2)$. Let us denote it by ψ_L.

We consider now the particular function ϕ, defined by

$$\phi(x) = -\frac{1}{2\pi}\int_\Gamma \operatorname{Log}|x-y|\, d\gamma(y).$$

ϕ is not in $W^1(\mathbf{R}^2)$. Let $\tilde{\phi}$ be the unique solution, in $W^1(\mathbf{R}^2)$, of the problem

$$\begin{cases} \Delta\tilde{\phi} = 0 & \text{in } \Omega \text{ and } \Omega', \\ \tilde{\phi} = \phi & \text{on } \Gamma. \end{cases} \tag{3.2}$$

$\phi - \tilde{\phi}$ is a nonzero function of $W^1_{-1}(\mathbf{R}^2)$. The following result, due to Giroire [7], enables us to find all the solutions of Problem (3.1):

THEOREM 3.1. *Each solution ψ of Problem* (3.1), *in the space $W^1_{-1}(\mathbf{R}^2)$, can be written as follows*:

$$\psi = \psi_L + \lambda(\phi - \tilde{\phi}). \tag{3.3}$$

The problem is then to determine the value of λ corresponding to the *physical solution* of the lifting flow problem.

By the regularity results of Theorem 2.1, we know that

$$\psi_L = a r^{1/\alpha} \sin\frac{\theta}{\alpha} + \phi_L \quad \text{in } \Omega',$$

$$\tilde{\phi} = b r^{1/\alpha} \sin\frac{\theta}{\alpha} + \tilde{\phi}_{\text{reg}} \quad \text{in } \Omega',$$

where $\alpha\pi$ is the exterior angle and $\phi_L, \tilde{\phi}_{\text{reg}} \in H^2_{\text{loc}}(\Omega')$. ϕ is regular (i.e., in $H^2_{\text{loc}}(\Omega')$ and $H^2(\Omega)$), since we have $[\partial\phi/\partial n] = 1$. (3.3) then takes the following form:

$$\psi = \phi_L + \lambda\left(\phi - \tilde{\phi}_{\text{reg}}\right) + (a - \lambda b) r^{1/\alpha} \sin\frac{\theta}{\alpha} \quad \text{in } \Omega'. \tag{3.4}$$

On the other hand, $\psi \in H^2(\Omega)$, since the interior angle is less than π.

The existence of the singular part in (3.4) leads to infinite velocities near the corner (the velocity behaves as $r^{(1/\alpha)-1}$), which are not physically feasible. The well-known Kutta-Joukowsky condition can then be stated as follows:

THEOREM 3.2. *There exists a unique flow, determined by the value of its lift, which leads to a finite velocity at the corner.*

The proof of this theorem can be found in the Appendix.

This result means that $b \neq 0$ and $\lambda = a/b$. The first way to calculate the solution is to solve both Problems (3.1) and (3.2) in $W^1(\mathbf{R}^2)$, deduce the value of λ, and then, by (3.4), the solution ψ of the lifting flow problem.

But, in order to avoid division by b (which can be small) and to solve one problem instead of two, we shall use another way.

Let us consider the jump of the normal derivative of the solution

$$\left[\frac{\partial \psi}{\partial n}\right] = \left[\frac{\partial \psi_L}{\partial n}\right] + \lambda\left[\frac{\partial \phi}{\partial n}\right] - \lambda\left[\frac{\partial \tilde{\phi}}{\partial n}\right].$$

We know that $[\partial\phi/\partial n] = 1$, and that

$$\left[\frac{\partial \psi_L}{\partial n}\right], \left[\frac{\partial \tilde{\phi}}{\partial n}\right] \in H_0^{-1/2}(\Gamma).$$

We then have $[\partial\psi/\partial n] = v + \lambda$, where we denote by v the quantity $[\partial\psi_L/\partial n] - \lambda[\partial\tilde{\phi}/\partial n]$. v is then the unique solution of the integral equation problem, in $H_0^{-1/2}(\Gamma)$,

$$-\frac{1}{2\pi}\int_\Gamma \int_\Gamma v(x)v'(y)\mathrm{Log}|x-y|\, d\gamma_x d\gamma_y = \langle v', \psi_0 - \lambda\phi\rangle.$$

We use the discretization introduced in Subsection 2.2. The discrete problem can be written as follows:

$$(3.5)\qquad \begin{cases} \text{Find } v_h \in V_h \text{ such that} \\ -\dfrac{1}{2\pi}\displaystyle\int_{\Gamma_h}\int_{\Gamma_h} v_h(x)v_h'(y)\mathrm{Log}|x-y|d\gamma_h(x)d\gamma_h(y) \\ \qquad\qquad = \langle v_h', \psi_{0h} - \lambda\phi\rangle, \quad \forall v_h' \in V_h. \end{cases}$$

Since $V_h = \{v_0\} \oplus W_h$, let us denote by $\{v_i\}_{i=1,\ldots,n-1}$ a basis of W_h,

$$v_h = \alpha_0 v_0 + \sum_{i=1}^{n-1} \alpha_i v_i.$$

By writing (3.5) for each element of the basis of V_h (i.e., for each v_j, $j = 0, 1, \ldots, n-1$), we have

$$(3.6)\qquad \sum_{i=0}^{n-1} \alpha_i a_h(v_i, v_j) + \lambda\langle v_j, \phi\rangle = \langle v_j, \psi_{0h}\rangle, \quad \text{for } j = 0, 1, \ldots, n-1.$$

The discrete problem is then a linear system of n equations with $n+1$ unknowns, since λ is unknown. But, by the Kutta-Joukowsky condition, we know that α_0 has to vanish since a nonvanishing α_0 would lead to infinite velocities at the corner. By setting $\alpha_0 = 0$, we get a linear system of n equations with n unknowns, but we lose the symmetry of the problem.

Remark 3.1. The flow studied in this paper is the *perturbation flow*. The stream function of the total flow ψ_T is then

$$\psi_T = \psi + \psi_\infty.$$

The tangential component of the velocity (which represents the entire velocity since the normal component has to vanish at the boundary) is given by the normal derivative of the stream function. Let us denote it by u:

$$u = \frac{\partial \psi_\infty}{\partial n} + \frac{\partial \psi}{\partial n} = \frac{\partial \psi_\infty}{\partial n} + \left[\frac{\partial \psi}{\partial n} \right] + \frac{\partial \psi_{\text{int}}}{\partial n},$$

where ψ_{int} is the stream function of the interior flow. ψ_{int} is the solution of the following problem:

$$\begin{cases} \Delta \psi = 0 & \text{in } \Omega, \\ \psi = -\psi_\infty & \text{on } \Gamma. \end{cases}$$

We have then $\psi_{\text{int}} = -\psi_\infty$ in Ω, because of the uniqueness of the interior Dirichlet problem. Thus,

$$u = \frac{\partial \psi_\infty}{\partial n} + \left[\frac{\partial \psi}{\partial n} \right] - \frac{\partial \psi_\infty}{\partial n}, \qquad u = \left[\frac{\partial \psi}{\partial n} \right].$$

The jump of the normal derivative of the stream function is the velocity of the flow.

This very important result shows that, by this method, we compute both the velocity of the flow and its lift, by solving a unique problem.

4. Some Numerical Results. The greatest numerical difficulties arise in the calculation of the first row (and first column) matrix coefficients. These coefficients, which depend on the singular basic function v_0, lead to the calculation of the following integrals:

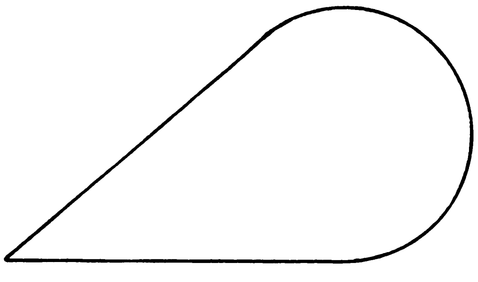
$$\int_{AB} \operatorname{Log}|s - t| \, dt. \tag{4.1}$$

$$\int_{AB} t^{(1/\alpha)-1} \operatorname{Log}|s - t| \, dt. \tag{4.2}$$

These integrals have to be calculated as precisely as possible, even exactly if possible, since we have to integrate these quantities over a segment a second time. We also note that the error estimates we obtained in Section 2 assume that there is no error in the computation of the coefficients.

The details of these calculations can be found in [3]. Let us just say that we use an analytic calculation when it can be performed (for (4.1)), and a very precise numerical integration–with mesh refinements–otherwise. These calculations constitute the greatest part of the computational time.

We first solved numerically our problem in a domain Ω for which the boundary Γ was constructed by connecting smoothly two segments making an angle with a circular arc.



The solution ψ_0 being given on the boundary Γ, we cannot generally know the exact solution anywhere else. We have no error estimates on the boundary.

Estimates (2.19) and (2.20), and a previous work [8], allow us to think that the greatest imprecision will affect the computed values on the boundary.

Thus, we compare the exact values and the computed ones on some points of the boundary, and we compare as well the exact values and the values computed without the use of a singular basis function. In this last case, the error is much worse on the corner point than on the other points. By using the singular basis function, we bring the value of the error on the corner point near its value on the other points.

TABLE 4.1

Angle of 18 *degrees*

	with the singular basis function		without the singular basis function	
Nb. of F. E. in the discretization	error on the corner	average error	error on the corner	average error
15	3.3%	0.5%	14.6%	1.6%
25	1.3%	0.17%	10.6%	0.75%
50	2.5%	0.1%	7.5%	0.26%

The improvement due to the introduction of a singular basis function is then very important on the corner. This can be easily seen for an angle of 9 degrees: the error with 8 Finite Elements using a singular basis function is the same as that with 100 Finite Elements without using the singular basis function.

TABLE 4.2

Angle of 9 *degrees. Error on the corner point*

Nb. of F. E.	with sing. basis funct.	without sing. basis funct.
8	5%	19.4%
15	0.5%	15%
25	0.71%	10.5%
50	0.58%	7.8%
100	0.43%	5%

The second type of domain is a Karmann-Trefftz one, obtained by transforming a circle by a conformal mapping. Trivial harmonic solutions of exterior problems are given by logarithms. Their transformation by a conformal mapping leads to singular solutions.

The use of the singular basis function greatly improves the computed solution; see Table 4.3.

TABLE 4.3

Karmann-Trefftz profile with interior angle of 9 degrees

	solution with sing. basis funct.	solution without sing. basis funct.
error on the corner point	5.3%	24.6%
error on the next point	0.76%	4.1%
average error	0.7%	2.6%

In conclusion, we can say that this method gives a precision which is comparable near or far from the corner.

The numerical results show very important improvements near the corner. These improvements are more important when the corner is sharp, which is the case of the physical problem. Therefore, it is worthwhile to carry on these numerical tests.

Appendix. Proof of Theorem 3.2. We obtain the desired result if we prove that $\tilde{\phi} \notin H^2_{\mathrm{loc}}(\Omega')$, (i.e., that $b \neq 0$).

By denoting $\Phi = \phi - \tilde{\phi}$, we notice that Φ is a solution of the following problem, in the space $W^1_{-1}(\Omega')$,

$$\text{(P)} \qquad \begin{cases} \Delta\Phi = 0 & \text{in } \Omega', \\ \Phi = 0 & \text{on } \Gamma. \end{cases}$$

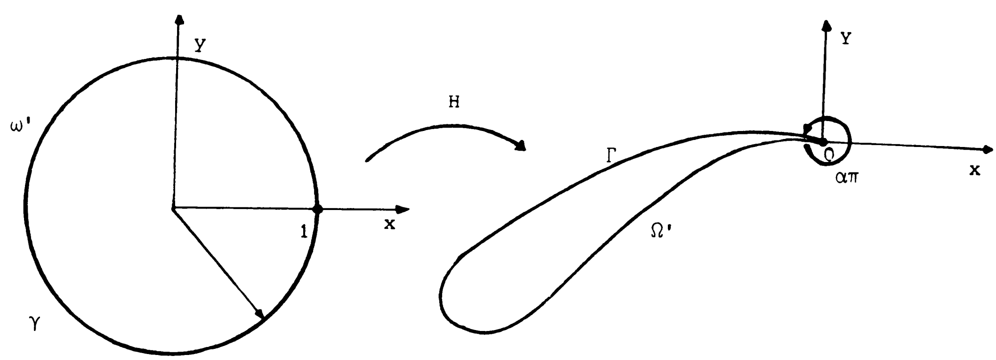
By applying Theorem 3.1, we can easily see that all the solutions in $W^1_{-1}(\Omega')$, of Problem (P), are multiples of Φ. Actually, if Φ_1 is a solution of (P), Theorem 3.1 shows that $\Phi_1 = \Phi_L + \lambda\Phi$, where Φ_L is the solution of (P) in $W^1_0(\Omega')$. Thus, $\Phi_L \equiv 0$ and $\Phi_1 = \lambda\Phi$.

If we consider now the unit circle γ and its exterior domain ω', we have the function ρ defined by $\rho(z) = \operatorname{Log}|z|$, which is a solution of the problem

$$\text{(p)} \qquad \begin{cases} \Delta\rho = 0 & \text{in } \omega', \\ \rho = 0 & \text{on } \gamma. \end{cases}$$

ρ is the real part of the holomorphic function $\eta(z) = \operatorname{Log} z$.

There exists a conformal mapping H from the exterior domain ω' onto the exterior domain Ω'. Moreover, we can expand H^{-1} at the vicinity of the corner $(z = 0)$ in the following way [9]:



$$H^{-1}(z) = 1 + a_1 z^{1/\alpha} + \cdots, \quad \text{with } a_1 \neq 0.$$

The function $\beta(z) = \eta \circ H^{-1}(z)$ is a holomorphic function. Its real part is then harmonic and vanishes on the boundary Γ, since η vanishes on γ. Then

$$\tilde{\Phi} = \operatorname{Re}(\beta)$$

is a solution of Problem (P), in the space $W^1_{-1}(\Omega')$. Thus, $\tilde{\Phi} = \lambda\Phi$, $\lambda \neq 0$. We can even assume that $\tilde{\Phi} = \Phi$. Thus,

$$\Phi = \operatorname{Re}(\eta \circ H^{-1}(z)) = \operatorname{Re}\big(\operatorname{Log}(1 + a_1 z^{1/\alpha} + \dots)\big),$$

$$\Phi = \operatorname{Re}\big(a_1 z^{1/\alpha} + \dots\big).$$

The singularity coefficient of the function does not vanish since $a_1 \neq 0$. Theorem 3.2 is then proved. □

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