MATHEMATICS OF COMPUTATION
VOLUME 36, NUMBER 154
APRIL 1981

The Jacobi-Perron Algorithm in Integer Form

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Abstract. We present an alternative expression of the Jacobi-Perron algorithm on a set of $n - 1$ independent numbers of an algebraic number field of degree n, where computation of real valued (nonrational) numbers is avoided. In some instances this saves the need to compute with high levels of precision. We also demonstrate a necessary and sufficient condition for the algorithm to cycle. The paper is accompanied by several numerical examples.

1. Introduction. The Jacobi-Perron algorithm (JPA) is a generalization of the continued fraction algorithm. From the JPA we can recursively obtain a series of simultaneous rational approximations to a set of real numbers known as convergents, with the numerators and denominator at each stage expressed as a linear combination of the corresponding numerators or denominators of several previous stages. The coefficients of these linear combinations which we will refer to as the *determining sequence*, are derived by the algorithm.

The JPA performs rational transformations on the original set of real numbers. Generally, to obtain convergents approximating the set of real numbers to an accuracy of 10^{-k} requires at least $O(k)$ steps and initial decimal approximations of these numbers with an accuracy of at least 10^{-k}. This restricts the algorithm in practical computing to small values of k, requiring multiprecision arithmetic, so that some questions remain open (Bernstein [2, p. 69]).

The continued fraction expansion of real quadratic numbers in a field $Q(\sqrt{d})$, $d > 0$, is ultimately periodic, the continued fraction coefficients (determining sequence) can be calculated using an adaption of the continued fraction algorithm due partly to Lagrange; see Chrystal [4, Chapter 33]. This adaption has the advantage that it operates with small bounded integers and requires only a very crude approximation to $\sqrt{d}$. (For example, for the expansion of $\sqrt{d}$, the terms are all positive integers, bounded above by $2\sqrt{d}$, and $[\sqrt{d}]$ is an adequate approximation to $\sqrt{d}$.)

By treating the algebraic numbers in a field of degree n as a ratio of elements of a Z-module, we generalize this adaption for any JPA expansion of $n - 1$ independent numbers of this field. Further, we note that the JPA is ultimately periodic if and only if the integer coefficients of the terms are bounded. We also develop explicit forms for this algorithm in the cases $n = 2$ and $n = 3$ which are illustrated by several examples.

Received April 2, 1979; revised August 14, 1980.
1980 *Mathematics Subject Classification.* Primary 10A30, 10F20, Secondary 12A30, 12A45.
Key words and phrases. Jacobi-Perron algorithm, multiprecision arithmetic, continued fractions, fundamental unit, cubic fields.

2. The General Algorithm. Let δ be a real algebraic integer of degree n, with minimal polynomial $p(x) = 0$, and let $\mathcal{O}_K$ be the ring of integers of $K = Q(\delta)$. Let $\Lambda = Z[1, \delta, \ldots, \delta^{n-1}]$ be the Z-module generated by the powers of δ. Let $\alpha_{0,1}, \ldots, \alpha_{0,n-1}$ be $n - 1$ rationally independent elements of $K \setminus Q$, and set $\boldsymbol{\alpha}_0 = (\alpha_{0,1}, \ldots, \alpha_{0,n-1}) \in K^{n-1}$. Following Bernstein [2], the sequence $(\boldsymbol{\alpha}_m)$, $m = 0, 1, \ldots,$ of vectors in R^{n-1} is called a Jacobi-Perron Algorithm (JPA) of $\boldsymbol{\alpha}_0$ if there exists a function f: $R^{n-1} \to R^{n-1}$ such that if $f(\boldsymbol{\alpha}_m) = \mathbf{a}_m \in R^{n-1}$, then

$$(1)\qquad \begin{aligned} &\text{(a)}\quad a_{m,1} \neq \alpha_{m,1},\\ &\text{(b)}\quad \boldsymbol{\alpha}_{m+1} = (\alpha_{m,1} - a_{m,1})^{-1}(\alpha_{m,2} - a_{m,2}, \ldots, \alpha_{m,n-1} - a_{m,n-1}, 1)\end{aligned}$$

for $m = 0, 1, \ldots$.

We shall refer to $(\mathbf{a}_m)$ as the *determining sequence* for the JPA of $\boldsymbol{\alpha}_0$. The nature of f is critical in the JPA. Below we restrict f to integer values, i.e. $f(\boldsymbol{\alpha}_m) \in Z^{n-1}$, so condition (a) of (1) will always be satisfied, and further $\alpha_{m,i} \in K \setminus Q$, $m = 0, 1, \ldots, i = 1, \ldots, n - 1$.

The usual method of calculating the first m terms of the determining sequence is to express the $\alpha_{0,i}$ in decimal expansion to a sufficient degree of accuracy so that significance is not lost in the first m steps of the JPA of $\boldsymbol{\alpha}_0$. Unfortunately, as the number of significant digits required for this accuracy is usually $O(m)$, real multiprecision arithmetic subroutines are frequently needed for computation, slowing down the algorithm and providing a restrictive barrier for large m.

Below we outline an alternative computational procedure, where all calculations (except possibly computing $f(\boldsymbol{\alpha}_m)$) are carried out as integer calculations, and in some instances for some f these integers remain small, avoiding the need for multiprecision arithmetic.

As $\alpha_{0,i} \in K$, we can find $\beta_{0,i}, \gamma_0 \in \mathcal{O}_K$ such that, for $i = 1, \ldots, n - 1$, $\alpha_{0,i} = \beta_{0,i}\gamma_0^{-1}$. However, for computational convenience, whenever necessary we can multiply numerator and denominator by a suitable rational integer so that $\beta_{0,i}$, $\gamma_0 \in \Lambda$.

Recurrence relation (1) can now be rewritten as

$$(2)\qquad \alpha_{m,i} = \beta_{m,i}\gamma_m^{-1}, \qquad i = 1, \ldots, n - 1,$$

where

$$(3)\qquad \begin{aligned} \beta_{m+1,i} &= \beta_{m,i+1} - a_{m,i+1}\gamma_m, \qquad i = 1, \ldots, n - 2,\\ \beta_{m+1,n-1} &= \gamma_m, \end{aligned}$$

and

$$(4)\qquad \gamma_{m+1} = \beta_{m,1} - a_{m,1}\gamma_m,$$

with $\mathbf{a}_m = f(\boldsymbol{\beta}_m\gamma_m^{-1})$.

Let $Q_m = N(\gamma_m)$. Then, as $Q_m\gamma_m^{-1}$ is the product of the $n - 1$ remaining conjugates of γ_m,

$$(5)\qquad \theta_{m,i} = Q_m\alpha_{m,i} = \beta_{m,i}Q_m\gamma_m^{-1}$$

is an element of Λ. If we let

$$(6)\qquad \phi_{m,i} = \theta_{m,i} - a_{m,i}Q_m,$$

we can obtain another expression for the recurrence relation. For $\lambda \in Z[1, \delta, \ldots, \delta^{n-1}]$, we will use the notation $\lambda' = N(\lambda)\lambda^{-1}$, so λ' is the product of the remaining conjugates of λ.

Now

$$Q_{m+1} = N(\gamma_{m+1}) = N(\beta_{m,1} - a_{m,1}\gamma_m) = N(\phi_{m,1})Q_m^{1-n}, \tag{7}$$

$$\begin{aligned} \theta_{m+1,n-1} &= \alpha_{m+1,n-1}Q_{m+1} = (\alpha_{m,1} - a_{m,1})^{-1}Q_{m+1} \\ &= \phi_{m,1}^{-1}Q_m Q_{m+1} = \phi'_{m,1}Q_m^{2-n}. \end{aligned} \tag{8}$$

Using (8), we can reexpress (7) to avoid using the norm function,

$$Q_{m+1} = \phi_{m,1}\phi'_{m,1}Q_m^{1-n} = \theta_{m+1,n-1}\phi_{m,1}Q_m^{-1}, \tag{9}$$

and, for $i = 1, \ldots, n-2$,

$$\theta_{m+1,i} = \phi_{m,i+1}\theta_{m+1,n-1}Q_m^{-1}. \tag{10}$$

Thus,

$$\mathbf{a}_{m+1} = f(\boldsymbol{\alpha}_{m+1}) = f(\boldsymbol{\theta}_{m+1}Q_{m+1}^{-1}), \tag{11}$$

and

$$\phi_{m+1,i} = \theta_{m+1,i} - a_{m+1,i}Q_{m+1} \quad \text{for } i = 1, \ldots, n-1. \tag{12}$$

Equations (8)–(12) specify the algorithm with initial values, Q_0 known, $\mathbf{a}_0 = f(\boldsymbol{\alpha}_0)$, $\boldsymbol{\theta}_0 = Q_0\boldsymbol{\alpha}_0$, $\boldsymbol{\phi}_0 = \boldsymbol{\theta}_0 - Q_0\mathbf{a}_0$. Except for (11), the evaluation of f of which may require the expression of $\boldsymbol{\theta}_{m+1}Q_{m+1}^{-1}$ as a real valued vector, all other calculations in this algorithm can be computed using integer expressions alone, if we express $\boldsymbol{\theta}_{m,i}$, $\boldsymbol{\phi}_{m,i}$, and $\phi'_{m,1}$ in terms of their coefficients as elements of Λ. The coefficients of $\phi'_{m,1}$ can be determined as a function of the coefficients of $\phi_{m,1}$ by multiplying the $n-1$ conjugates together and applying the elementary symmetric functions on the coefficients of $p(x) = 0$.

Suppose the JPA is periodic in the sense that there are positive integers M and k such that, for all $m > M$,

$$\boldsymbol{\alpha}_{m+k} = \boldsymbol{\alpha}_m.$$

For all $m \geqslant 0$, let I_m be the $\mathcal{O}_K$-module generated by $\{1, \alpha_{m,1}, \ldots, \alpha_{m,n-1}\}$. We see

$$\gamma_m I_m = \langle \gamma_m, \beta_{m,1}, \ldots, \beta_{m,n-1} \rangle$$

is an ideal, and from the recurrence relations (3) and (4) we find

$$\gamma_m I_m = \gamma_{m+1} I_{m+1}$$

and so, by induction, $\gamma_m I_m = \gamma_0 I_0$, and for all $m > M$

$$\gamma_m I_m = \gamma_{m+k} I_{m+k} = \gamma_{m+k} I_m$$

by the periodicity of $\boldsymbol{\alpha}_m$. Now

$$\begin{aligned} \gamma_m I_m = \gamma_{m+k} I_m &\Rightarrow \gamma_m(\gamma_m I_m) = \gamma_{m+k}(\gamma_m I_m) \\ &\Rightarrow |N(\gamma_m)| N(\gamma_m I_m) = |N(\gamma_{m+k})| N(\gamma_m I_m) \end{aligned}$$

and, since $N(\gamma_m I_m) = N(\gamma_0 I_0) \neq 0$, we have

$$|N(\gamma_m)| = |N(\gamma_{m+k})|.$$

Thus, $Q_m = sQ_{m+k}$, where $s = \pm 1$, and, consequently,

$$\boldsymbol{\theta}_m = Q_m \boldsymbol{\alpha}_m = sQ_{m+k}\boldsymbol{\alpha}_{m+k} = s\boldsymbol{\theta}_{m+k}.$$

Since

$$\begin{aligned} Q_{m+1} &= N(\phi_{m,1})Q_m^{1-n} = N(\theta_{m,1} - a_{m,1}Q_m)Q_m^{1-n} \\ &= N(s\theta_{m+k,1} - a_{m+k,1}sQ_{m+k})(sQ_{m+k})^{1-n} \\ &= sN(\phi_{m+k,1})Q_{m+k}^{1-n} = sQ_{m+k+1}, \end{aligned}$$

by induction we have

$$Q_m = sQ_{m+k} = s^2 Q_{m+2k}.$$

Thus, if $s = 1$, we have Q_m, $\boldsymbol{\theta}_m$ periodic of length k, and if $s = -1$, we have Q_m, $\boldsymbol{\theta}_m$ periodic of length $2k$. (See Example 2 for a case in which the length of the period of Q_m, $\boldsymbol{\theta}_m$ is twice the length of the period of $\boldsymbol{\alpha}_m$.)

Alternatively, if there are integers M and k such that, for all $m > M$,

$$Q_m = Q_{m+k} \quad \text{and} \quad \boldsymbol{\theta}_m = \boldsymbol{\theta}_{m+k},$$

then, as $\boldsymbol{\alpha}_m = \boldsymbol{\theta}_m Q_m^{-1}$, $\boldsymbol{\alpha}_m = \boldsymbol{\alpha}_{m+k}$. Hence

THEOREM 1. *The Jacobi-Perron algorithm expansion of* $\boldsymbol{\alpha}_0 \in K^{n-1}$ *is periodic if and only if the values* Q_m, $\boldsymbol{\theta}_m$ *are periodic.*

Further, if the values Q_m, $\boldsymbol{\theta}_m$ do cycle, then the integer coefficients of $\boldsymbol{\theta}_m$ and the integers Q_m are bounded in absolute value by their extreme values in the preperiod and the first period. Alternatively, if the integer coefficients of $\boldsymbol{\theta}_m$ and the integers Q_m are bounded in absolute value, there will only be a finite number of combinations of Q_m, $\boldsymbol{\theta}_m$ possible, so there will exist $M, k \in Z$ such that $Q_M = Q_{M+k}$ and $\boldsymbol{\theta}_M = \boldsymbol{\theta}_{M+k}$. Thus,

$$\boldsymbol{\alpha}_M = \boldsymbol{\theta}_M / Q_M = \boldsymbol{\theta}_{M+k} / Q_{M+k} = \boldsymbol{\alpha}_{M+k},$$

and, consequently,

$$\mathbf{a}_M = f(\boldsymbol{\alpha}_M) = f(\boldsymbol{\alpha}_{M+k}) = \mathbf{a}_{M+k}.$$

From (1) it is clear that $\boldsymbol{\alpha}_{m+k} = \boldsymbol{\alpha}_m$, for all $m > M$, that is, the algorithm cycles. So we have

COROLLARY. *The Jacobi-Perron algorithm expansion of* $\boldsymbol{\alpha}_0 \in K^{n-1}$ *is periodic if and only if the integers* Q_m *and the integer coefficients of* $\theta_{m,i}$, *expressed as elements of the Z-module* $[1, \delta, \ldots, \delta^{n-1}]$, *are bounded.*

For some f these integer coefficients remain bounded and multiprecision arithmetic is avoided.

For suitably chosen f, the JPA can be used to derive a series of rational approximations $p_{m,i}/q_m$ to $\alpha_{0,i}$, where $p_{m,i}, q_m \in Z$, in the following way. Generalizing the convergents of continued fractions, we write

$$\mathbf{p}_m = (q_m, p_{m,1}, \ldots, p_{m,n-1}) \in Z^n \tag{13}$$

recursively defined by

$$P_m = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & a_{m,1} & a_{m,2}, & \ldots, & a_{m,n-1} \end{bmatrix} P_{m-1},$$

where $P_{-1} = I_n$, and $\mathbf{p}_m$ is the nth row of P_m.

3. The Algorithm for Quadratic Numbers. If K is a real quadratic field, we can find a δ satisfying $x^2 - d = 0$, i.e. $\delta^2 = d > 1$, so that $K = Q(\delta)$. For simplification, as $n - 1 = 1$, we will reduce the double suffix notation to single suffices on all variables, identifying $\alpha_{m,i}$ as α_m, etc. Thus, for $\alpha_0 \in K$, $\exists a, b, c, d \in Z$, with $(a, b, c) = 1$, such that

$$\alpha_0 = (a + b\sqrt{d})c^{-1} = \beta_0\gamma_0^{-1}.$$

Hence,

$$Q_0 = N(c) = c^2 \quad \text{and} \quad \theta_0 = (a + b\sqrt{d})c,$$

and we have

$$\alpha_0 = \left(A_0 + B\sqrt{d}\right)Q_0^{-1}, \quad \text{where } A_0 = ac \text{ and } B = bc. \tag{14}$$

If

$$\theta_m = A_m + (-1)^m B\sqrt{d}\,, \tag{15}$$

Eq. (8) gives us,

$$\theta_{m+1} = \phi'_m = \theta'_m - a_m Q_m = (A_m - a_m Q_m) - (-1)^m B\sqrt{d}$$

so

$$A_{m+1} = A_m - a_m Q_m, \tag{16}$$

and

$$\theta_{m+1} = A_{m+1} + (-1)^{m+1} B\sqrt{d}\,.$$

Hence (15) is established by induction. Now, by (9),

$$\begin{aligned} Q_{m+1} &= \theta_{m+1}\phi_m Q_m^{-1} \\ &= \left(A_{m+1} + (-1)^{m+1}B\sqrt{d}\right)\left(A_{m+1} + (-1)^m B\sqrt{d}\right)Q_m^{-1} \\ &= \left(A_{m+1}^2 - B^2 d\right)Q_m^{-1} = Q_{m-1} - 2a_m A_m + a_m^2 Q_m \quad (Q_{-1} = Q_0 N(\alpha_0)) \end{aligned} \tag{17}$$

and so, of course,

$$a_m = f\left(\left(A_m + (-1)^m B\sqrt{d}\right)Q_m^{-1}\right). \tag{18}$$

If we let

$$\varepsilon_m = p_m - q_m\alpha_0, \tag{19}$$

where $(p_m, q_m) = \mathbf{p}_m$ is defined by (13), we find

$$\varepsilon_{m+1} = a_m\varepsilon_m + \varepsilon_{m-1} \tag{20}$$

and, by induction,

$$N(\varepsilon_m) = Q_{m+1}Q_0^{-1}. \tag{21}$$

Thus, if $\alpha_0 \in \mathcal{O}_K$ and $Q_{m+1} = \pm Q_0$ for some $m > 0$, ε_m will be a unit of $\mathcal{O}_K$.

For $f(\alpha_m) = [\alpha_m]$, the integer value of α_m, the JPA becomes the simple continued fraction algorithm (SCF) for α_0, with the determining sequence (a_m) being the sequence of continued fraction coefficients, and $\mathbf{p}_m = (q_m, p_m)$ giving the convergents p_m/q_m to α_0. The integral algorithm (15), (16), (18) is very close to the Lagrange algorithm for SCF coefficients (see Chrystal [4, Chapter 33]). The SCF for a quadratic surd is always ultimately periodic, so after a finite number of steps the complete determining sequence (SCF coefficients) is known. If we expand $\sqrt{d}$ (or $\frac{1}{2} + \frac{1}{2}\sqrt{d}$ for $d \equiv 1 \pmod 4$), the first cycle is complete when $\pm Q_m = Q_0$, so that by (21) ε_{m-1} is a unit of $Q(\sqrt{d})$ and in fact (Hendy [5, pp. 167–168]) ε_{m-1} is the fundamental unit of $Q(\sqrt{d})$.

Example 1. Let $\alpha_0 = \sqrt{211}$. For $f(\alpha_m) = [\alpha_m]$ the JPA becomes the SCF. We list below the first 16 values of A_m, Q_m, a_m. It is easily seen from the algorithm that whenever $A_m = -A_{m+1}$ or $Q_m = -Q_{m+1}$ we reach the midpoint of the cycle with the subsequent values being reflected until the $2m$th or $2m-1$th term where $Q_{2m} = Q_0$ or $Q_{2m-1} = -Q_0$.

m	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
A_m	0	−14	1	−13	14	−11	7	−11	9	−4	11	−13	8	−13	13	−8
Q_m	1	−15	14	−3	5	−18	9	−10	13	−15	6	−7	21	−2	21	−7
a_m	14	1	1	9	5	1	2	2	1	1	4	3	1	13	1	3

In this case $Q_{26} = Q_0 = 1$, so by (21) ε_{25} is the fundamental unit of $Q(\sqrt{211})$. The sequence (a_m) cycles with cycle length 26, after the first term:

$$(a_m) = (14, \dot{1}, 1, 9, 5, 1, 2, 2, 1, 1, 4, 3, 1, 13, 1, 3, 4, 1, 1, 2, 2, 1, 5, 9, 1, 1, \dot{28}).$$

Using (13) or the standard recurrence formulae for SCF convergents, we find

$$\varepsilon_{25} = P_{25} + q_{25}\sqrt{211} = 278354373650 + 19162705353\sqrt{211}$$

as the fundamental unit of $Q(\sqrt{211})$. It is only in this final computational step that calculations involving large integers may be required.

Example 2. Let $\alpha_0 = (2 + \sqrt{10})/(1 + \sqrt{10}) \in Q(\sqrt{10})$. With $f(\alpha_m) = [\alpha_m]$ we have the SCF.

$$Q_0 = N(1 + \sqrt{10}) = -9, \qquad \theta_0 = (2 + \sqrt{10})(1 - \sqrt{10}) = -8 - \sqrt{10}.$$

m	0	1	2	3	4	...
A_m	−8	1	−3	3	−3	...
Q_m	−9	1	−1	1	−1	...
a_m	1	4	6	6	6	...

Thus $(a_m) = (1, 4, \dot{6})$.

If we use the nearest integer approximation, $f(\alpha) = \{\alpha\} = [\alpha + \frac{1}{2}]$, then the JPA becomes the nearest integer algorithm (NIA), which is observed in general to require about 69% of the number of steps required by the SCF to obtain the fundamental unit (Williams [10], Adams [1]). We illustrate the NIA below in calculating the fundamental unit of $\sqrt{211}$. In comparison with the SCF, Example 1, which required 26 steps, the NIA requires only 18 steps.

Example 3. $f(\alpha_m) = \{\alpha_m\}$, $\alpha_0 = \sqrt{211}$.

m	0	1	2	3	4	5	6	7	8	9	10	11	12	...
A_m	0	−15	13	−14	16	−11	19	−11	13	−15	15	−13	17	...
Q_m	1	14	−3	5	9	−10	−15	6	−7	−2	−7	6	13	...
a_m	15	−2	−9	−6	3	3	−2	−4	−4	15	−4	−5	2	...

$$(a_m) = (15, -\dot{2}, -9, -6, 3, 3, -2, -4, -4, 15, -4, -5, 2, 2, 3, -6, -10, 2, 2\dot{9})$$

with period length 18. The calculation of ε_{17}, using (13), gives the fundamental unit as before, but with less computation.

4. The Algorithm for Cubic Numbers. Let $K = Q(\delta)$, where δ is a real root of the irreducible polynomial $x^3 + rx + s = 0$, $r, s \in Z$. As before, we can express $\theta_{m,i}$, $\phi_{m,i}$, as linear integral combinations of $1, \delta, \delta^2$. Suppose

$$\theta_{m,1} = B_m + C_m\delta + D_m\delta^2, \tag{22}$$

$$\theta_{m,2} = F_m + G_m\delta + H_m\delta^2, \tag{23}$$

$B_m, C_m, D_m, F_m, G_m, H_m \in Z$ and let

$$A_m = B_m - a_{m,1}Q_m, \tag{24}$$

$$E_m = F_m - a_{m,2}Q_m, \tag{25}$$

so that

$$\phi_{m,1} = A_m + C_m\delta + D_m\delta^2, \tag{26}$$

$$\phi_{m,2} = E_m + G_m\delta + H_m\delta^2. \tag{27}$$

Inserting these values for $\theta_{m,i}$, $\phi_{m,i}$ in Eqs. (8), (10), and (9), we obtain recurrence relations amongst these coefficients, viz.

$$\tag{28}\begin{cases} F_{m+1} = (A_m^2 - 2rA_mD_m + rC_m^2 + sC_mD_m + r^2D_m^2)Q_m^{-1}, \\ G_{m+1} = -(A_mC_m + sD_m^2)Q_m^{-1}, \\ H_{m+1} = (C_m^2 + rD_m^2 - A_mD_m)Q_m^{-1}, \\ B_{m+1} = (E_mF_{m+1} - s(G_mH_{m+1} + H_mG_{m+1}))Q_m^{-1}, \\ C_{m+1} = (E_mG_{m+1} + G_mF_{m+1} - r(G_mH_{m+1} + H_mG_{m+1}) - sH_mH_{m+1})Q_m^{-1}, \\ D_{m+1} = (E_mH_{m+1} + G_mG_{m+1} + H_mF_{m+1} - rH_mH_{m+1})Q_m^{-1}, \\ Q_{m+1} = (F_{m+1}A_m - s(G_{m+1}D_m + H_{m+1}C_m))Q_m^{-1}. \end{cases}$$

A_{m+1} and E_{m+1} are obtained from (24), (25) and

$$\mathbf{a}_{m+1} = f(\boldsymbol{\theta}_{m+1}Q_{m+1}^{-1}).$$

A more involved determinantal form of formulae (28) has been developed by O. Perron [7].

Example 4. Set $\alpha_{0,1} \doteq 3.3186$, the real root of the cubic $x^3 - 8x - 10 = 0$, and $\alpha_{0,2} = \alpha_{0,1}^2$. $\alpha_{0,1}$ is the number whose intriguing continued fraction expansion was noted by John Brillhart and explained by H. M. Stark [8]. In contrast, with $f(\alpha_{m,1}, \alpha_{m,2}) = ([\alpha_{m,1}], [\alpha_{m,2}])$ the JPA is periodic with period 2 and a preperiod also of length 2.

m	A_m	B_m	C_m	D_m	E_m	F_m	G_m	H_m	Q_m	$a_{m,1}$	$a_{m,2}$
0	−3	0	1	0	−11	0	0	1	1	3	11
1	19	19	1	−2	−20	1	3	1	7	0	3
2	−3	0	1	0	−20	3	3	1	1	3	23
3	19	40	1	−2	−20	1	3	1	7	3	3
4	−3	0	1	0	−20	3	3	1	1	3	23
5	19	40	1	−2	−20	1	3	1	7	3	3
6	−3	0	1	0	−20	3	3	1	1	3	23
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Using Eq. (13) we get the following sequence of simultaneous rational approximations for $(\alpha_{0,1}, \alpha_{0,2})$:

$$\left(\frac{3}{1}, \frac{11}{1}\right), \left(\frac{10}{3}, \frac{33}{3}\right), \left(\frac{239}{72}, \frac{793}{72}\right), \left(\frac{750}{226}, \frac{2489}{226}\right), \left(\frac{17977}{5417}, \frac{59659}{5417}\right),$$
$$\left(\frac{56420}{17001}, \frac{187237}{17001}\right), \left(\frac{1352341}{407500}, \frac{4487917}{407500}\right).$$

In some instances the JPA can be applied to the problem of finding the fundamental unit of pure cubic fields. If $\delta^3 = d > 1$, a cubefree integer, then $r = 0$, $s = -d$ so that Eqs. (28) simplify to

$$(29)\qquad \begin{cases} F_{m+1} = (A_m^2 - dC_mD_m)Q_m^{-1}, \\ G_{m+1} = (dD_m^2 - A_mC_m)Q_m^{-1}, \\ H_{m+1} = (C_m^2 - A_mD_m)Q_m^{-1}, \\ B_{m+1} = (E_mF_{m+1} + dG_mH_{m+1} + dH_mG_{m+1})Q_m^{-1}, \\ C_{m+1} = (E_mG_{m+1} + G_mF_{m+1} + dH_mH_{m+1})Q_m^{-1}, \\ D_{m+1} = (E_mH_{m+1} + G_mG_{m+1} + H_mF_{m+1})Q_m^{-1}, \\ Q_{m+1} = (A_mF_{m+1} + dC_mH_{m+1} + dD_mG_{m+1})Q_m^{-1}. \end{cases}$$

(Formulae (29) also appear in a different form in Bernstein [3].)

Example 5. Let $\delta^3 = 71$, and set $\alpha_{0,1} = \delta$, $\alpha_{0,2} = \delta^2$, with $f(\alpha_{m,1}, \alpha_{m,2}) = (\{\alpha_{m,1}\}, \{\alpha_{m,2}\})$, the nearest integer approximation. Some values of the coefficients are:

m	A_m	B_m	C_m	D_m	E_m	F_m	G_m	H_m	Q_m	$a_{m,1}$	$a_{m,2}$
0	−4	0	1	0	−17	0	0	1	1	4	17
1	5	12	3	−1	−33	16	4	1	7	1	7
2	−10	2	2	0	−66	34	8	2	4	3	25
3	−66	−93	3	6	−29	25	5	1	−27	−1	−2
4	369	117	45	9	972	−486	−108	−27	−243	−2	6
5	68	348	−2	−2	818	−442	−92	−22	−140	−2	9
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
22	−2	142	2	−2	−561	159	57	15	144	1	5
23	11	7	−7	1	−8	2	2	0	−2	2	−5
24	113	113	13	−2	481	−309	−74	−19	−395	0	2
25	56	42	−14	0	5	−37	3	−1	14	−1	−3
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮

Again, using (13), we find

$$\varepsilon_{25} = p_{25,2} + p_{25,1}\delta + q_{25}\delta^2$$
$$= 1788355606552816482 + 431884645684316172\delta + 104299361097095425\delta^2,$$

which we see from Wada's table [9, p. 1136] is the fundamental unit of $Q(\delta)$. Directly, we could note the approximation

$$N(\varepsilon_{25}) \doteqdot \frac{3}{2}\left((x\delta^{-1} - y)^2 + (y - z\delta)^2 + (z\delta - x\delta^{-1})^2\right) dy,$$

where the error $O(y^{-3/2})$ is sufficiently accurate to show $N(\varepsilon_{25}) = 1$, so ε_{25} is a unit. Then, following the testing procedure of Jeans and Hendy [6], it is readily shown that this unit is fundamental. We note, however, that similar calculations with $f(\boldsymbol{\alpha}_m) = ([\alpha_{m,2}], [\alpha_{m,2}])$ do not attain the coefficients of the fundamental unit of $Q(\sqrt[3]{71})$.

Example 6. Let $\delta^3 = 4$ and set $\alpha_{0,1} = \delta$, $\alpha_{0,2} = \delta^2$. Bernstein [2, p. 69] makes the following comments about the vector $\boldsymbol{\alpha}_0$,

"This vector seems to occupy a most magic, and most annoying, place in the theory of JPA with $f(a^{(k)}) = [a^{(k)}]$. No human effort nor the capability of any computer have ever succeeded to find even a small hint of periodicity, tests having been pushed up to $a^{(0)}T^{(150)}$. Of course, this does not yet disprove periodicity".

Using (29) with $f(\boldsymbol{\alpha}_m) = ([\alpha_{m,1}], [\alpha_{m,2}])$, the first 170 steps of the JPA of $\boldsymbol{\alpha}_0$ were obtained with the aid of a B6700 computer. This was achieved without resorting to multiprecision arithmetic programs as the standard double precision (24-digit mantissa) available on the B6700 was sufficient to perform the required calculations. However, any further steps would require the use of multiprecision arithmetic programs. These 170 steps required approximately three seconds of processing time. The additional steps beyond Bernstein's do not reveal any periodicity. On the contrary, the Corollary to Theorem 1 suggests that periodicity will not occur. If periodicity is to occur eventually, the coefficients B_m, C_m, D_m, F_m, G_m, H_m, Q_m must be bounded. For the 170 steps calculated the coefficients show no sign of being bounded and appear generally to grow at an exponential rate over this interval. At the 170th step the coefficients range in magnitude from

$$H_{170} = 699434216343936663 21$$

to

$$F_{170} = 27955246331140829 1981.$$

In contrast to the above, we find that for the JPA of $\boldsymbol{\alpha}_0$ with $f(\boldsymbol{\alpha}_m) = (\{\boldsymbol{\alpha}_{m,1}\}, \{\alpha_{m,2}\})$ the nearest integer approximation is periodic with a preperiod of length 3 and a period of length 12, and the largest valued coefficient being 12.

5. Conclusion. Specific algorithms such as (29) could be developed for fields of degree greater than three, and indeed the authors have done so for the fields $Q(\delta)$ where δ satisfies the minimal polynomial $x^4 - d = 0$. However, the equations rapidly grow in complexity, and it seems pointless to derive them until a specific purpose is at hand. The equations (8), (9) and (10) together with the symmetric functions on the roots of the minimal polynomial suffice to create the appropriate equations.

It appears, in investigating periodicity of the JPA on pairs of cubic integers, that when periodicity occurs the coefficients all remain small integers (< 500), and in all other instances the coefficients eventually begin to increase rapidly in value. We are currently investigating the appropriate f function to apply in order to keep the coefficients small.

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