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A Numerical Investigation Into the Length of the Period of the Continued Fraction Expansion of $\sqrt{D}$

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Abstract. Let $p(D)$ be the length of the period of the continued fraction expansion of $\sqrt{D}$, where D is a positive square-free integer. In this paper it is suggested that $p(D) = O(\sqrt{D} \log \log D)$ and several tables of numerical results, which support this suggestion, are provided.

1. Introduction. Let θ be any positive irrational and let

$$\theta = [q_0, q_1, q_2, \ldots, q_{n-1}, \theta_n] \tag{1}$$

be the continued fraction expansion of $\sqrt{D}$. If $\theta = \sqrt{D}$, where D is a square-free positive integer, then

$$\sqrt{D} = [q_0, \overline{q_1, q_2, \ldots, q_p}],$$

where $p = p(D)$ is the length of the shortest period in this expansion of $\sqrt{D}$. In Stanton, Sudler, and Williams [8] it is shown that if $D > 7$, then

$$p(D) < .72\sqrt{D} \log D.$$

In Cohn [2] this result is improved to

$$p(D) < cD^{1/2} \log D + O(D^{1/2}),$$

where $c = 7/2\pi^2 \approx .3546$.

The brief numerical study on the size of $p(D)$ in Beach and Williams [1] seems to indicate that these bounds are not very good for larger values of D. The purpose of this note is to suggest why this should be the case and to provide numerical evidence for the belief that $p(D)$ can be bounded by an expression of the form $CD^{1/2} \log \log D$, where C is a constant.

2. Theoretical Considerations. Let $\theta = \sqrt{D}$ and let ε_0 be the fundamental unit of $Q(\sqrt{D})$. From Williams and Broere [9], we have

$$\varepsilon_0 = \prod_{i=1}^{p} \theta_i,$$

unless some $Q_r = 4$. If this latter case occurs, there must exist a pair of odd integers x, y such that $x^2 - Dy^2 = 4$ is solvable. Thus, we must have $D \equiv 5 \pmod 8$; also

$$\varepsilon_0^3 = \prod_{i=1}^{p} \theta_i.$$

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Now, from (2.8) of [8], we have

$$\prod_{i=1}^{p} \theta_i > \alpha^p,$$

where $\alpha = (1 + \sqrt{5})/2$. Hence, if we define $\lambda = 1/3$ when $D \equiv 5 \pmod 8$ and $\lambda = 1$ otherwise, we have $R > \lambda p \log \alpha$, where $R = \log \varepsilon_0$ is the regulator of $Q(\sqrt{D})$.

Levy [3] has shown that, for almost all irrationals θ, we have

$$\lim_{n\to\infty} \sqrt[n]{\theta_1\theta_2\theta_3 \cdots \theta_n} = e^m,$$

where $m = \pi^2/12 \log 2 \approx 1.18656911$. Thus, if p is large, we would expect $R \approx \lambda pm$. For example, when $D = 26437680473689$, we have $R \approx 21737796.43$ (Shanks [5]), $p = 18331889$ (see [7]), and $R/p \approx 1.185791406$.

If h is the class number of $Q(\sqrt{D})$, it is well known that

$$2Rh = \sqrt{d}\, L(1, \chi_d), \tag{2}$$

where d is the discriminant of $Q(\sqrt{D})$; that is,

$$d = \begin{cases} D & \text{when } D \equiv 1 \pmod 4, \\ 4D & \text{otherwise.} \end{cases}$$

Here

$$L(1, \chi_d) = \sum_{n=1}^{\infty} \left(\frac{d}{n}\right)\frac{1}{n} = \prod_{q=2}^{\infty}\left(\frac{q}{q - (d/q)}\right), \tag{3}$$

where $\chi_d(n) = (d/n)$ is the Kronecker symbol and the (Euler) product is evaluated over all the primes q. Littlewood [4] showed that if the Extended Riemann Hypothesis (ERH) is true for $\chi_d(n) = (d/n)$, then

$$L(1, \chi_d) < \{1 + o(1)\}2e^{\gamma}\log\log d,$$

where γ is Euler's constant. Further, Shanks [6] noted that under the same hypotheses,

$$L_{-D}(1) = \sum_{m=1}^{\infty}\left(\frac{4D}{m}\right)\frac{1}{m} < \{1 + o(1)\}e^{\gamma}\log\log 4D.$$

From the Euler product in (3), we see that

$$L_{-D}(1) = \begin{cases} \frac{1}{2}L(1, \chi_d) & \text{when } D \equiv 1 \pmod 8, \\ \frac{3}{2}L(1, \chi_d) & \text{when } D \equiv 5 \pmod 8, \\ L(1, \chi_d) & \text{otherwise;} \end{cases}$$

hence, Shanks' result is stronger than Littlewood's except when $D \equiv 1 \pmod 8$.

If we define

$$f(D) = \begin{cases} \sqrt{D}\log\log D & \text{for } D \equiv 1 \pmod 8, \\ \sqrt{D}\log\log 4D & \text{otherwise,} \end{cases}$$

we have

$$Rh < \{1 + o(1)\}\lambda e^{\gamma} f(D). \tag{4}$$

From the above remarks we would certainly expect that

$$G(D) = p(D)/f(D) < K_1 + o(1), \tag{5}$$

and even that

$$G(D) < K_2 + o(1), \tag{6}$$

where $K_1 = e^{\gamma}/\log \alpha \approx 3.701223297$ and

$$K_2 = e^{\gamma}m = 12e^{\gamma}\log 2/\pi^2 \approx 1.501027123, \quad \text{provided the ERH is true.}$$

3. Numerical Results. By using the algorithm mentioned in [9], $p(D)$ was evaluated for all square-free D where $2 < D \leqslant 2 \times 10^7$. In Table 1 we present only those values of D such that $p(D) > p(E)$ for all $E < D$. We also show the prime factorization of these D values together with $p(D)$ and $G(D)$ in (5).

TABLE 1

D	FACTORS	PERIOD	G(D)	D	FACTORS	PERIOD	G(D)
2	2	1	0.965862	13126	2.6563	262	0.958487
3	3	2	1.268574	15031	15031	268	0.911469
7	7	4	1.256078	16669	79.211	280	0.900769
13	13	5	1.009257	17341	17341	281	0.884992
19	19	6	0.939114	17494	2.8747	290	0.909037
31	31	8	0.913538	17614	2.8807	300	0.936937
43	43	10	0.930715	18379	18379	322	0.982945
46	2.23	12	1.071314	19231	19231	332	0.989114
94	2.47	16	0.927143	21319	21319	348	0.981006
139	139	18	0.828021	23599	23599	352	0.939685
151	151	20	0.876523	25939	25939	374	0.949117
166	2.83	22	0.912368	27589	47.587	380	0.933031
211	211	26	0.938215	28414	2.14207	388	0.937773
331	331	34	0.947446	31606	2.15803	394	0.899571
421	421	37	0.899215	32839	32839	400	0.894780
526	2.263	40	0.857079	32971	32971	438	0.977686
571	571	42	0.859239	34654	2.17327	440	0.956370
604	151	44	0.872132	38119	38119	444	0.917183
631	631	48	0.928307	39439	39439	448	0.908783
751	751	52	0.912086	39901	39901	449	0.905168
886	2.443	54	0.863544	40429	40429	451	0.902845
919	919	60	0.940111	40639	40639	456	0.910334
1291	1291	62	0.804128	42046	2.21023	464	0.909638
1324	331	64	0.818534	42571	42571	474	0.923112
1366	2.683	70	0.879911	43261	43261	489	0.944193
1516	379	76	0.901798	46006	2.23003	506	0.945490
1621	1621	79	0.903331	48799	48799	544	0.985062
1726	2.863	88	0.971966	53299	53299	566	0.977858
2011	2011	94	0.954351	60811	60811	574	0.924468
2311	2311	96	0.902889	61051	61051	614	0.986821
2566	2.1283	102	0.905738	67846	2.33923	618	0.939040
2671	2671	104	0.903403	72934	2.36467	642	0.938725
3004	751	108	0.879654	73516	18379	644	0.937681
3019	3019	114	0.925995	76651	76651	654	0.931348
3334	2.1667	118	0.907828	78094	2.39047	692	0.975748
3691	3691	122	0.887854	78439	78439	696	0.979093
3931	3931	130	0.914096	82471	82471	716	0.980770
4174	2.2087	136	0.925506	85999	85999	720	0.964559
4846	2.2423	152	0.953606	90931	90931	734	0.954639
5119	5119	156	0.949949	95131	95131	750	0.952358
6211	6211	158	0.866183	100291	100291	754	0.930986
6451	6451	170	0.912993	102859	102859	790	0.962446
6679	6679	172	0.906496	106591	106591	808	0.965950
6694	2.3347	174	0.915922	111094	2.55547	834	0.975404
7606	2.3803	194	0.952913	127219	127219	838	0.912178
8254	2.4127	196	0.921061	131884	32971	864	0.922716
8779	8779	202	0.918119	133519	133519	876	0.929449
8941	8941	207	0.931590	139591	139591	904	0.936842
9739	9739	210	0.902425	145006	2.72503	944	0.958789
9949	9949	217	0.921824	155299	155299	958	0.938340
10399	10399	228	0.945694	162094	2.81047	1016	0.972865
10651	10651	234	0.958119	166846	2.83423	1028	0.969430
10774	2.5387	238	0.968478	173671	173671	1040	0.960178
12541	12541	239	0.896083	187366	2.93683	1106	0.980963
12919	12919	248	0.915071	189814	2.94907	1110	0.977781

TABLE 1 (*continued*)

D	FACTORS	PERIOD	G(D)
196771	196771	1122	0.969735
217519	217519	1152	0.944325
221659	221659	1174	0.952828
230239	230239	1224	0.973692
241894	2.120947	1262	0.978097
253621	23.11027	1272	0.961529
256699	256699	1274	0.956931
257371	257371	1318	0.988617
285574	2.142787	1358	0.964268
289111	289111	1400	0.987661
294694	2.147347	1438	1.004292
313699	313699	1442	0.974454
333334	2.166667	1454	0.951632
339091	339091	1458	0.945681
351751	351751	1484	0.944140
358471	358471	1524	0.959974
363379	363379	1538	0.961879
375631	375631	1548	0.951377
383839	383839	1560	0.947903
395014	2.197507	1578	0.944465
395509	395509	1585	0.948029
399499	399499	1654	0.984087
424414	2.212207	1716	0.988982
438166	2.219083	1738	0.984996
462214	2.231107	1742	0.959902
462331	462331	1798	0.990628
493399	493399	1836	0.977552
505411	505411	1866	0.981039
528334	2.264167	1900	0.975891
553414	2.276707	1918	0.961414
576046	2.288023	1920	0.942359
580759	580759	1956	0.955926
583339	583339	1986	0.968330
584791	584791	2016	0.981675
609046	2.304523	2026	0.965705
630919	630919	2052	0.960138
636574	2.318287	2104	0.979866
647854	2.323927	2120	0.978252
675151	675151	2124	0.959086
677791	677791	2156	0.971543
698086	2.349043	2166	0.961047
710854	2.355427	2186	0.960736
749974	2.374987	2270	0.969992
784039	784039	2280	0.951817
784939	784939	2338	0.975443
799621	799621	2383	0.984597
861799	861799	2432	0.966135
909451	909451	2522	0.974006
966211	966211	2526	0.945070
969406	2.484703	2664	0.994978
1018879	1018879	2672	0.972264
1059829	431.2459	2750	0.980193
1089334	2.544667	2778	0.976028
1138999	1138999	2872	0.985756
1174429	251.4679	2882	0.973442
1202191	1202191	2908	0.970277
1214326	2.607163	2922	0.969833
1234531	1234531	3030	0.997025
1336141	1336141	3117	0.984040
1365079	1365079	3196	0.997726
1427911	1427911	3308	1.008648
1526086	2.763043	3310	0.974740
1526431	1526431	3336	0.982280
1627861	1627861	3395	0.966564
1660411	1660411	3526	0.993516
1702639	1702639	3548	0.986668
1834309	1834309	3609	0.965285
1890079	1890079	3732	0.982673
1957099	1957099	3898	1.007854
2139349	139.15391	3900	0.962516
2151451	2151451	3934	0.968049
2185726	2.1092863	4012	0.979122
2237134	2.1118567	4212	1.015519
2494606	2.1247303	4236	0.964806
2519911	2519911	4364	0.988734
2631511	2631511	4480	0.992302
2765239	2765239	4504	0.972127
2797414	2.1398707	4522	0.970134
2810014	2.1405007	4600	0.984556
2847079	2847079	4784	1.016956
3082699	3082699	4938	1.007016
3318214	2.1659107	5030	0.987116
3462229	3462229	5237	1.005206
3573574	2.1786787	5250	0.991198
3574411	3574411	5326	1.005424
3874126	2.1937063	5520	0.999189
4103719	4103719	5652	0.992823
4144534	2.2072267	5730	1.001344
4355311	4355311	5916	1.007453
4760926	2.2380463	5968	0.970215
4836679	4836679	5992	0.966136
4840159	4840159	6000	0.967063
4865974	2.2432987	6018	0.967279
4938124	1234531	6036	0.962760
4960069	2027.2447	6054	0.963402
4966966	2.2483483	6182	0.983059
5046094	2.2523047	6252	0.986036
5280031	5280031	6256	0.963643
5281054	2.2640527	6396	0.985109
5290399	5290399	6472	0.995897
5679886	2.2839943	6604	0.979291
5715319	5715319	6892	1.018692
6308311	6308311	7072	0.992918
6663526	2.3331763	7150	0.975642
6810301	6810301	7413	1.000120
6938779	6938779	7514	1.003932
6980191	6980191	7616	1.014413
7501939	7501939	7630	0.978856
7522519	7522519	7728	0.990016
7764766	2.3882383	7836	0.987431
8038111	8038111	8104	1.002982
8277019	8277019	8134	0.991471
8462854	2.4231427	8146	0.981531
8658589	8658589	8227	0.979569
8797006	2.4398503	8228	0.971640
8845951	8845951	8388	0.987679
9208879	9208879	8432	0.972312
9234619	9234619	8642	0.995082
9674239	9674239	8684	0.976024
9747061	9747061	8801	0.985325
9931639	9931639	8952	0.992501
10073779	10073779	9082	0.999501
10933399	10933399	9376	0.988851
11358559	11358559	9564	0.988874
11366371	11366371	9602	0.992448
11629531	11629531	9710	0.991743
11823991	11823991	9852	0.997611
11971639	11971639	9920	0.998039
12675646	2.6337823	10000	0.976653
12737734	2.6368867	10022	0.976320
12765349	12765349	10023	0.975319
12843814	2.6421907	10442	1.012861
13523134	2.6761567	10484	0.990066
13893751	13893751	10624	0.989294
14024851	14024851	10630	0.985035
14236819	14236819	10738	0.987320
14669251	14669251	10886	0.985492
14768191	14768191	10972	0.989816
14823559	14823559	11120	1.001220
14841766	2.7420883	11226	1.010119
15094291	15094291	11282	1.006302
16122934	2.8061467	11410	0.983467
16178374	2.8089187	11426	0.983092
16656319	16656319	11464	0.971563
16843891	16843891	11846	0.998118
17827891	17827891	11886	0.972400
17844766	2.8922383	11956	0.977647
18191086	2.9095543	12132	0.982190
18293311	18293311	12380	0.999356
18331699	18331699	12446	1.003591
18461899	18461899	12542	1.007624
19049671	19049671	12680	1.002273
19951759	19951759	12720	0.981581

Note that each number D in Table 1 is either q, $2q_1$, or q_1q_2, where q is a prime and q_1, q_2 are primes of the form $4k - 1$. Since h for $Q(\sqrt{D})$ can be 1 only for such values of D, this, in view of (4), is just what we would expect. In order to save computer time, Table 1 was extended up to 5×10^7 by examining only numbers of the above forms. These results are presented in Table 2.

TABLE 2

D	FACTORS	PERIOD	G(D)	D	FACTORS	PERIOD	G(D)
20041891	20041891	12942	0.996378	31277731	31277731	16638	1.016891
20289091	20289091	13358	1.021884	32483851	32483851	16838	1.009129
21548011	21548011	13546	1.004397	34156651	34156651	17134	1.000490
22365379	22365379	13642	0.992165	34225591	34225591	17272	1.007495
22748014	2.11374007	13668	0.985346	37260094	2.18630047	17648	0.985098
22861276	5715319	13800	0.992302	37290031	37290031	17716	0.988482
23196079	23196079	13812	0.985703	37479751	37479751	17752	0.987890
23345326	2.11672663	14348	1.020554	38010751	38010751	17924	0.990219
23817271	23817271	14488	1.019868	38839579	38839579	18218	0.995276
25609159	25609159	14632	0.991974	39803611	39803611	19002	1.025004
25959781	25959781	14771	0.994360	40781911	40781911	19396	1.033181
26030239	26030239	14876	1.000023	43503931	43503931	19510	1.005049
26796526	2.13398263	15104	1.000187	44450701	44450701	19743	1.005774
27512731	27512731	15354	1.002931	44488159	44488159	19868	1.011701
27892771	27892771	15514	1.006200	47710909	47710909	20121	0.988142
28473454	2.14236727	15876	1.018737	48924019	48924019	20398	0.988807
30343366	2.15171683	15894	0.986810	49196359	49196359	20608	0.996119
30345319	30345319	16300	1.011984	49241494	2.24620747	20826	1.006179

In Table 2 no value of D of the form q_1q_2 appears. Also, for each value of D in the table, except 22365379 and 22748014, we have $(D/r_i) = 1$ $(i = 1, 2, 3, \ldots, 10)$, where $r_1 = 3$, $r_2 = 5$, $r_3 = 7, \ldots, r_{10} = 31$, the first ten odd primes. This again is what we would expect. If we examine (3), we see that $L(1, \chi_d)$ will be largest when its leading terms are positive. The larger $L(1, \chi_d)$ is, the larger R and hence p will be. With this in mind, further calculations were performed for values of D $(> 5 \times 10^7)$ of the form q and $2q_1$, such that $(D/r_i) = 1$ $(i = 1, 2, 3, \ldots, 10)$. In Table 3 we present these calculations for $D = q \equiv -1 \pmod 4$, in Table 4 for $D = q \equiv 1 \pmod 4$, and in Table 5 for $D = 2q_1$. Each of these tables was computed for all values of such D up to (and in some cases somewhat beyond) 2×10^9. Again, only the values of D for which $p(D)$ is a local maximum are printed.

TABLE 3

D	PERIOD	G(D)	D	PERIOD	G(D)
51562471	20828	0.982565	82557931	27166	1.004541
52022359	21128	0.992146	83256391	27360	1.007318
52192891	21206	0.994124	83808631	27468	1.007842
52981699	21390	0.994993	85181539	27866	1.013891
54467659	21654	0.992954	88720459	28170	1.003604
54983839	21868	0.997883	93016291	28794	1.001060
55953970	21914	0.990970	96045511	28820	0.985501
56283439	22084	0.995628	96910531	28866	0.982510
59028979	22910	1.007717	97544899	29818	1.011496
59495419	22958	1.005723	101164519	29864	0.994155
62290099	23526	1.006413	102808099	29986	0.989936
63709759	23592	0.997536	103113739	30170	0.994483
64463239	23656	0.994176	104671891	30174	0.986934
65014819	23686	0.991058	105947011	30566	0.993519
65290471	23716	0.990144	106347151	31368	1.017602
67392439	24412	1.002627	108075991	31392	1.009929
71338411	24822	0.989894	109032751	31724	1.015971
73124731	24934	0.981721	113215411	31962	1.003873
73461259	25354	0.995889	116189371	32358	1.002784
74382499	25398	0.991208	117480379	32446	0.999786
75442579	25962	1.005829	117678031	32652	1.005260
76746679	26156	1.004401	119749939	32862	1.002643
77288311	26332	1.007489	121135459	32878	0.997186
78986779	26946	1.019456	121935199	33072	0.999665

TABLE 3 (*continued*)

D	PERIOD	G(D)	D	PERIOD	G(D)
124936639	33164	0.989930	574536691	73682	1.001096
125026771	33710	1.005853	579967411	73830	0.998258
126281731	33782	1.002813	580836811	73910	0.998568
128841619	34970	1.027371	582740911	74036	0.998586
134746819	35042	1.005927	583284199	74936	1.010240
136716259	35098	1.000010	589990699	75230	1.008248
137984851	36150	1.025083	591735379	75566	1.011212
138590299	36578	1.034876	592987651	76362	1.020752
146076571	36734	1.011429	598799419	77330	1.028512
151529431	36968	0.998787	626715391	79300	1.030250
154417519	37244	0.996480	660832351	80504	1.017726
156888559	38116	1.011484	677285359	81984	1.023393
158359639	38468	1.015917	692253871	83164	1.026502
164021971	38502	0.998535	716863291	83206	1.008712
164641591	38708	1.001925	721892011	83238	1.005474
165840931	39010	1.005964	724290499	84690	1.021268
168432091	39122	1.000809	727729531	85154	1.024362
169362691	39342	1.003578	735869731	85806	1.026310
171982651	39518	1.000109	762091639	86948	1.021389
172216951	39584	1.001076	793854931	87454	1.005962
178159591	40700	1.011428	796549471	90020	1.033673
180628639	41828	1.032100	850231099	90782	1.008006
191892919	41932	1.002853	866740351	92248	1.014194
192922339	42290	1.008627	873013051	92678	1.015146
201340099	43662	1.018644	895519531	93278	1.008420
210114139	44018	1.004588	909554671	93900	1.007051
216242959	44156	0.992896	914728291	94570	1.011281
218739511	44880	1.003215	916539199	94720	1.011854
225158179	45338	0.998439	919842751	94980	1.012754
226899271	45340	0.994522	934063519	95656	1.011941
230044291	45818	0.997894	937065691	97742	1.032302
232153531	46366	1.005084	984025771	98738	1.016906
237410419	46838	1.003653	986444911	99560	1.024076
240652591	47248	1.005377	998910091	99822	1.020158
242544691	47638	1.009588	1022553211	100706	1.016877
243959431	48004	1.014297	1061460271	100944	0.999881
254434051	49214	1.017552	1069135279	102760	1.014103
264274399	49396	1.001516	1088391859	105014	1.026871
267143431	49864	1.005389	1140963079	106980	1.021012
270009139	50330	1.009215	1174803919	107140	1.007277
273892459	50434	1.003879	1174868179	107450	1.010163
278314591	50668	1.000238	1201507519	108924	1.012277
279272239	51352	1.011948	1219074739	109562	1.010632
286793959	52440	1.019319	1241094859	110330	1.008387
292011619	52474	1.010538	1250691919	110464	1.005618
306832639	52484	0.985250	1251417619	110814	1.008504
309953851	53138	0.992334	1263816031	110872	1.003927
315210919	54296	1.005202	1272774259	110962	1.001098
321398191	54324	0.995691	1276693531	111646	1.005677
322164571	54834	1.003804	1277026591	112948	1.017269
322503091	56006	1.024704	1310500591	113412	1.007943
330571459	56038	1.012308	1315753399	114804	1.018217
336015859	56670	1.015139	1334575771	116518	1.025896
339840439	56680	1.009410	1337079979	117590	1.034337
340193071	57924	1.031013	1382899519	119212	1.030588
344983291	58118	1.027035	1435746139	120450	1.021399
365093611	59518	1.021496	1464061351	123328	1.035353
384597019	59662	0.996861	1492180699	125154	1.040452
392948131	60190	0.994609	1545871471	125224	1.022281
394935451	62610	1.031910	1596398959	126640	1.016882
416881039	62812	1.006782	1627221751	126684	1.007282
426014131	64322	1.019533	1632003619	127026	1.008478
439466239	65036	1.014465	1639266379	127102	1.006781
447095191	65956	1.019732	1650241399	131476	1.037860
452980999	66020	1.013865	1697725591	132180	1.028309
457940851	66286	1.012253	1721637919	132912	1.026594
463106359	66996	1.017199	1760271451	132950	1.015238
465390979	67458	1.021619	1782611671	133656	1.014032
475477759	68836	1.031034	1830656851	136270	1.019825
505313251	71166	1.033026	1839183571	137830	1.029038
526895191	71660	1.018023	1885728079	137928	1.016624
540112351	72004	1.009936	1903736431	138648	1.016951
556140979	72430	1.000721	1939256251	139890	1.016356
556390279	73052	1.009081	1957723639	142220	1.028262
569657911	73196	0.998871	2041222639	144064	1.019471
570773659	73330	0.999692	2061860971	145542	1.024618

TABLE 4

D	PERIOD	G(D)	D	PERIOD	G(D)
53255029	20899	0.969567	395347261	58949	0.971050
53458981	20993	0.972003	395479309	60929	1.003493
54632341	21189	0.970114	416813461	61583	0.987166
55749541	21513	0.974681	419699149	62197	0.993469
60609949	21689	0.941050	433164229	63909	1.004334
61419829	21699	0.935039	461090029	63961	0.973305
61856701	22607	0.970600	477948949	65951	0.985186
62982781	22953	0.976298	478568869	66437	0.991784
65271229	23241	0.970460	483892501	66911	0.993182
66375709	23307	0.964803	488064901	67321	0.994857
66792301	23335	0.962840	509362309	67583	0.976991
68463781	23581	0.960628	510949429	69069	0.996873
69118261	24323	0.985990	538544029	69695	0.979016
72280261	24789	0.981894	544327501	71319	0.996331
74563141	25181	0.981506	554156989	72057	0.997402
76653229	25249	0.970184	559701949	72295	0.995577
77658109	25527	0.974281	559956541	72571	0.999144
77835949	25683	0.979076	590067349	73087	0.979462
78940261	26411	0.999519	594811309	75897	1.012933
84482941	26637	0.973310	623957629	75969	0.989218
86262829	26973	0.975019	635189461	77911	1.005226
90853669	26997	0.950071	642187309	77957	1.000160
91050829	27369	0.962084	662283301	78025	0.985273
92408941	27971	0.975747	664057549	79429	1.001621
92919061	28871	1.004281	690716581	79605	0.983699
99890389	28965	0.970565	707740069	79833	0.974225
100460221	29243	0.977004	711035389	80255	0.977035
100685341	29685	0.990624	721277701	81365	0.983281
108168061	30147	0.969449	730905421	82761	0.993346
108414541	30343	0.974604	733665949	83019	0.994510
109582069	30411	0.971396	757946029	83511	0.983774
109866901	30631	0.977112	758248261	83551	0.984043
112609429	31035	0.977465	759586909	84313	0.992116
118214581	31143	0.956551	773509669	84533	0.985447
119157061	31155	0.953002	787094389	84639	0.977878
119221909	32253	0.986311	794812789	85099	0.978266
121872661	32535	0.983694	800466781	85523	0.979559
122515429	33365	1.006050	800547829	86895	0.995221
126788869	33465	0.991349	813539269	87149	0.989893
127165789	33609	0.994089	826006861	88871	1.001580
137228701	34235	0.973540	852510541	89801	0.995741
137870749	34435	0.976869	868915909	92353	1.014040
140670469	34969	0.981770	929633461	93741	0.994112
142308709	35009	0.977032	942744709	94199	0.991794
146129701	35093	0.966065	957481981	94893	0.991157
146748781	35245	0.968133	981114709	97069	1.001243
147041101	35767	0.981462	1021948981	98089	0.990754
147280501	35869	0.983435	1027927861	98419	0.991107
150407581	35941	0.974773	1034999221	98891	0.992353
152513461	36581	0.985033	1036649821	100845	1.011131
158992549	36587	0.964253	1053014869	100977	1.004328
160649941	37821	0.991452	1060023589	101223	1.003344
163384621	37945	0.986070	1087703941	101255	0.990437
166840909	38203	0.982101	1116143989	101683	0.981502
170493229	38455	0.977588	1134596941	103337	0.989087
173826181	39081	0.983620	1160379949	105909	1.002053
177739501	39767	0.989448	1186354621	107933	1.009638
183931141	41121	1.005208	1230572821	110741	1.016585
197142709	42161	0.994380	1289956669	111311	0.997344
204895909	42867	0.991101	1301211781	112117	1.000086
212672821	43909	0.995861	1304158909	113671	1.012769
217235461	44627	1.001118	1385195821	113699	0.982090
226175821	45567	1.001150	1397731141	116007	0.997393
236610949	45837	0.983914	1432794661	116593	0.989739
236690749	47297	1.015077	1442507509	117483	0.993835
251342701	47695	0.992384	1492904869	119175	0.990500
262831501	48065	0.977287	1506539941	119421	0.987915
266208541	48395	0.977537	1517705149	119617	0.985786
274963789	51973	1.032431	1541965429	121121	0.990073
303728461	52161	0.984335	1553226229	121357	0.988297
309156901	52843	0.988136	1566687301	125153	1.014698
316660789	53383	0.985964	1660069909	127093	1.000202
321437869	54795	1.004259	1668111901	129747	1.018555
346477069	56947	1.004102	1760056741	130317	0.995195
373293229	57637	0.977951	1769158141	133575	1.017373
380960869	58389	0.980381	1855452421	137415	1.021306
382399021	58919	0.987360	1895437261	140655	1.033991

Table 5

D	PERIOD	G(D)	D	PERIOD	G(D)
51013174	21042	0.998179	328257526	55110	0.999158
52184014	21360	1.001431	335152126	56836	1.019464
52796734	21616	1.007327	335962246	57054	1.022101
54487294	21696	0.994694	352467886	57112	0.998150
55839694	22536	1.020176	354601846	58466	1.018639
59321326	22776	0.999266	371087326	59312	1.009451
60029566	23004	1.003088	379441126	60538	1.018559
63437014	23322	0.988312	398301814	60566	0.993865
63443494	23394	0.991310	402964006	61390	1.001361
65259286	23434	0.978612	405675526	61770	1.004083
66140974	23696	0.982705	418270966	62282	0.996576
67020886	24534	1.010525	420160126	63160	1.008280
69246934	24678	0.999418	420721414	63266	1.009277
71363854	24732	0.986123	425326126	63644	1.009627
72020974	24988	0.991618	439157086	65900	1.028316
72663886	25408	1.003660	468262414	67484	1.018779
74409934	25452	0.993126	487067494	69342	1.025804
75619294	26068	1.008715	515758006	69642	1.000304
79420366	26160	0.986921	517107046	70094	1.005442
81829654	26894	0.999051	522478174	71128	1.014856
85433734	27134	0.985750	535409326	71964	1.013933
87163246	27276	0.980692	544115014	72370	1.011216
87322174	27732	0.996149	548033014	73510	1.023355
88986934	28178	1.002334	575251966	73748	1.001350
91358446	28728	1.008096	575903134	73808	1.001581
98073166	28816	0.974781	600950494	74700	0.991699
98205286	29014	0.980795	616398766	75376	0.987677
99553294	29636	0.994786	617808766	75588	0.989290
100993006	29932	0.997293	620858254	75872	0.990492
101044414	30168	1.004891	626545726	75992	0.987410
103345246	30720	1.011438	633787774	76448	0.987473
111039646	30724	0.974715	637951246	77776	1.001245
111669286	31026	0.981424	643888774	79650	1.020489
112283614	31424	0.991199	671879014	81738	1.024541
114517846	32018	0.999705	698613094	82862	1.017970
115036366	32688	1.018245	726654646	84282	1.014644
119255614	32880	1.005338	728538574	85520	1.028176
123753694	33280	0.998287	738915046	86494	1.032342
128303926	35178	1.035717	788809774	87724	1.012384
136119526	35282	1.007527	796562974	87976	1.010194
138592414	36248	1.025532	803297686	89058	1.018195
152580334	36432	0.980799	842788774	92518	1.031941
153233326	38016	1.021187	860788366	92896	1.024948
158728894	38072	1.004251	893423686	93050	1.007170
161158726	38738	1.013833	907162246	93406	1.003113
169739014	39054	0.995091	908397814	95538	1.025290
171596734	39268	0.994936	910511926	95930	1.028266
172800406	39886	1.006953	946711054	96308	1.011810
173690806	40214	1.012544	970397926	97786	1.014356
177566614	40242	1.001768	989718766	98004	1.006354
177878614	40298	1.002254	996328414	99476	1.017977
182871406	41276	1.012012	998660854	99522	1.017222
183346606	41344	1.012321	1022146486	100726	1.017287
191083894	41378	0.991765	1025302534	101278	1.021241
191161006	41800	1.001671	1071565174	101650	1.001977
196017886	42420	1.003448	1085931526	102774	1.006139
202411966	43648	1.015531	1089373366	102966	1.006378
215606374	44006	0.991029	1105475086	104636	1.015009
216246094	44548	1.001702	1138979326	105800	1.010654
221844814	44564	0.988931	1143452326	106130	1.011765
225900574	44788	0.984653	1192002694	108390	1.011438
228599134	46064	1.006518	1225662694	108490	0.997973
231643294	46356	1.006008	1239568366	109920	1.005276
239011774	46800	0.999366	1290271966	110692	0.991673
239925214	47196	1.005840	1292542486	110886	0.992513
240909694	47660	1.013585	1296454366	112308	1.003679
246221446	47690	1.002874	1312891486	113388	1.006785
250433206	47846	0.997390	1341438814	113456	0.996305
256060366	48288	0.995130	1350442846	114320	1.000443
260465734	49366	1.008432	1363092886	115698	1.007658
263286046	50348	1.022794	1365708094	116476	1.013434
283538854	51194	1.000975	1402581574	116662	1.001239
287315326	51288	0.995992	1406521726	117804	1.009583
297273454	51484	0.982383	1424254654	118260	1.006981
299553094	52702	1.001669	1440761326	118464	1.002757
305730814	54264	1.020556	1441607974	119722	1.013100
319601494	54342	0.998904	1456962406	119914	1.009211
319852894	54800	1.006914	1467125566	120360	1.009350

TABLE 5 (*continued*)

D	PERIOD	G(D)	D	PERIOD	G(D)
1475776054	121126	1.012707	1687335094	132490	1.033978
1524382414	122088	1.003881	1798654894	132856	1.003331
1537069174	122206	1.000577	1799775286	133022	1.004263
1545714934	124850	1.019280	1816733494	136858	1.028254
1546343254	125110	1.021190	1896380326	137390	1.009731
1569268726	125946	1.020264	1915369054	140016	1.023773
1595259766	127426	1.023570	2015561326	140576	1.001278
1678290574	129424	1.012846	2024387374	143612	1.020608

Note that in all of these tables (6) is satisfied.

If we compare our function $G(D)$ to Shanks' [6] upper Littlewood index ULI, we see that we would have $G(D) = K_2 \cdot \mathrm{ULI}/h$ for almost all D. Shanks suggests that the ULI can be expected to slowly increase (on the average) up to an average value of about 3/4. This average value may then begin to deteriorate. He also gives values of D ($> 10^{10}$) such that the ULI exceeds .7 and becomes as large as .713. It seems from this that we could expect to have many values of D outside of the range of these tables for which $G(D)$ is as large as $3/4\,(1.501) = 1.126$.

The largest value of $G(D)$ in our tables here occurs for $D_1 = 1492180699$, where $G(D_1) = 1.040452$. It is interesting to note that for $D_2 = 26437680473689$ we have $G(D_2) = 1.039159$. This latter D value, however, is a quadratic residue of all primes less than 151 and therefore we would expect it to have a large value for its L function. (In fact, $L(1, \chi_{d_2}) = 8.455394$; whereas, $L(1, \chi_{d_1}) = 3.831228$.) The reason that D_1 is stronger than D_2 here is that $D_2 \equiv 1 \pmod 8$ is a pseudosquare and $D_1 \equiv 3 \pmod 8$ is not. In [6, p. 275 and Figure 1] Shanks points out why the ULI for the pseudosquares tends to lag behind those of numbers like D_1. (This tendency will diminish as the values of D become larger.) From this reasoning we would not anticipate finding many values of $D \equiv 1 \pmod 8$ in the above tables and, in fact, none occurs.

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