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Some Results Concerning Voronoi's Continued Fraction Over $\mathcal{Q}(\sqrt[3]{D})$

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Abstract. Let D be a cube-free integer and let ε_0 be the fundamental unit of the pure cubic field $\mathcal{Q}(\sqrt[3]{D})$. It is well known that Voronoi's algorithm can be used to determine ε_0. In this work several results concerning Voronoi's algorithm in $\mathcal{Q}(\sqrt[3]{D})$ are derived and it is shown how these results can be used to increase the speed of calculating ε_0 for many values of D. Among these D values are those such that D (> 3) is not a prime $\equiv 8 \pmod 9$ and the class number of $\mathcal{Q}(\sqrt[3]{D})$ is not divisible by 3. A frequency table of all class numbers not divisible by 3 for all $\mathcal{Q}(\sqrt[3]{D})$ with $D < 2 \times 10^5$ is also presented.

1. Introduction. It is well known (see, for example, Perron [9]) that if ϕ is a given real number and if we define $\phi_0 = \phi$, $q_0 = [\phi_0]$,* $\phi_{n+1} = (\phi_n - q_n)^{-1}$, $q_{n-1} = [\phi_{n+1}]$ $(n = 0, 1, 2, 3, \dots)$, then

$$\phi = q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{q_3 \; \ddots \; + \cfrac{1}{q_{n-1} + \cfrac{1}{\phi_n}}}}}$$

is the continued fraction expansion of ϕ. We denote this by the less cumbersome

$$\phi = \langle q_0, q_1, q_2, \dots, q_{n-1}, \phi_n \rangle.$$

If $A_{-1} = B_{-2} = 1$, $A_{-2} = B_{-1} = 0$ and $A_{r+1} = q_{r+1}A_r + A_{r-1}$, $B_{r+1} = q_{r+1}B_r + B_{r-1}$ $(r = -1, 0, 1, 2, 3, \dots)$, we have

$$\frac{A_n}{B_n} = \langle q_0, q_1, q_2, \dots, q_n \rangle.$$

Let d be a square free positive integer and let $\phi = \sqrt{d}$. In this case we have

$$\phi_n = \frac{P_n + \sqrt{d}}{Q_n},$$

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*We use $[\alpha]$ to denote that integer such that $\alpha - 1 < [\alpha] < \alpha$.

where

$$Q_0 = 1, \quad P_0 = 0, \quad q_0 = [\sqrt{d}\,] \quad \text{and}$$

$$P_{r+1} = q_r Q_r - P_r, \qquad Q_{r+1} = (d - P_{r+1}^2)/Q_r \qquad (r = 0, 1, 2, 3, \dots).$$

Let $\mathfrak{Q}(\sqrt{d}\,)$ be the quadratic field formed by adjoining $\sqrt{d}$ to the rationals $\mathfrak{Q}$. If $N(\alpha)$ is the norm of $\alpha \in \mathfrak{Q}(\sqrt{d}\,)$ ($N(\alpha) = \alpha\bar{\alpha}$, where $\bar{\alpha}$ is the conjugate of α), then

$$N(A_n + \sqrt{d}\, B_n) = A_n^2 - dB_n^2 = (-1)^{n+1} Q_{n+1}.$$

Also, there always exists a least nonnegative integer s such that $Q_{s+1} = 1$. If η_0 (> 1) is the fundamental unit of $\mathfrak{Q}(\sqrt{d}\,)$, then

$$\eta_0 = A_s + \sqrt{d}\, B_s \quad \text{or} \quad \eta_0^3 = A_s + \sqrt{d}\, B_s.$$

The latter case occurs only when $d > 5$, $d \equiv 5 \pmod 8$ and $Q_{k+1} = 4$ for some $k < s/2$. In this event

$$\eta_0 = (A_r + \sqrt{d}\, B_r)/2, \tag{1.1}$$

where r is the least positive integer such that $Q_{r+1} = 4$.

When $N(\eta) = +1$ (and no r as defined above exists), it is known that $P_j = P_{j+1}$ for a minimal $j \geq 1$. In this case we have $s = 2j$ and

$$\eta_0 = (A_{j-1} + \sqrt{d}\, B_{j-1})^2 / Q_j \tag{1.2}$$

(see Williams and Broere [12]). Thus, in order to determine η_0 it is never necessary to go beyond $q_{s/2}$ in the determination of the continued fraction expansion of $\sqrt{d}$. Note that $Q_j \mid 2P_j$ and therefore [9, p. 107] $Q_j \mid 2d$. In fact, Q_j is a principal factor of the discriminant of $\mathfrak{Q}[\sqrt{d}\,]$ (see Barrucand and Cohn [1]).

In this paper we consider the pure cubic field $\mathfrak{Q}(\sqrt[3]{D}\,)$, where $D = ab^2$ is cube free and a, b are coprime integers. If $\theta \in \mathfrak{Q}(\sqrt[3]{D}\,)$, then $\theta = (c_1 + c_2\delta + c_3\bar{\delta})/c_4$, where $\delta^3 = ab^2$, $\bar{\delta}^3 = a^2b$, $c_1, c_2, c_3, c_4 \in \mathfrak{Z}$ (the set of rational integers). Also, we define the norm of θ (written $N(\theta)$) to be $N(\theta) = \theta\theta'\theta''$, where

$$\theta' = (c_1 + c_2\omega\delta + c_3\omega^2\bar{\delta})/c_4, \qquad \theta'' = (c_1 + c_2\omega^2\delta + c_3\omega\bar{\delta})/c_4,$$

and ω is a primitive cube root of unity, i.e., an arbitrary but fixed zero of $x^2 + x + 1$.

Let ε_0 (> 1) be the fundamental unit of $\mathfrak{Q}(\delta)$. The usual continued fraction algorithm described above is not very useful for determining ε_0. (It can be used to find ε_0, however, when any nontrivial unit is known; see Jeans and Hendy [8]). In 1896 Voronoi [11] described an extension of the continued fraction algorithm which can be used to find ε_0. A version of this algorithm is described in detail in Williams, Cormack, and Seah [13]. In this paper we extend the earlier work of Williams [14] by developing some further results concerning Voronoi's continued fraction algorithm which are analogous to the results (1.1) and (1.2) above. It will be seen that these developments allow us to increase the speed of calculating ε_0 for many values of D and certainly for those values of D (> 3) such that D is not a prime $\equiv 8 \pmod 9$ and the class number of $\mathfrak{Q}(\sqrt[3]{D}\,)$ is not divisible by 3. We also present a frequency table of all class numbers not divisible by three for all $\mathfrak{Q}(\sqrt[3]{D}\,)$ such that $D < 2 \times 10^5$.

2. Preliminary Results Concerning $\mathfrak{Q}(\delta)$. We first require some well-known results on pure cubic fields.

If $D \not\equiv \pm 1 \pmod 9$, then $[1, \delta, \bar{\delta}]$ is a *basis* of the ring of integers $\mathfrak{Q}[\delta]$ of $\mathfrak{Q}(\delta)$, and the discriminant Δ of $\mathfrak{Q}(\delta)$ is $-27a^2b^2$. If $D \equiv \pm 1 \pmod 9$, then $[1, \delta, (1 + a\delta + b\bar{\delta})/3]$ is a basis of $\mathfrak{Q}[\delta]$ and $\Delta = -3a^2b^2$. Thus, if x_1, x_2, x_3, $\sigma \in \mathfrak{Z}$, g.c.d.$(x_1, x_2, x_3, \sigma) = 1$ and $\alpha = (x_1 + x_2\delta + x_3\bar{\delta})/\sigma \in \mathfrak{Q}[\delta]$, then $\sigma = 1$ when $D \not\equiv \pm 1 \pmod 9$ and $\sigma = 3$, $x_1 \equiv ax_2 \equiv bx_3 \pmod 3$ when $D \equiv \pm 1 \pmod 9$. Further, $N(\alpha) \in \mathfrak{Z}$ and

$$\sigma^3 N(\alpha) = x_1^3 + ab^2x_2^3 + a^2bx_3^3 - 3abx_1x_2x_3. \tag{2.1}$$

If $\varepsilon \in \mathfrak{Q}[\delta]$ and $N(\varepsilon) = 1$, we say that ε is a unit of $\mathfrak{Q}(\delta)$. Further, $\varepsilon = \pm\varepsilon_0^n$, where $n \in \mathfrak{Z}$. If $3 \mid D$, put $S = |\Delta|/27$; otherwise put $S = |\Delta|/3$. S is simply the square of the product of all primes $p \in \mathfrak{Z}$ such that the principal ideal $[p] = P^3$, where P is a prime ideal of $\mathfrak{Q}[\delta]$ and the norm of P, $N(P)$, is p. It should also be noted that if $3 \nmid S$, then $[3] = PQ^2$, where P, Q are distinct prime ideals of $\mathfrak{Q}[\delta]$ and $N(P) = N(Q) = 3$.

We now present three simple lemmas which will be needed in the work that follows

LEMMA 2.1. *Let* $\alpha = x_1 + x_2\delta + x_3\bar{\delta}$, *where* $x_1, x_2, x_3 \in \mathfrak{Z}$.

(a) *If* $3 \nmid D$, *then* $3 \mid N(\alpha)$ *if and only if* $x_1 + ax_2 + bx_3 \equiv 0 \pmod 3$.

(b) *If* $3 \nmid D$ *and* $D \not\equiv \pm 1 \pmod 9$, *then* $9 \mid N(\alpha)$ *if and only if* $x_1 \equiv ax_2 \equiv bx_3 \pmod 3$.

(c) *If* $D \equiv \pm 1 \pmod 9$, *then* $3 \mid \alpha$ *if and only if* $x_1 \equiv ax_2 \equiv bx_3 \pmod 3$; *also, if* $27 \mid N(\alpha)$, *then* $\alpha'\alpha''/3 \in \mathfrak{Q}[\delta]$.

Proof. The first of (a) follows easily from (2.1) with $\sigma = 1$. To prove (b) we first note that $\alpha'\alpha'' \in \mathfrak{Q}[\delta]$ and

$$\alpha'\alpha'' = (x_1^2 - abx_2x_3) + (ax_3^2 - x_1x_2)\delta + (bx_2^2 - x_1x_3)\bar{\delta} \equiv \alpha^2 \pmod 3.$$

Since $[3] = P^3$, we have $\alpha\alpha'\alpha'' \equiv 0 \pmod{P^6}$, and $\alpha'\alpha'' \equiv \alpha^2 \pmod{P^3}$; thus, $\alpha \equiv 0 \pmod{P^2}$ and $3 \mid \alpha'\alpha''$. It follows that $3 \mid x_1^2 - abx_1x_3$, $3 \mid ax_3^2 - x_1x_2$, $3 \mid bx_2^2 - x_1x_3$, and therefore $x_1 \equiv ax_1 \equiv bx_3 \pmod 3$.

The proof of the first part of (c) follows easily from our previous remarks concerning the integers in $\mathfrak{Q}[\delta]$. To prove the second part, we note that $[3] = PQ^2$, $\alpha\alpha'\alpha'' \equiv 0 \pmod{P^3Q^6}$, $\alpha'\alpha'' \equiv \alpha^2 \pmod{PQ^2}$. Thus, $\alpha'\alpha'' \equiv 0 \pmod{PQ^2}$ and $\alpha'\alpha''/3 \in \mathfrak{Q}[\delta]$. □

LEMMA 2.2. *If* $\alpha = (x_1 + x_2\delta + x_3\bar{\delta})/\sigma \in \mathfrak{Q}[\delta]$, $t^3 \mid N(\alpha)$, *and* $t \mid S$, *then* g.c.d.$(x_1, x_2, x_3) \equiv 0 \pmod t$.

Proof. See Lemma 2 of [14].

LEMMA 2.3. *Let* $\alpha \in \mathfrak{Q}[\delta]$ *and let* $t = ef^2$, *where* $e = e_1e_2$, $f = f_1f_2$, *and* $e_1f_1 \mid a$, $e_2f_2 \mid b$. *If* $t \mid N(\alpha)$, *then*

$$\frac{\delta\alpha}{e_2 f}, \frac{\bar{\delta}\alpha}{e_1 f} \in \mathfrak{Q}[\delta].$$

Proof. From (2.1) and the fact that $ef \mid ab$, we have $f \mid x_1$, $e \mid x_1$, $f_1 \mid x_2$, $f_2 \mid x_3$; thus

$$\sigma\delta\alpha = x_1\delta + x_2 b\bar{\delta} + x_3 ab \equiv 0 \pmod{e_2 f},$$

$$\sigma\bar{\delta}\alpha = x_1\bar{\delta} + x_2 ab + x_3 a\delta \equiv 0 \pmod{e_1 f}.$$

Since $(\sigma, ab) = 1$, the lemma follows. □

It is easy to see that if $\alpha \in \mathfrak{Q}[\delta]$ and $N(\alpha) \mid S$, then $\alpha^3/N(\alpha) \in \mathfrak{Q}[\delta]$. If such an α exists and $|N(\alpha)| \neq 1$, ab^2, a^2b, we say that $|N(\alpha)|$ is a *principal factor* of the discriminant Δ; cf. [1]. Since $N(\alpha^3/N(\alpha)) = 1$, we see that the determination of a unit of $\mathfrak{Q}(\delta)$ is a simple matter when such an α is known. Several results concerning the existence of these principal factors can be found in Barrucand and Cohn [1], [2], Brunotte, Klingen, and Steurich [4], and Halter-Koch [6].

LEMMA 2.4. *Let $\alpha \in \mathfrak{Q}[\delta]$ and suppose $N(\alpha) \mid S$. Put $N(\alpha) = 3^\tau d_1 d_4 d_2^2 d_5^2$, where $a = d_1 d_2 d_3$, $b = d_4 d_5 d_6$. If*

$$\lambda^3 = 3^\tau \min\{d_1 d_4 d_2^2 d_5^2,\ d_3 d_5 d_1^2 d_6^2,\ d_2 d_6 d_4^2 d_3^2\},$$

then $\beta = \lambda\alpha/N(\alpha)^{1/3} \in \mathfrak{Q}[\delta]$. Further, $N(\beta) \mid S$ and, if we put $N(\beta) = 3^\tau mn^2$, where $m = m_1 m_2$, $n = n_1 n_2$, $m_1 n_1 \mid a$, $m_2 n_2 \mid b$, then

$$\frac{\delta}{m_2 n},\ \frac{\bar{\delta}}{m_1 n} > 1.$$

Proof. Let $\kappa_1 = d_1 d_3 d_6^2/d_4 d_5 d_2^2 = (\delta/d_2 d_4 d_5)^3$, $\kappa_2 = d_3^2 d_5 d_6/d_1 d_2 d_5^2 = (\bar{\delta}/d_1 d_2 d_5)^3$. We have $N(\beta) = \lambda^3$ and $\lambda^3/N(\alpha) = \min\{1, \kappa_1, \kappa_2\}$; hence, by Lemma 2.3, we have $\beta \in \mathfrak{Q}[\delta]$. Also, $\{(\delta/m_2 n)^3, (\bar{\delta}/m_1 n)^3\}$ is one of $\{\kappa_1, \kappa_2\}$, $\{\kappa_2\kappa_1^{-1}, \kappa_1^{-1}\}$, $\{\kappa_2^{-1}, \kappa_1\kappa_2^{-1}\}$. If $\lambda^3/N(\alpha) = 1$, then $(\delta/m_2 n)^3 = \kappa_1 > 1$ and $(\bar{\delta}/m_1 n)^3 = \kappa_2 > 1$; if $\lambda^3/N(\alpha) = \kappa_1$, then $(\delta/m_2 n)^3 = \kappa_2\kappa_1^{-1} > 1$ and $(\bar{\delta}/m_1 n)^3 = \kappa_1^{-1} > 1$; if $\lambda^3/N(\alpha) = \kappa_2$, then $(\delta/m_2 n)^3 = \kappa_2^{-1} > 1$ and $(\bar{\delta}/m_1 n)^3 = \kappa_1\kappa_2^{-1} > 1$. □

Thus, if we can find $\alpha \in \mathfrak{Q}[\delta]$ such that $N(\alpha) \mid S$, we can easily find $\beta \in \mathfrak{Q}[\delta]$ such that $N(\beta) \mid S$, $N(\beta) = 3^\tau mn^2$, $m = m_1 m_2$, $n = n_1 n_2$, $m_1 n_1 \mid a$, $m_2 n_2 \mid b$, and $\delta/m_2 n$, $\bar{\delta}/m_1 n > 1$. Since $\mathfrak{Q}(\sqrt[3]{ab^2}) = \mathfrak{Q}(\sqrt[3]{a^2 b})$, we can assume without loss of generality that a and b are such that

$$\gamma_1 = \delta/m_2 n < \gamma_2 = \bar{\delta}/m_1 n.$$

For, if this is not the case, we can simply interchange the values of a and b, m_1 and m_2, and n_1 and n_2.

We conclude this section by pointing out that if $\alpha \in \mathfrak{Q}[\delta]$, $d = N(\alpha) \mid S$ and $N(\alpha) = 3^\tau d_1 d_2^2 d_4 d_5^2$, where $d_1, d_2, d_3, d_4, d_5, d_6$ are defined as above, then, by Lemmas 2.2 and 2.3, the six numbers

$$\alpha,\quad \frac{\delta\alpha}{d_2 d_4 d_5},\quad \frac{\bar{\delta}\alpha}{d_1 d_2 d_5},\quad \frac{\alpha^2}{d_2 d_5},\quad \frac{\delta\alpha^2}{d_2 d_5^2 d_1 d_4},\quad \frac{\bar{\delta}\alpha^2}{d_2^2 d_5 d_1 d_4}$$

are all in $\mathfrak{Q}[\delta]$ and their norms all divide S; thus, as noted in [1], each of the elements of the set

(2.2) $\{3^\tau d_1 d_2^2 d_4 d_5^2,\ 3^\tau d_1^2 d_3 d_5 d_6^2,\ 3^\tau d_2 d_3^2 d_4^2 d_6,\ 3^\nu d_1^2 d_2 d_4^2 d_5,\ 3^\nu d_2^2 d_3 d_4 d_6^2,\ 3^\nu d_1 d_3^2 d_5^2 d_6\},$

where

$$\nu = \begin{cases} 0 & \text{when } \tau = 0, \\ 1 & \phantom{\text{when }} \tau = 2, \\ 2 & \phantom{\text{when }} \tau = 1, \end{cases}$$

is a principal factor whenever d is. We call this set a *principal factor set*. If t is the number of distinct prime factors of S, there are $(3^t - 3)/6$ sets of the form (2.2) but there can be at most one principal factor set; see [1].

3. Relative Minima. We first summarize some of the basic ideas concerning relative minima over a pure cubic lattice. For a more detailed discussion of these ideas see [13], [11], Delone and Faddeev [5] and Steiner [10].

Let $\alpha = \mathfrak{Q}(\delta)$ and consider the ordered triple

$$A = \left(\alpha, \frac{\alpha' - \alpha''}{2i}, \frac{\alpha' + \alpha''}{2}\right),$$

where $i^2 = -1$. Since A is uniquely determined once α is known, we often identify A with α and write $A \approx \alpha$ or $\alpha \approx A$ where the lower-case letter refers to the element of $\mathfrak{Q}(\delta)$ and the upper-case letter to the corresponding ordered triple. Let $\mu, \nu \in \mathfrak{Q}(\delta)$ and let

$$\mathfrak{R} = \{A \mid A \approx x + y\mu + z\nu, x, y, z \in \mathfrak{Z}\}.$$

We say that $\mathfrak{R}$ is a lattice with basis $[1, \mu, \nu]$.

We say that $\Theta \approx \theta \in \mathfrak{Q}(\delta)$ is a *relative minimum* of $\mathfrak{R}$ if $\Theta \in \mathfrak{R}$ and there does not exist $\Phi\ (\neq (0, 0, 0)) \in \mathfrak{R}$ such that $|\phi| < |\theta|$ and $\phi'\phi'' < \theta'\theta''$. If Θ and Φ are relative minima of $\mathfrak{R}$ with $\theta > \phi$, we say they are *adjacent* relative minima of $\mathfrak{R}$ when there does not exist $\Psi\ (\neq (0, 0, 0)) \in \mathfrak{R}$ such that $|\psi| < |\theta|$ and $\psi'\psi'' < \phi'\phi''$. If $\theta_i \approx \Theta_i \in \mathfrak{R}$ $(i = 1, 2, 3, \ldots, n, \ldots)$, $\theta_{i+1} > \theta_i$, and Θ_i, Θ_{i+1} are adjacent relative minima, we call the sequence

$$\Theta_1, \Theta_2, \Theta_3, \ldots, \Theta_n, \ldots \tag{3.1}$$

a *chain* of relative minima. If Θ_i precedes Θ_j in such a chain we say that Θ_i is less than Θ_j. It is easy to see that if Φ is any relative minimum of $\mathfrak{R}$ and $\phi > \theta_1$, then $\Phi = \Theta_k$ for some k.

In [11] Voronoi presented a method of finding a chain of relative minima when $\Theta_1 = (1, 0, 1)$ is a relative minimum of $\mathfrak{R}$. This technique is simply a means of finding in any such lattice a relative minimum Θ_g adjacent to (1, 0, 1). Here we shall concern ourselves with finding $\Theta_g \approx \theta_g$ such that $\theta_g > 1$. Let $\mathfrak{R}_1 = \mathfrak{R}$ and let $\Theta_g^{(1)} \approx \theta_g^{(1)}$ be the relative minimum adjacent to (1, 0, 1) in $\mathfrak{R}_1$ with $\theta_g^{(1)} > 1$. Embed $1, \theta_g^{(1)}$ in a basis of $\mathfrak{R}_1$ and let this basis be $[1, \theta_g^{(1)}, \theta_h^{(1)}]$. Let $\mathfrak{R}_2$ have basis $[1, 1/\theta_g^{(1)}, \theta_h^{(1)}/\theta_g^{(1)}]$. We see that (1, 0, 1) is a relative minimum of $\mathfrak{R}_2$ and find the relative minimum $\Theta_g^{(2)} \approx \theta_g^{(2)} > 1$ adjacent to (1, 0, 1) in $\mathfrak{R}_2$. We continue this process by defining $\mathfrak{R}_{i+1}$ to be the lattice with basis $[1, 1/\theta_g^{(i)}, \theta_h^{(i)}/\theta_g^{(i)}]$, where $\Theta_g^{(i)} \approx \theta_g^{(i)} > 1$ is the relative minimum adjacent to (1, 0, 1) in $\mathfrak{R}_i$ and $[1, \theta_g^{(i)}, \theta_h^{(i)}]$ is a basis of $\mathfrak{R}_i$. It follows that $\Theta_n \approx \theta_n$, where

$$\theta_n = \prod_{i=1}^{n-1} \theta_g^{(i)}, \qquad \theta_g^{(r)} = \left(m_1 + m_2\delta + m_3\bar{\delta}\right)/\sigma_r,$$

$$\theta_h^{(r)} = \left(n_1 + n_2\delta + n_3\bar{\delta}\right)/\sigma_r,$$

$m_1, m_2, m_3, n_1, n_2, n_3, \sigma_r \in \mathfrak{Z}$, $\sigma_r > 0$ and g.c.d.$(\sigma_r, m_1, m_2, m_3, n_1, n_2, n_3) = 1$.

From now on we shall assume that $\mathfrak{R}_1$ is the lattice with basis $[1, \mu, \nu]$, where $[1, \mu, \nu]$ is an integral basis of $\mathfrak{Q}[\delta]$. We note that (1, 0, 1) is a relative minimum of $\mathfrak{R}$ and so is $E \approx \varepsilon$, where ε is any unit of $\mathfrak{Q}(\delta)$. Thus, since this algorithm gives us a method of finding all relative minima Θ such that $\theta > 1$, it can be used to find ε_0. If we put $e_r = m_2n_3 - n_2m_3$, by Theorem 3.1 of [13], we have $N(\theta_r) = \sigma_r^2/|e_r|\sigma$. Thus, if r (> 1) is the least integer such that $\sigma_r^2 = |e_r|\sigma$, then $\varepsilon_0 = \theta_r$.

When $D \equiv \pm 1 \pmod 9$, let $\overline{\mathfrak{R}}_1$ be the lattice with basis $[1, \delta, \bar{\delta}]$. If $E_0 \approx \varepsilon_0$ and $E_0 \in \overline{\mathfrak{R}}_1$, then E_0 is certainly a relative minimum of $\overline{\mathfrak{R}}_1$. If, however, $E_0 \notin \overline{\mathfrak{R}}_1$, then $K_0 \approx \kappa_0 = 3\varepsilon_0$ is always in $\overline{\mathfrak{R}}_1$, but it is not clear whether or not K_0 is a relative minimum of $\overline{\mathfrak{R}}_1$. In fact, if $D = 44$, then $\varepsilon_0 = (4007 + 1135\delta + 643\bar{\delta})/3$ and $K_0 \approx \kappa_0 = 4007 + 1135\delta + 643\bar{\delta}$ is not a relative minimum of $\overline{\mathfrak{R}}_1$. In Theorem 3.1 we show that $D = 44$ is the largest D value with $a > b$ such that K_0 is not a relative minimum of $\overline{\mathfrak{R}}_1$.

THEOREM 3.1. *If $D \equiv \pm 1 \pmod 9$, $\theta/3 \in \mathfrak{Q}[\delta]$ and $N(\theta) = 27$, then Θ ($\approx \theta$) is a relative minimum of $\overline{\mathfrak{R}}_1$ whenever $a > b$ and $D > 44$.*

Proof. If Θ is not a relative minimum of $\overline{\mathfrak{R}}_1$, then there must exist $\gamma \approx \Gamma \in \overline{\mathfrak{R}}_1$ such that $0 < \gamma < \theta$ and $\gamma'\gamma'' < \theta'\theta''$. Since $\theta/3 \in \mathfrak{Q}[\delta]$, we have $\theta'\theta''/9 \in \mathfrak{Q}[\delta]$; therefore, if $\rho = \theta'\theta''\gamma/3 = 9\gamma/\theta$, then $\rho = x_1 + x_2\delta + x_3\bar{\delta}$, where $x_1, x_2, x_3 \in \mathfrak{Z}$. Also, since $\rho \in \mathfrak{Q}[\delta]$ and $3 \mid \rho$, we have $x_1 \equiv ax_2 \equiv bx_3 \pmod 3$ by Lemma 2.1. Since $|\rho| < 9$ and $|\rho'| = |\rho''| < 9$, we have $|x_1| < 9$, $\delta|x_2| < 9$, $\bar{\delta}|x_3| < 9$. It follows that if $\bar{\delta} > 9$ and $\delta > 3$, then $x_2 = x_3 = 0$ and $x_1\theta = 9\gamma$; that is, $9 \mid x_1$ and therefore $x_1 = 0$ and Θ is a relative minimum of $\overline{\mathfrak{R}}_1$. This will certainly be the case when $b > 9$; thus, there are only four possible values for b such that Θ might not be a relative minimum of $\overline{\mathfrak{R}}_1$. These are 1, 2, 5, 7. If $b = 1$, then $D > 44$ means that $a > 44$, $\delta > 3$ and $\bar{\delta} > 9$. If $b = 2$ and $a > 11$, then, since $a \equiv \pm 2 \pmod 9$, we must have $a \geqslant 29$; if $b = 5$, then $a \equiv \pm 4 \pmod 9$ and $a > 14$; if $b = 7$, then $a \equiv \pm 2 \pmod 9$ and $a \geqslant 11$. In all of these cases we see that $\bar{\delta} > 9$ and $\delta > 3$, and the theorem now follows. □

If $D \equiv \pm 1 \pmod 9$, Θ ($\approx \theta$) is a relative minimum of $\overline{\mathfrak{R}}_1$ and $N(\theta) = 27$, it does not necessarily follow that $\theta/3 \in \mathfrak{Q}[\delta]$. For example, when $D = 62$, and $\theta = 15 + 4\delta + \bar{\delta}$, we have Θ ($\approx \theta$) a relative minimum of $\overline{\mathfrak{R}}_1$ and $\theta/3 \notin \mathfrak{Q}[\delta]$. There is, however, a simple method of determining when a given Θ_r ($\approx \theta_r$) in the chain (3.1) of $\overline{\mathfrak{R}}_1$ is such that $\theta_r/3 \in \mathfrak{Q}[\delta]$. We give this as

THEOREM 3.2. *If Θ_r ($\approx \theta_r$) is in the chain (3.1) of relative minima of $\overline{\mathfrak{R}}_1$ with $\Theta_1 = (1, 0, 1)$ and $27 \mid N(\theta)$, then $\theta_r/3 \in \mathfrak{Q}[\delta]$ if and only if $3 \mid (\sigma_r/|e_r|)$.*

Proof. The proof of this result makes use of the methods of Theorem 3.1 of [13]. Let $\gamma = \theta_r'\theta_r'' = g_1 + g_2\delta + g_3\bar{\delta}$ and let $[1, \mu_r, \nu_r]$ be a basis of $\overline{\mathfrak{R}}_r$. We have

$$\begin{pmatrix} 1 \\ \mu_r \\ \nu_r \end{pmatrix} = \frac{1}{\theta_r} T \begin{pmatrix} 1 \\ \delta \\ \bar{\delta} \end{pmatrix},$$

where T is a 3×3 matrix $(t_{ij})_{3\times 3}$ with $t_{ij} \in \mathfrak{Z}$ and $|T| = \pm 1$. Thus,

$$\theta_r = t_{11} + t_{12}\delta + t_{13}\bar{\delta}, \qquad \theta_r\mu_r = t_{21} + t_{22}\delta + t_{23}\bar{\delta},$$

$$\theta_r\nu_r = t_{31} + t_{32}\delta + t_{33}\bar{\delta},$$

and

$$N(\theta_r) = \left(g_1 + g_2\delta + g_3\bar{\delta}\right)\left(t_{11} + t_{12}\delta + t_{13}\bar{\delta}\right),$$
$$N(\theta_r)\mu_r = \left(g_1 + g_2\delta + g_3\bar{\delta}\right)\left(t_{21} + t_{22}\delta + t_{23}\bar{\delta}\right) = u_1 + u_2\delta + u_3\bar{\delta},$$
$$N(\theta_r)\nu_r = \left(g_1 + g_2\delta + g_3\bar{\delta}\right)\left(t_{31} + t_{32}\delta + t_{33}\bar{\delta}\right) = v_1 + v_2\delta + v_3\bar{\delta}.$$

This can be written as

$$\begin{bmatrix} 27k & 0 & 0 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} = T \begin{bmatrix} g_1 & g_2 & g_3 \\ Dg_3 & g_1 & g_2 \\ Dg_2 & Dg_3 & g_1 \end{bmatrix}, \tag{3.2}$$

where $k = N(\theta_r)/27$. Taking determinants of both sides of (3.2), we get

$$27k(u_2v_3 - u_3v_2) = \pm N(\theta_r)^2 = \pm 27^2k^2$$

and $u_2v_3 - u_3v_2 = \pm 27k$. If we put

$$d = \text{g.c.d.}\,(u_1, u_2, u_3, v_1, v_2, v_3, 27k),$$

we have $\sigma_r = 27k/d$ and $e_r = \pm 27k/d^2$.

If $\theta_r/3 \in \mathfrak{Q}[\delta]$, then $g_1 \equiv g_2 \equiv g_3 \equiv 0 \pmod 3$ and therefore $3 \mid d$. Since $d = \sigma_r/|e_r|$, we have $3|(\sigma_r/|e_r|)$.

If, on the other hand, $3 \mid (\sigma_r/|e_r|)$, then $3 \mid u_3$ and $3 \mid v_3$. Hence, from (3.2) we have $TG \equiv 0 \pmod 3$, where

$$G = \begin{bmatrix} g_3 \\ g_2 \\ g_1 \end{bmatrix}.$$

Since 27 is a divisor of $N(\theta_r)$, we must have $3 \mid \theta_r'\theta_r''$; hence, $g_1 \equiv ag_2 \equiv bg_3 \pmod 3$ by Lemma 2.1. If $3 \nmid g_1$, then $|T| \equiv 0 \pmod 3$, which is not true; thus $g_1 \equiv g_2 \equiv g_3 \equiv 0 \pmod 3$. It follows that $t_{11} \equiv at_{12} \equiv bt_{13} \pmod 3$ and $\theta_r/3 \in \mathfrak{Q}[\delta]$. □

We now have a result analogous to (1.1) in the following

COROLLARY. *If* $D \equiv \pm 1 \pmod 9$ $(D > 44)$ *and* Θ_r $(\approx \theta_r)$ *is the first element of the chain* (3.1) *of relative minima of* $\mathfrak{R}_1$ *with* $\Theta_1 = (1, 0, 1)$ *such that* $N(\theta_r) = 27$ *and* $3|(\sigma_r/|e_r|)$, *then* $\varepsilon_0 = \theta_r/3$.

If we could find a relative minimum Θ $(\approx \theta)$ of $\mathfrak{R}_1$ such that $N(\theta) \mid S$, then, since $N(\theta^3/N(\theta)) = 1$, we could possibly shorten the labor needed in determining ε_0. There are, however, certain divisors of S such that if $N(\alpha)$ is one of these divisors, then $A \approx \alpha$ cannot be a relative minimum of $\mathfrak{R}_1$. As we see in Lemma 3.3. two of these divisors of S are ab^2 and a^2b.

LEMMA 3.3. *If* $\alpha \in \mathfrak{Q}[\delta]$ *and* $N(\alpha) = ab^2$ *or* a^2b, *then* A $(\approx \alpha)$ *cannot be a relative minimum of* $\mathfrak{R}_1$.

Proof. If $N(\alpha) = ab^2$, put $\beta = \alpha\bar{\delta}/ab$; if $N(\alpha) = a^2b$, put $\beta = \alpha\delta/ab$. By Lemma 2.3, we must have $\beta \in \mathfrak{Q}[\delta]$. Since $\delta\bar{\delta} = ab$, we have $0 < \beta < \alpha$ and $\beta'\beta'' < \alpha'\alpha''$. It follows that A cannot be a relative minimum of $\mathfrak{R}_1$. □

Indeed, if $N(\alpha) \mid S$ and β is defined as in Lemma 2.4, then A $(\approx \alpha)$ cannot be a relative minimum of $\mathfrak{R}_1$ whenever $\alpha \neq \beta$. In Theorem 2 of [14] it was shown that if

$D \not\equiv \pm 1 \pmod 9$ and $\tau = 0$, then $B \approx \beta$ is a relative minimum of $\mathfrak{R}_1$. In much of the work that follows we will assume that either $\tau > 0$ or $\sigma = 3$ when $\tau = 0$.

THEOREM 3.4. *Suppose there exists* $\alpha\ (>0) \in \mathfrak{Q}[\delta]$ *such that* $N(\alpha) \mid S$, $3^\tau \parallel N(\alpha)$, *and either* $\tau > 0$ *or* $\sigma = 3$ *when* $\tau = 0$. *Let* $\beta, m, n, \gamma_1, \gamma_2, (\gamma_2 > \gamma_1)$ *be defined as in Lemma* 2.4. *Then* $B\ (\approx \beta)$ *is not a relative minimum of* $\mathfrak{R}_1$ *if and only if there exists a nonzero* $\mu \in \mathfrak{Q}[\delta]$ *such that* $\mu = mn^2\chi$, *where* $\chi = X_1 + X_2\gamma_1 + X_3\gamma_2$ $(X_1, X_2, X_3 \in \mathfrak{Z})$ *and*

$$(3.3) \quad \begin{cases} \text{a)} \quad X_1 + am_2nX_2 + bm_1nX_3 \equiv 0 \pmod 3 \text{ when } \tau = 2, \\ \text{b)} \quad X_1 \equiv am_2nX_2 \equiv bm_1nX_3 \pmod 3 \text{ when } \tau = 1 \text{ or } 0, \end{cases}$$

$$(3.4) \qquad 0 < \chi < 3,$$

$$(3.5) \quad F(\chi) = X_1^2 + \gamma_1^2X_2^2 + \gamma_2^2X_3^2 - \gamma_1X_1X_2 - \gamma_2X_1X_3 - \gamma_1\gamma_2X_2X_3 < 9.$$

Proof. If B is not a relative minimum of $\mathfrak{R}_1$, there must exist $\phi \in \mathfrak{Q}[\delta]$ such that $0 < \phi < \beta$ and $\phi'\phi'' < \beta'\beta''$. If we put $\rho = N(\beta)\phi/\beta \in \mathfrak{Q}[\delta]$, we have

$$(3.6) \qquad 0 < \rho < N(\beta),$$

$$(3.7) \qquad |\rho'| = |\rho''| < N(\beta),$$

$$(3.8) \qquad N(\rho) = N(\beta)^2N(\phi).$$

If

$$\rho = (x_1 + x_2\delta + x_3\bar{\delta})/\sigma, \qquad (x_1, x_2, x_3 \in \mathfrak{Z}),$$

we have $mn^2 \mid x_1$, $m_1n \mid x_2$ and $m_2n \mid x_3$ by (3.8), Lemma 2.2, and Lemma 2.3.

If $\sigma = 3$, then $D \equiv \pm 1 \pmod 9$ and $\tau = 0$. If we put $X_1 = x_1/mn^2$, $X_2 = x_2/m_1n$, $X_3 = x_3/m_2n$, we see that $\mu = 3\rho$, $X_1 \equiv am_2nX_2 \equiv bm_1nX_3 \pmod 3$ (Lemma 2.1) and $0 < \chi < 3$, $F(\chi) < 9$ by (3.6) and (3.7).

If $\sigma = 1$ and $\tau = 1$, we find that $\mu = \rho$ and $0 < \chi < 3$, $F(\chi) < 9$, where X_1, X_2, X_3 are defined as above. Further, since $9 \mid N(\mu)$ (from (3.8)), we must have $X_1 \equiv am_2nX_2 \equiv bm_1nX_3$ by Lemma 2.1.

If $\sigma = 1$ and $\tau = 2$, then $81 \mid N(\mu)$ and $3 \mid x_1$, $3 \mid x_2$, $3 \mid x_3$ by Lemma 2.2. Putting $X_1 = x_1/3mn^2$, $X_2 = x_2/3m_1n$, $X_3 = x_3/3m_2n$, we get $\mu = \rho/3$, $0 < \chi < 3$, $F(\chi) < 9$. Since $N(\mu) = N(\rho)/27$ and $81 \mid N(\mu)$, we have $3 \mid N(\mu)$; hence, $X_1 + am_2nX_2 + bm_1nX_3 \equiv 0 \pmod 3$ by Lemma 2.1.

Now suppose that μ as described by the theorem exists. Define $\theta = 3^{\tau-1}\mu\beta/N(\beta)$. By Lemma 2.1, we see that $3^{3-\tau} \mid N(\mu)$. Since $m^2n^4 \mid N(\mu)$, we have $N(\beta)^2 \mid N(3^{\tau-1}\mu)$ and $N(\beta)^3 \mid N(3^{\tau-1}\mu\beta)$. Since $N(\beta) \mid S$, we see by Lemma 2.2 that $\theta \in \mathfrak{Q}[\delta]$. Also, since $0 < \chi < 3$ and $F(\chi) < 9$, we have $0 < \theta < \beta$ and $\theta'\theta'' < \beta'\beta''$; thus, B cannot be a relative minimum of $\mathfrak{R}_1$. □

COROLLARY. *If* B *above is not a relative minimum of* $\mathfrak{R}_1$, *then* $\Theta \approx \theta = 3^{\tau-1}\mu\beta/N(\beta)$ *is a relative minimum of* $\mathfrak{R}_1$ *when* $\mu = mn^2\chi$ *and* χ *is the least value of* $X_1 + X_2\gamma_1 + X_3\gamma_2$ *satisfying* (3.3), (3.4), *and* (3.5).

In the next section we shall limit the possible values for the minimum χ which satisfies (3.3), (3.4), and (3.5).

4. Some Lemmas Concerning χ. We first give a lemma which limits the possible values of X_1, X_2, X_3 with g.c.d.$(X_1, X_2, X_3) = 1$ for which χ satisfies (3.4), (3.5), and $3 \mid N(mn^2\chi)$.

LEMMA 4.1. *Let* $\chi = X_1 + X_2\gamma_1 + X_3\gamma_2$, *where* $X_1, X_2, X_3 \in \mathfrak{Z}$, g.c.d.$(X_1, X_2, X_3) = 1$, $0 < \chi < 3$, $F(\chi) < 9$. *If* $\mu = mn^2\chi$, $\mu \in \mathfrak{Q}[\delta]$ *and* $3 \mid N(\mu)$, *then values for* X_1, X_2, X_3 *must come from the* 21 *possible cases given in Tables* 1, 2, *and* 3 *below.*

TABLE 1

X_1	1	1	-1	1	2	-1	2	-1	1
X_2	-1	2	1	1	-1	2	1	1	-1
X_3	2	-1	1	-1	1	1	-1	2	1

TABLE 2

X_1	-1	0	0	2	2	-1
X_2	0	-1	2	-1	0	2
X_3	2	2	-1	0	-1	0

TABLE 3

X_1	-1	-1	0	0	1	1
X_2	0	1	-1	1	0	1
X_3	1	0	1	1	1	0

Proof. Since $F(\chi) < 9$, we have $|X_1 + \omega X_2\gamma_1 + \omega^2 X_3\gamma_2| = |X_1 + \omega^2 X_2\gamma_1 + \omega X_3\gamma_2| < 3$. We also have $|\chi| < 3$; hence, $|X_1|, |X_2\gamma_1|, |X_3\gamma_2| < 3$, and, since γ_1, $\gamma_2 > 1$, we have $|X_1|, |X_2|, |X_3| < 3$. Thus, there are at most $5 \times 5 \times 5 = 125$ possible values for χ. Since $3 \mid N(\mu)$, we have $X_1 + am_2nX_2 + bm_1nX_3 \equiv 0 \pmod 3$. If any two of X_1, X_2, X_3 are zero, the third must be zero. It follows that we can discard 13 of the 125 possible χ values. Also, since g.c.d.$(X_1, X_2, X_3) = 1$, we can eliminate 20 more of these possibilities. Since $\gamma_2 > \gamma_1 > 1$ and $\chi > 0$, we can eliminate 35 more cases and 14 additional ones can be deleted by noticing that $\chi < 3$.

Since $F(\chi) = 3X_1^2 - 3X_1\chi + \chi^2 - 3X_2X_3\gamma_1\gamma_2$, it is clear that if $X_2X_3 = -2$, then $F(\chi) > 9$ whenever $X_1 \neq 0, 1$. This allows us to reject 10 more cases. Since

$$4F(\chi) = (2X_1 - \gamma_1 X_2 - \gamma_2 X_3)^2 + 3(X_2\gamma_1 - X_3\gamma_2)^2 < 36,$$

we must have $|X_2\gamma_1 - X_3\gamma_2| < 2\sqrt{3}$. Thus, if $X_2X_3 < 0$, we have $|X_2|\gamma_1 + |X_2|\gamma_2 < 2\sqrt{3}$ and, consequently,

$$|X_2X_3|\gamma_1\gamma_1 < 3. \tag{4.1}$$

Therefore, we cannot have $X_2X_3 = -4$ and, as a result, we are able to eliminate three further cases. We also have $|2X_1 - \gamma_1X_2 - \gamma_2X_3| < 6$; hence, if $X_1 = -2$ and $X_2, X_3 > 0$, we must have $\gamma_1|X_2| + \gamma_2|X_3| < 2$. Since this is not possible, we can reject three more possibilities.

If $\chi = -2 + \gamma_2$ and $F(\chi) < 9$, then $\gamma_2 < \sqrt{6} - 1$. Since this means that $\chi < 0$, we cannot have $\chi = -2 + \gamma_2$. Similarly, it is not possible to have $\chi = -2 + \gamma_1$ or $\chi = -2\gamma_1 + \gamma_2$, and we are able to reject three more cases.

If, for $\chi = X_1 + X_2\gamma_1 + X_3\gamma_2$, we are to have $F(\chi) < 9$, then it is necessary and sufficient that

$$(4.2)\qquad \begin{aligned} X_1 + X_2\gamma_1 - \sqrt{36 - 3(X_1 - X_2\gamma_1)^2} \\ < 2X_3\gamma_2 < X_1 + X_2\gamma_1 + \sqrt{36 - 3(X_1 - X_2\gamma_1)^2}\,. \end{aligned}$$

Thus, if $\chi = 1 - 2\gamma_1 + \gamma_2$ and $F(\chi) < 9$, we have

$$2\gamma_2 < 1 - 2\gamma_1 + \sqrt{36 - 3(1 + 2\gamma_1)^2}\,.$$

Since $28\gamma_1^2 + 4\gamma_1 > 32$, we have

$$2\gamma_1 > 1 - 2\gamma_1 + \sqrt{36 - 3(1 + 2\gamma_1)^2} > 2\gamma_2,$$

which is a contradiction.

If $\chi = -1 - \gamma_1 + \gamma_2$ and $0 < \chi < 3$, then $\gamma_2 > \gamma_1 + 1$. But $12\gamma_1^2 + 12\gamma_1 > 24$ or $9(\gamma_1 + 1)^2 > 36 - 3(1 - \gamma_1)^2$; hence, $2(\gamma_1 + 1) > -\gamma_1 - 1 + \sqrt{36 - 3(1 - \gamma_1)^2} > 2\gamma_2$ when $F(\chi) < 9$. This is also a contradiction. Similarly, if $\chi = -2 - \gamma_1 + \gamma_2 > 0$, then $\gamma_2 > 2 + \gamma_1$. Here we have $2(2 + \gamma_1) > -2 - \gamma_1 + \sqrt{36 - 3(2 - \gamma_1)^2} > 2\gamma_2$ when $F(\chi) < 9$. We have eliminated three more cases, and only the 21 cases given in Tables 1, 2, 3 remain. □

For the case in which we must have $X_1 \equiv am_2nX_2 \equiv bm_1nX_3 \pmod 3$, we can limit the minimum value of χ yet further. We do this in

LEMMA 4.2. *Let* $\mu = mn^2\chi \in \mathcal{Q}[\delta]$ *and let* χ *be the least positive value of* $X_1 + X_2\gamma_1 + X_3\gamma_2$ *such that* $X_1, X_2, X_3 \in \mathcal{Z}$, $X_1 \equiv am_2nX_2 \equiv bm_1nX_3 \pmod 3$, *and* $F(\chi) < 9$. *We can have* $\chi < 3$ *if and only if one of the following is true.*

(i) $am_2n \equiv 1$, $bm_1n \equiv -1 \pmod 3$,

$$2\gamma_2 < -1 - \gamma_1 + \sqrt{36 - 3(\gamma_1 - 1)^2}\,, \qquad \chi = 1 + \gamma_1 - \gamma_2.$$

(ii) $am_2n \equiv -1$, $bm_1n \equiv 1 \pmod 3$,

$$2\gamma_2 < 1 - \gamma_1 + \sqrt{36 - 3(1 + \gamma_1)^2}\,, \qquad \chi = 1 - \gamma_1 + \gamma_2.$$

(iii) $am_2n \equiv bm_1n \equiv -1 \pmod 3$,

$$2\gamma_2 < -1 + \gamma_1 + \sqrt{36 - 3(1 + \gamma_1)^2}\,, \qquad \chi = -1 + \gamma_1 + \gamma_2.$$

Proof. Clearly g.c.d.$(X_1, X_2, X_3) = 1$ and none of the X_i's can be zero; thus, the only possibilities for X_1, X_2, X_3 are those given in Table 1.

Suppose $X_3 > 0$; if $F(\chi) < 9$, then by (4.2) we must have

$$(4.3)\qquad 2\gamma_2X_3 - X_1 - X_2\gamma_1 < \sqrt{36 - 3(X_1 - X_2\gamma_1)^2}\,.$$

On the other hand, if (4.3) is true for $(X_1, X_2) = (1, -1), (-1, 1), (2, -1), (-1, 2)$, the left-hand side of (4.3) must exceed zero; hence,

$$4F(\chi) = (2\gamma_2X_3 - X_1 - X_2\gamma_1)^2 + 3(X_1 - X_2\gamma_1)^2 < 36.$$

Also, for the above values of X_1 and X_2 it is easy to verify that

$$(X_2\gamma_1 + X_1)^2 - X_1X_2\gamma_1 > 3(X_1 + X_2\gamma_1);$$

thus,

$$\sqrt{36 - 3(X_1 - \gamma_1X_2)^2} < 6 - 3X_1 - 3X_2\gamma_1,$$

and $X_3\gamma_2 < 3 - X_1 - X_2\gamma_1$ or $\chi < 3$. Hence, since $\chi > 0$ for the values of X_1 and X_2 in Table 1 when $X_3 > 0$, we see that, for these values of the X's, we have $F(\chi) < 9$ and $0 < \chi < 3$ whenever (4.3) is true.

Suppose $X_3 < 0$ and

$$2\gamma_2X_3 > X_1 + \gamma_1X_2 - \sqrt{36 - 3(X_1 - \gamma_1X_2)^2}\,. \tag{4.4}$$

We have

$$X_1 + X_2\gamma_1 - 2X_3\gamma_2 < \sqrt{36 - 3(X_1 - \gamma_1X_2)^2}\,.$$

For $(X_1, X_2, X_3) = (1, 1, -1), (1, 2, -1), (2, 1, -1)$, we also have $X_1 + X_2\gamma_1 - 2X_3\gamma_2 > 0$; hence,

$$4F(\chi) = (2X_3\gamma_2 - X_1 - X_2\gamma_1)^2 + 3(X_1 - X_2\gamma_1)^2 < 36.$$

Since $X_2X_3 < 0$, we see from (4.1) that $|X_2X_3|\gamma_1\gamma_2 < 3$. Consequently, $0 < X_2\gamma_1 < \sqrt{6}$ $(X_2X_3 = -1, -2)$ and $\chi < 3$. We also have $X_1^2 + X_1X_2\gamma_1 + X_2^2\gamma_1 > 3$; therefore,

$$9(X_1 + \gamma_1X_2)^2 > 36 - 3(X_1 - \gamma_1X_2)^2 \quad \text{and} \quad \chi > 0.$$

Thus, if $X_3 < 0$ and (4.4) is true, then $0 < \chi < 3$ and $F(\chi) < 9$.

If $am_2n \equiv bm_1n \equiv 1 \pmod 3$, we must have $X_1 \equiv X_2 \equiv X_3 \pmod 3$, and no such case exists in Table 1.

If $am_2n \equiv bm_1n \equiv -1 \pmod 3$, then χ must be one of $\chi_1 = 1 - \gamma_1 + 2\gamma_2$, $\chi_2 = 1 + 2\gamma_1 - \gamma_2$, $\chi_3 = -1 + \gamma_1 + \gamma_2$. Put $4r_1 = -\gamma_1 + 1 + \sqrt{36 - 3(1 + \gamma_1)^2}$, $2r_2 = -1 - 2\gamma_1 + \sqrt{36 - 3(1 - 2\gamma_1)^2}$, $2r_3 = -1 + \gamma_1 + \sqrt{36 - 3(1 + \gamma_1)^2}$. From the results proved above, we see that if $\gamma_2 < r_i$, then $F(\chi) < 9$ and $0 < \chi_i < 3$. Since $\gamma_1 > 1$, it can be verified that $r_3 > r_1$, $r_3 > r_2$ and $r_2 < 1 + \gamma_1/2$. If $\chi_3 > \chi_2$, then $\gamma_2 > 1 + \gamma_1/2 > r_2$ and $F(\chi_2) > 9$ by (4.2). If $\chi_3 > \chi_1$, then $\gamma_2 < 2\gamma_1 - 2$; hence, $\gamma_1 < 2\gamma_1 - 2$ and $\gamma_1 > 2$. But, if $F(\chi_3) < 9$, we must have $\gamma_2\gamma_1 < 3$, by (4.1), and therefore $\gamma_1 < \sqrt{3}$, which is a contradiction. It follows that, if either of χ_1 or χ_2 is such that $0 < \chi_i < 3$ and $F(\chi_i) < 9$, then $0 < \chi_3 < \chi_i$ and $F(\chi_3) < 9$; thus, $\chi = \chi_3$.

The values of χ for $am_2n \equiv 1$, $bm_1n \equiv -1 \pmod 3$ and $am_2n \equiv -1$, $bm_1n \equiv 1 \pmod 3$ can be verified in a similar fashion. □

As an example of this result, we notice that, if $\alpha = 4 + 2\delta + \bar{\delta}$ when $D = 10$, then $N(\alpha) = 4$ and $4 \mid S$. We have $d_1 = d_4 = d_5 = d_6 = 1$, $d_2 = 2$, $d_3 = 5$ and $\lambda^3 = m_1m_2n^2 = \min\{4, 5, 20\} = 4$; hence, $\beta = \alpha$. Also, $am_2n = 20 \equiv -1 \pmod 3$ and $bm_1n = 2 \equiv -1 \pmod 3$, $\gamma_1 = \sqrt[3]{10}/2 \simeq 1.08$, $\gamma_2 = \sqrt[3]{100}/2 \simeq 2.32$ and $2\gamma_2 < -1 + \gamma_1 + \sqrt{36 - 3(1 + \gamma_1)^2}$. Thus, B ($\approx \beta = \alpha$) is not a relative minimum of $\mathfrak{R}_1$. In fact, if $\theta = (11 + 5\delta + 2\bar{\delta})/3 = (-4 + 2\delta + 2\bar{\delta}) \cdot (4 + 2\delta + \bar{\delta})/12$, then Θ ($\approx \theta$) is a relative minimum of $\mathfrak{R}_1$.

We now limit the possibilities for χ when (3.3a) is true but (3.3b) is not.

LEMMA 4.3. *Let* $\mu \in \mathfrak{Q}[\delta]$ *and* $\mu = mn^2\chi$, *where* χ *is the least positive value of* $X_1 + \gamma_1 X_2 + \gamma_2 X_3$ *such that* $X_1, X_2, X_3 \in \mathfrak{Z}$, $F(\chi) < 9$ *and* $X_1 + am_2nX_2 + bm_1nX_3 \equiv 0 \pmod 3$. *If it is not the case that* $X_1 \equiv am_2nX_2 \equiv bm_1nX_3 \pmod 3$, *then* $\chi < 3$ *if and only if one of the following is true.*

(i) $am_2n \equiv bm_1n \equiv 1 \pmod 3$, $\gamma_1 < (\sqrt{33} - 1)/2$. *In this case* χ *is one of* $\gamma_1 - 1$ *or* $\gamma_2 - \gamma_1$;

(ii) $am_2n \equiv bm_1n \equiv -1 \pmod 3$, $\gamma_1 < 2$. *In this case* χ *is one of* $\gamma_2 - \gamma_1$, $2 - \gamma_1$, $1 + \gamma_1$;

(iii) $am_2n \equiv 1$, $bm_1n \equiv -1$, $\gamma_1 < (\sqrt{33} - 1)/2$. *In this case* $\chi = -1 + \gamma_1$;

(iv) $am_2n \equiv -1$, $bm_1n \equiv 1 \pmod 3$, $\gamma_1 < (\sqrt{33} - 1)/2$ *or* $\gamma_1 < 2$. *In this case* χ *is one of* $-1 + \gamma_2$, $2 - \gamma_1$ *or* $1 + \gamma_1$.

Proof. We can assume once again that g.c.d.$(X_1, X_2, X_3) = 1$. Thus, from Lemma 4.1 we can only have values for X_1, X_2, X_3 given by those in Tables 2 and 3. Also, if the values of X_1, X_2, X_3 ($X_3 \neq 0$) are selected from Table 2, then, in order for $F(\chi) < 9$, we must have $\gamma_2 < 2$.

The proofs of the remaining cases are similar to the following proof of case (iii); thus, we will only prove this case of the lemma here. If $am_2n \equiv 1$, $bm_1n \equiv -1 \pmod 3$, the only possible values for X_1, X_2, X_3 are given in Table 4 below.

TABLE 4

X_1	-1	0	1	-1	0	0	2
X_2	1	1	0	0	-1	2	0
X_3	0	1	1	2	2	-1	-1

If $\gamma_1 > (\sqrt{33} - 1)/2$, then $F(-1 + \gamma_1) > 9$ and $\chi \neq -1 + \gamma_1$; also, since $\gamma_2 > \gamma_1$, we see that χ cannot be given by any of the remaining possibilities for X_1, X_2, X_3 in Table 4.

If $\gamma_1 < (\sqrt{33} - 1)/2$, then $F(-1 + \gamma_1) < 9$ and $0 < -1 + \gamma_1 < 3$; thus, χ could be $-1 + \gamma_1$. Since $-1 + \gamma_1 < 1 + \gamma_2 < \gamma_1 + \gamma_2$ and $-1 + \gamma_1 < -\gamma_1 + 2\gamma_2 < -1 + 2\gamma_2$, we see that $\chi \neq 1 + \gamma_2$, $\gamma_1 + \gamma_2$, $-\gamma_1 + 2\gamma_2$ or $-1 + 2\gamma_2$. Also, if $F(2\gamma_1 - \gamma_2) < 9$ or $F(2 - \gamma_2) < 9$, then $\gamma_2 < \sqrt{6} - 1$. If this is so, then $\gamma_1 + \gamma_2 < 3$ and $-1 + \gamma_1 < 2 - \gamma_2 < 2\gamma_1 - \gamma_2$. Thus, we can only have $\gamma = -1 + \gamma_1$ when $\gamma_1 < (\sqrt{33} - 1)/2$. □

As an example, we mention that if $D = 22$ and $\alpha = 196 + 70\delta + 25\bar{\delta}$, then $N(\alpha) = 36$ and $36 \mid S$. We find here that $\tau = 2$, $d_1 = d_4 = d_5 = d_6 = 1$, $d_3 = 11$, $d_2 = 2$. Also, $\lambda^3 = 3^2 \min\{4, 11, 2.11^2\} = 36$, $\beta = \alpha$, $m = 1$, $n = 2$, $\gamma_1 = \sqrt[3]{22}/2 \simeq 1.40$, $\gamma_2 = \sqrt[3]{484}/2 \simeq 3.93$, $am_2n = 44 \equiv -1 \pmod 3$, $bm_1n = 2 \equiv -1 \pmod 3$. Since $\gamma_1 < 2$, we have the result that $B \simeq \beta$ cannot be a relative minimum of $\mathfrak{R}_1$. Further, $2 - \gamma_1 < 1 + \gamma_1 < \gamma_2 - \gamma_1$ and $F(2 - \gamma_1) < 9$; hence, $\chi = 2 - \gamma_1$. If $\theta = 3 \cdot 4 \cdot (2 - \gamma_1)\beta/n(\beta)$, then $\theta = 39 + 14\delta + 5\bar{\delta}$ and Θ ($\approx \theta$) is a relative minimum of $\mathfrak{R}_1$.

5. The Main Results. Results for the pure cubic case which are analogous to (1.2) in the quadratic case will be presented in Theorem 5.4; however, we must first prove

THEOREM 5.1. *Let* $\alpha \in \mathfrak{Q}[\delta]$ *and* $N(\alpha) \mid S$. *Put* $N(\alpha) = 3^{\tau} d_1 d_4 d_2^2 d_5^2$, *where* $a = d_1 d_2 d_3$, $b = d_4 d_5 d_6$ *and let* $\lambda^3/3^{\tau} = m_1 m_2 n^2 = \min\{d_1 d_4 d_2^2 d_5^2,\ d_3 d_5 d_1^2 d_6^2,\ d_2 d_6 d_4^2 d_3^2\}$, *where* $m_1 \mid a$, $m_2 \mid b$. *If* $\gamma_1 = \min\{\delta/m_2 n, \bar{\delta}/m_1 n\}$ *and* $\gamma_2 = \max\{\delta/m_2 n, \bar{\delta}/m_1 n\}$, *then* B $(\approx \beta = \lambda\alpha/N(\alpha)^{1/3})$ *is a relative minimum of* $\mathfrak{R}_1$ *if*

(i) $D \not\equiv \pm 1 \pmod 9$ *and* $\tau = 0$,
(ii) $D \equiv \pm 1 \pmod 9$ *and* $\gamma_2 > \sqrt{6}$,
(iii) $D \not\equiv \pm 1 \pmod 9$, $\tau = 1$, *and* $\gamma_2 > \sqrt{6}$,
(iv) $D \not\equiv \pm 1 \pmod 9$, $\tau = 2$, *and* $\gamma_1 > (\sqrt{33} - 1)/2$.

Proof. We saw in Section 3 that (i) is true. If B is not a relative minimum of $\mathfrak{R}_1$, there must exist a χ as described in Theorem 3.4. It is a simple matter to verify that if $\gamma_1 > 1$, we must have

$$-1 - \gamma_1 + \sqrt{36 - 3(\gamma_1 - 1)^2} < 4, \qquad 1 - \gamma_1 + \sqrt{36 - 3(\gamma_1 + 1)^2} < 2\sqrt{6},$$
$$-1 + \gamma_1 + \sqrt{36 - 3(\gamma_1 + 1)^2} < 2\sqrt{6}.$$

Thus, in cases (ii) and (iii) above, we see, by Lemma 4.2, that we cannot have a χ value as specified by Theorem 3.4. Hence B must be a relative minimum of $\mathfrak{R}_1$.

If $\gamma_1 > (\sqrt{33} - 1)/2$, we have $-1 + \gamma_1 + \gamma_2 > 3$. If $F(\chi) < 9$ and $\chi = 1 + \gamma_1 - \gamma_2$ or $1 - \gamma_1 + \gamma_2$, then $\gamma_1 < \sqrt{3}$ by (4.1). Thus, in case (iv), we cannot have $X_1 \equiv am_2 nX_2 \equiv bm_1 nX_3 \pmod 3$. But, if it is not true that $X_1 \equiv am_2 nX_2 \equiv bm_1 nX_3 \pmod 3$, we see, by Lemma 4.3, that we cannot have a χ value satisfying the properties specified by Theorem 3.4. It follows that if $\tau = 2$ and $\gamma_1 > (\sqrt{33} - 1)/2$, then B is a relative minimum of $\mathfrak{R}_1$. □

It should be mentioned here that conditions (ii), (iii), and (iv) of Theorem 5.4 are only sufficient conditions for B to be a relative minimum of $\mathfrak{R}_1$. In any individual case one should consult the more detailed results of Lemmas 4.2 and 4.3.

We will now attempt to describe when we have $\varepsilon_0 = \theta_k^3/N(\theta_k)$, where Θ_k $(\approx \theta_k)$ is a member of the chain (3.1) of relative minima of $\mathfrak{R}_1$ with $\Theta_1 = (1, 0, 1)$. In order to do this we require Theorem 5.3; however, we first prove

LEMMA 5.2. *If* Θ $(\approx \theta > 0)$ *is a relative minimum of* $\mathfrak{R}_1$, *then* Φ $(\approx \phi = \varepsilon_0^n \theta$, $n \in \mathfrak{Z})$ *is also a relative minimum of* $\mathfrak{R}_1$.

Proof. If Φ is not a relative minimum of $\mathfrak{R}_1$, there must exist a γ (> 0) such that $\gamma \in \mathfrak{Q}[\delta]$ and $\gamma < \phi$ and $\gamma'\gamma'' < \phi'\phi''$. That is, $\gamma < \varepsilon_0^n \theta$ and $\gamma'\gamma'' < (\varepsilon_0'\varepsilon_0'')^n \theta'\theta''$. If we put $\rho = \varepsilon_0^{-n}\gamma \in \mathfrak{Q}[\delta]$, we see that $0 < \rho < \theta$ and $\rho'\rho'' < \theta'\theta''$. This contradicts the given fact that Θ is a relative minimum of $\mathfrak{R}_1$; thus, Φ is a relative minimum of $\mathfrak{R}_1$. □

Note that since $N(\pm\varepsilon_0^n \theta) = \pm N(\theta)$, we can say that, if we have any relative minimum Θ of $\mathfrak{R}_1$ $(\Theta \approx \theta)$, then there exists a relative minimum Φ $(\approx \phi)$ in the chain (3.1) of relative minima of $\mathfrak{R}_1$ such that $N(\phi) = |N(\theta)|$.

The following theorem is an extension of Lemma 6 of [14].

THEOREM 5.3. *Suppose* Ψ $(\approx \psi)$ *and* Φ $(\approx \phi)$ *are relative minima of* $\mathfrak{R}_1$ *such that* $N(\phi) \neq N(\psi)$, $N(\phi)$, $N(\psi) \neq 1$, $N(\phi) \mid S$ *and* $N(\psi) \mid S$. *If* Θ_k $(\approx \theta_k)$ *is the first element in the chain* (3.1) *of relative minima of* $\mathfrak{R}_1$ *with* $\Theta_1 = (1, 0, 1)$ *such that* $N(\theta_k) \mid S$, *then* $\varepsilon_0 = \theta_k^3/N(\theta_k)$.

Proof. Since $N(\psi)$, $N(\phi) \neq 1$, we must have $\varepsilon_0^{m_1} < \psi < \varepsilon_0^{m_1+1}$, $\varepsilon_0^{m_2} < \phi < \varepsilon_0^{m_2+1}$ for some $m_1, m_2 \in \mathfrak{Z}$. Thus, if $\psi^* = \varepsilon_0^{-m_1}\psi$, $\phi^* = \varepsilon_0^{-m_2}\phi$, then $\psi^*, \phi^* \in \mathfrak{Q}[\delta]$ and $1 < \psi^* < \varepsilon_0$, $1 < \phi^* < \varepsilon_0$. By Lemma 5.2, Θ^* ($\approx \theta^*$) and Φ^* ($\approx \phi^*$) are relative minima of $\mathfrak{R}_1$; hence they must be in the chain (3.1). Further, there must exist a least Θ_k ($\approx \theta_k$) in the chain (3.1) such that $N(\theta_k) \mid S$ and $1 < \theta_k < \varepsilon_0$. Now

$$N(\theta_k^3/N(\theta_k)) = N(\psi^{*3}/N(\psi^*)) = N(\phi^{*3}/N(\phi^*)) = 1;$$

hence,

$$\theta_k^3/N(\theta_k) = \varepsilon_0^{n_1}, \quad \psi^{*3}/N(\psi^*) = \varepsilon_0^{n_2}, \quad \phi^{*3}/N(\phi^*) = \varepsilon_0^{n_3},$$

where $n_1, n_2, n_3 \in \mathfrak{Z}$. Since, $\theta_k, \psi^*, \phi^* < \varepsilon_0$, we see that $n_i \leqslant 2$ ($i = 1, 2, 3$). Also, $\theta_k'\theta_k''$, $\psi^{*\prime}\psi^{*\prime\prime}$, $\phi^{*\prime}\phi^{*\prime\prime} < 1$; thus, $\theta_k^3/N(\theta_k)$, $\psi^{*3}/N(\psi^*)$, $\phi^{*3}/N(\phi^*) > 1$ and $n_i \geqslant 1$ ($i = 1, 2, 3$).

Since $N(\psi^*) \neq N(\phi^*)$, we may assume with no loss of generality that $\psi^* < \phi^*$. Thus, since Φ^* is a relative minimum of $\mathfrak{R}_1$, we must have $\psi^{*\prime}\psi^{*\prime\prime} > \phi^{*\prime}\phi^{*\prime\prime}$ and $\psi^{*3}/N(\psi^*) < \phi^{*3}/N(\phi^*)$. It follows that $\psi^{*3}/N(\psi^*) = \varepsilon_0$ and $\phi^*/N(\phi^*) = \varepsilon_0^2$. By definition of θ_k, we must have $\theta_k \leqslant \psi^*$. Since $\theta_k^3/N(\theta_k) \leqslant \psi^{*3}/N(\psi^*)$ and $\theta_k^3/N(\theta_k)$ cannot be less than ε_0, we see that $\varepsilon_0 = \theta_k^3/N(\theta_k)$. □

We remark here that we have shown that there can be at most two elements Θ_i ($\approx \theta_i$) and Θ_j ($\approx \theta_j$) in the chain (3.1) of relative minima of $\mathfrak{R}_1$ with $\Theta_1 = (1, 0, 1)$ such that $\theta_i, \theta_j < \varepsilon_0$ and $N(\theta_i) \mid S$, $N(\theta_j) \mid S$.

If Θ_k ($\approx \theta_k$) is the least element in the chain (3.1) of relative minima of $\mathfrak{R}_1$ with $\Theta_1 = (1, 0, 1)$ such that $N(\theta_k) \mid S$, then it can occur that $\varepsilon_0 \neq \theta_k^3/N(\theta_k)$. In these cases we get $\varepsilon_0^2 = \theta_k^3/N(\theta_k)$. For example, this occurs when $D = 14, 52, 77, 92$, etc. However, we are able to prove

THEOREM 5.4. *Let Θ_k ($\approx \theta_k$) be the least relative minimum in the chain* (3.1) *of relative minima of $\mathfrak{R}_1$ with $\Theta_1 = (1, 0, 1)$ such that $N(\theta_k) \mid S$ and $N(\theta_k) \neq 1$. If $N(\theta_k) = 3^\tau d_1 d_4 d_2^2 d_5^2$, where $a = d_1 d_2 d_3$, $b = d_4 d_5 d_6$, let*

$$m_1 m_2 n^2 = \min\{d_2 d_5 d_1^2 d_4^2,\ d_3 d_4 d_2^2 d_6^2,\ d_1 d_6 d_3^2 d_5^2\},$$

where $m_1 \mid a$ and $m_2 \mid b$. Put $\gamma_1 = \min\{\delta/m_2 n, \bar{\delta}/m_1 n\}$, $\gamma_2 = \max\{\delta/m_2 n, \bar{\delta}/m_1 n\}$. We have $\varepsilon_0 = \theta_k^3/N(\theta_k)$ if any of the following is true:

(i) $D \not\equiv \pm 1 \pmod 9$, $\tau = 0$,
(ii) $D \equiv \pm 1 \pmod 9$, $\gamma_2 > \sqrt{6}$,
(iii) $D \not\equiv \pm 1 \pmod 9$, $\tau = 2$, $\gamma_2 > \sqrt{6}$,
(iv) $D \not\equiv \pm 1 \pmod 9$, $\tau = 1$, $\gamma_1 > (\sqrt{33} - 1)/2$.

Proof. Put $\alpha = \theta_k^2/d_2 d_5$ when $\tau = 0, 1$ and put $\alpha = \theta_k^2/3 d_2 d_5$ when $\tau = 2$. We have $\alpha \in \mathfrak{Q}[\delta]$, $N(\alpha) = 3^\nu d_2 d_5 d_1^2 d_4^2$, where

$$\nu = \begin{cases} 0 & \text{when } \tau = 0, \\ 1 & \text{when } \tau = 2, \\ 2 & \text{when } \tau = 1, \end{cases}$$

and $N(\alpha) \mid S$. If we define β as in Lemma 2.4, we have $N(\beta) \mid S$ and we see, by Theorem 5.1, that B ($\approx \beta$) must be a relative minimum of $\mathfrak{R}_1$. If $\tau > 0$, we have $(3, ab) = 1$ and therefore $N(\beta) \neq N(\theta_k)$, $N(\beta) \neq 1$. Suppose $\tau = 0$. Since $N(\beta) = m_1 m_2 n^2$, a and b are square free and $N(\theta_k) \neq 1$, we see that, if $N(\beta) = N(\theta_k)$ or $N(\beta) = 1$, we must have $N(\theta_k) = ab^2$ or $a^2 b$. By Lemma 3.3, this is not possible;

thus, $N(\theta_k) \neq N(\beta)$, $N(\beta) \neq 1$, $N(\theta_k) \neq 1$ and $N(\beta) \mid S$, $N(\theta_k) \mid S$. By Theorem 5.3, we get $\varepsilon_0 = \theta_k^3/N(\theta_k)$. □

6. Some Special Results. It has already been noted in [14] that, when ab is a prime or the triple of a prime, Voronoi's algorithm can be used to find values of $\alpha \in \mathfrak{Q}[\delta]$ such that $N(\alpha) \mid S$. We show in this section how the more general results of Sections 4 and 5 can be used to find such values of α when ab is the product of two distinct primes. In these cases we also characterize some values of a and b for which $\varepsilon_0 = \theta_k^3/N(\theta_k)$, where Θ_k ($\approx \theta_k$) is the least relative minimum of $\mathfrak{R}_1$ such that $\theta_k > 1$ and $N(\theta_k) \mid S$. In this section we use the symbols p and q to denote distinct primes in $\mathfrak{Z}$.

THEOREM 6.1. *Let $D = pq \equiv \pm 1 \pmod 9$. If $D > 10$ and $N(\alpha) = p$ is solvable for $\alpha \in \mathfrak{Q}[\delta]$, there exists a relative minimum B ($\approx \beta$) in $\mathfrak{R}_1$ such that $N(\beta) \mid S$. Further, if Θ_k ($\approx \theta_k$) is the least relative minimum in the chain* (3.1) *with $\Theta_1 =$ (1, 0, 1) such that $N(\theta_k) \mid S$, then $\varepsilon_0 = \theta_k^3/N(\theta_k)$.*

Proof. Since $N(\alpha) = p$, we have $d_1 = p$, $d_2 = 1$, $d_3 = q$, $d_4 = d_5 = d_6 = 1$, $\tau = 0$ and

$$m_1m_2n^2 = \lambda^3 = \min\{p, p^2q, q^2\} = \min\{p, q^2\}.$$

If $p < q^2$, then $\lambda^3 = p$, $m_1 = p_1m_2 = n = 1$ and $\gamma_2 = \max\{\sqrt[3]{pq}, \sqrt[3]{p^2q^2}/p\}$. Supposing $pq > 10$, we see that $\sqrt[3]{pq} > \sqrt{6}$ and $\gamma_2 > \sqrt{6}$. Thus, by Theorem 5.1, $B_1 \approx \beta_1 = \alpha$ is a relative minimum of $\mathfrak{R}_1$.

If $p > q^2$, we have $\lambda^3 = q^2$, $m_1 = m_2 = 1$, $n = q$, $\gamma_2 = \max\{\sqrt[3]{pq}/q, \sqrt[3]{p^2q^2}/q\}$ $= \sqrt[3]{p^2q^2}/q$. Since $p > q^2$, we have $p^2 > q^4 > (\sqrt{6})^3q$ when $q > \sqrt{6}$. If $q = 2$, then, since $pq > 10$, we must have $q \geqslant 13$ and $p^2 > 2(\sqrt{6})^3 = (\sqrt{6})^3q$.

Thus, we have $\gamma_2 > \sqrt{6}$, and B_2 ($\approx \beta_2 = \lambda\alpha/p^{1/3}$) is a relative minimum of $\mathfrak{R}_1$ and $N(\beta_2) \mid S$.

We next consider α^2. We have $N(\alpha^2) = p^2$ and $d_1 = 1$, $d_2 = p$, $d_3 = q$, $d_4 = d_5 = d_6 = 1$, $m_1m_2n^2 = \lambda^3 = \min\{p^2, q, q^2p\} = \min\{p^2, q\}$. If $p^2 < q$, then $\lambda^3 = p^2$, $m_1 = 1$, $m_2 = 1$, $n = p$ and $\gamma_2 = \max\{\sqrt[3]{pq}/p, \sqrt[3]{p^2q^2}/p\}$. We have already seen that $\sqrt[3]{p^2q^2}/p > \sqrt{6}$; hence, B_3 ($\approx \beta_3 = \alpha^2$) is a relative minimum of $\mathfrak{R}_1$.

If $p^2 > q$, then $\lambda^3 = q$, $m_1 = q$, $m_2 = 1$, $n = 1$,

$$\gamma_2 = \max\left\{\sqrt[3]{pq}, \sqrt[3]{pq}/q\right\} > \sqrt{6};$$

thus, B_4 ($\approx \beta_4 = \sqrt[3]{pq}\ \alpha/p$) is a relative minimum of $\mathfrak{R}_1$.

We have now shown that one of B_1 or B_2 and one of B_3 or B_4 are relative minima of $\mathfrak{R}_1$. Further, $N(\beta_1) = p$, $N(\beta_2) = q^2$, $N(\beta_3) = p^2$, $N(\beta_4) = q$; hence, no two of these norms are equal and none of them is 1. The theorem now follows from Theorem 5.3. □

We also have

THEOREM 6.2. *Let $D = pq^2 \equiv \pm 1 \pmod 9$. If $N(\alpha) = p$ is solvable for $\alpha \in \mathfrak{Q}[\delta]$, there exists a relative minimum B ($\approx \beta$) in $\mathfrak{R}_1$ such that $N(\beta) \mid S$. Further, if Θ_k ($\approx \theta_k$) is the least relative minimum in the chain* (3.1) *with $\Theta_1 = (1, 0, 1)$ such that $N(\theta_k) \mid S$, then $\varepsilon_0 = \theta_k^3/N(\theta_k)$.*

Proof. Similar to the proof of Theorem 6.1. □

In [2] it was shown that if D has no prime factor $\equiv 1 \pmod 3$ and D has at least one prime factor $\equiv 2$ or $5 \pmod 9$, then there exists a principal factor of Δ. Now if $D = pq \equiv \pm 1 \pmod 9$, the only possible principal factor set is

$$\{p, p^2q, q^2, p^2, q, pq^2\},$$

and if $D = pq^2 \equiv \pm 1 \pmod 9$, the only possible principal factor set is

$$\{p, q, p^2q^2, p^2, q^2, pq\};$$

thus, if $D = pq$ with $p \equiv 2$, $q \equiv 5 \pmod 9$ or if $D = pq^2$ with $p \equiv q \equiv 2$ or $5 \pmod 9$, we have a solution $\alpha \in \mathcal{Q}[\delta]$ such that $N(\alpha) = p$.

Those fields $\mathcal{Q}[\delta]$ for which 3 is not a divisor of the class number of $\mathcal{Q}(\delta)$ are given by (Honda [7])

(i) $D = 3$,

(ii) $D = p, p \equiv -1 \pmod 3$,

(iii) $D = 3p$ or $9p$, where $p \equiv 2, 5 \pmod 9$,

(iv) $D = pq$, where $p \equiv 2, q \equiv 5 \pmod 9$,

(v) $D = pq^2$, where $p \equiv q \equiv 2, 5 \pmod 9$.

If $D \neq p \equiv 8 \pmod 9$, we know from [14] that in cases (ii) and (iii) we have $\varepsilon_0 = \theta_k^3/N(\theta_k)$, where Θ_k ($\approx \theta_k$) is the least element of the chain (3.1) with $\Theta_1 = (1, 0, 1)$ such that $N(\theta_k) = 3$ or 9. We also know that such a Θ_k will always exist in these cases. We have now seen by Theorems 6.1 and 6.2 that if D is given by cases (iv) or (v), there always exists a least Θ_k ($\approx \theta_k$) in the chain (3.1) such that $N(\theta_k) \mid S$ and for this θ_k we have $\varepsilon_0 = \theta_k^3/N(\theta_k)$. This observation allows us to calculate the regulator of $\mathcal{Q}(\delta)$ (see [14]) about 3 times faster than it would take by using the method of going through the entire set of relative minima of (3.1) until Θ_n ($\approx \theta_n$) was found such that $N(\theta_n) = 1$. Once the regulator has been determined it is not very difficult to calculate the class number $h(D)$ of $\mathcal{Q}(\delta)$ (see Barrucand, Williams, and Baniuk [3]; the Euler product method was used here). In Table 5 below, we present the frequency $f(h)$ of each class number $h = h(D)$ for all 16843 $\mathcal{Q}(\sqrt[3]{D})$ such that $3 \nmid h(D)$, $D = ab^2 < 2 \times 10^5$, and $a > b$. In the third column of this table, we give the least D such that $\mathcal{Q}(\sqrt[3]{D})$ has the h in the first column as its class number.

TABLE 5

h	f(h)	D
1	8230	2
2	4136	11
4	1700	113
5	507	263
7	275	235
8	587	141
10	224	303
11	79	2348
13	47	1049
14	98	514
16	185	681

TABLE 5 (*continued*)

h	f(h)	D
17	27	8511
19	32	667
20	106	761
22	42	281
23	16	21241
25	14	10181
26	23	3403
28	59	509
29	9	12079
31	5	16553
32	37	2399
34	18	1719
35	9	37207
37	7	5545
38	13	12813
40	27	2733
41	7	6659
43	6	32847
44	16	4817
46	9	59975
47	1	198377
49	5	8171
50	14	14372
52	15	4793
53	4	38373
55	3	147257
56	14	857
58	7	6814
59	1	95905
61	2	36161
62	3	42407
64	12	9749
65	2	88169
67	4	14073
68	4	9521
70	4	3467
71	3	3539
73	2	133709
74	5	3581
76	7	23469
77	2	134189

TABLE 5 (*continued*)

h	f(h)	D
79	2	61741
80	10	4799
83	1	17362
85	3	10783
86	4	43403
88	2	132011
89	3	64882
92	2	15131
95	4	15797
97	1	131302
98	2	130859
100	6	31547
101	3	48767
104	7	11549
107	1	180298
110	5	17333
112	5	11665
115	1	99973
118	2	47093
119	1	197003
121	1	57543
122	1	160345
124	2	35349
125	1	189575
127	2	2741
128	4	5987
130	1	103429
136	4	3209
139	1	143326
140	4	36263
148	3	60149
149	2	52737
152	2	118113
154	2	9041
155	1	36107
158	1	66813
160	1	168092
161	3	95001
170	1	45321
173	1	139109
175	2	5711

TABLE 5 (*continued*)

h	f(h)	D
181	1	12251
182	1	115751
188	1	119921
190	1	193247
191	1	47639
193	2	46783
196	1	10522
200	4	12197
202	1	158867
214	3	16823
224	1	103627
230	1	4451
232	2	84093
248	1	194811
254	1	8002
259	1	148763
262	1	28979
263	1	164737
268	1	112757
280	1	35969
284	1	25913
296	1	26601
305	1	39821
316	2	39106
319	1	171629
329	1	183347
334	2	87257
340	1	18257
352	1	51549
358	1	27329
370	1	73779
389	1	24023
392	1	67157
400	1	53434
421	1	47303
431	1	114221
433	1	69539
490	1	169007
559	1	114833
581	1	192754
583	1	63766

TABLE 5 (*continued*)

h	f(h)	D
595	1	185957
628	1	61547
698	1	30867
706	1	26991
746	1	195581
748	1	17573
788	1	101539
827	1	97066
904	1	131084
920	1	17579
958	1	140897
980	1	38463
1190	1	74079
1201	1	128879
1312	1	133251
1442	1	32771
1484	1	79601
1640	1	54874
1760	1	125002
2327	1	141269
2380	1	54869
2599	1	167087
5431	1	161879
5623	1	125003

If $D = pq \not\equiv \pm 1 \pmod 9$ $(p, q \neq 3)$, there are four possible principal factor sets. These are

$$\{3, 3pq, 3p^2q^2, 9, 9pq, 9p^2q^2\},$$
$$\{p, p^2q, q^2, p^2, q, q^2p\},$$
$$\{3p, 3p^2q, 3q^2, 9p^2, 9q, 9q^2p\},$$
$$\{3q, 3q^2p, 3p^2, 9q^2, 9p^2q, 9p\}.$$

If one of p or q is $\equiv 2$ or $5 \pmod 9$ and the other is $\equiv -1 \pmod 3$, we know that there must exist a principal factor of Δ. If this principal factor is in either of the first two sets, then it is a simple matter to show that $\varepsilon_0 = \theta_k^3/N(\theta_k)$, where θ_k has the usual meaning assigned to it in this section. Also, such a θ_k must exist. We now describe what happens when the principal factor is in either of the other two sets.

THEOREM 6.3. *If D is given as above and $N(\alpha) = 3p$ $(3q)$ is solvable for some $\alpha \in \mathfrak{O}[\delta]$, there exists a relative minimum B $(\approx \beta)$ of $\mathfrak{R}_1$ such that $N(\beta) \mid S$. Further, if $q > 8p^2$ and Θ_k $(\approx \theta_k)$ is the first element of the chain* (3.1) *such that $N(\theta_k) \mid S$, then $\varepsilon_0 = \theta_k^3/N(\theta_k)$.*

Proof. The proof of the first part of this theorem is similar to that of Theorem 6.1. In fact, we show that one of B_1 ($\approx \alpha$) or B_2 ($\approx \beta_2 = 3\sqrt[3]{D^2}\,\alpha/N(\alpha)$) must be a relative minimum of $\mathfrak{R}_1$.

If we have $N(\alpha^2) = 9p^2$, then $d_1 = 1$, $d_2 = p$, $d_3 = q$, $d_4 = d_5 = d_6 = 1$, $\tau = 2$, $m_1 m_2 n^2 = \min\{p^2, q, q^2 p\} = p^2$ when $q > 8p^2$. Hence $m_1 = m_2 = 1$, $n = p$, and $am_2 n \equiv bm_1 n \equiv -1 \pmod 3$. We also have $\gamma_1 = \min\{\sqrt[3]{pq}/p, \sqrt[3]{p^2q^2}/p\} = \sqrt[3]{pq}/p > 2$. By Lemma 4.3, we see that B_3 ($\approx \alpha^2$) is a relative minimum of $\mathfrak{R}_1$. If we have $N(\alpha^2) = 9q^2$, then $d_1 = 1$, $d_2 = q$, $d_3 = p$, $d_4 = d_5 = d_6 = 1$, $\tau = 2$, $m_1 m_2 n^2 = \min\{q^2, p, p^2 q\} = p$. Also, $\gamma_1 = \min\{\sqrt[3]{pq}, \sqrt[3]{p^2q^2}/p\}$. Since

$$\sqrt[3]{pq} > 2p > (\sqrt{33} - 1)/2 \quad \text{and} \quad q^2 > 64p^4 > \left((\sqrt{33} - 1)/2\right)^3 p,$$

we have $\gamma_1 > (\sqrt{33} - 1)/2$, and B_4 ($\approx \beta_4 = \alpha^2\sqrt[3]{pq}/q$) is a relative minimum of $\mathfrak{R}_1$. Since $N(\beta_1) = 3p$ $(3q)$, $N(\beta_2) = 3q^2$ $(3p^2)$, $N(\beta_3) = 9p^2$, $N(\beta_4) = 9p$ are all distinct, the theorem follows from Theorem 5.3. □

Thus, we have seen that if $D = pq$, where $p \equiv q \equiv -1 \pmod 3$, one of $p, q \equiv 2, 5 \pmod 9$ and $q > 8p^2$, then there exists θ_k as described above and $\theta_k^3/N(\theta_k) = \varepsilon_0$. We remark here that restriction $q > 8p^2$ can be replaced by the restriction $q > 8p^2 - 3$. This is simply because q must be a prime and $q \equiv -1 \pmod 3$. This inequality is actually sharp for $p = 2$ and $p = 5$. For, when $D = 2 \cdot 29$ or $5 \cdot 197$, we find that $\theta_k^3/N(\theta_k) = \varepsilon_0^2$.

We can also show that if $D = pq^2$, where $p > ((\sqrt{33} - 1)/2)^3 q$, $p \equiv q \equiv -1 \pmod 3$ and one of p, q is congruent to 2 or 5 (mod 9), then there exists a least Θ_k ($\approx \theta_k$) of the chain (3.1) such that $N(\theta_k) \mid S$. Also, $\varepsilon_0 = \theta_k^3/N(\theta_k)$ here.

We conclude by pointing out that, although the ordinary continued fraction algorithm for $\sqrt{d}$ always finds a principal factor (as a norm of $A_{j-1} + \sqrt{d}\,B_{j-1}$) whenever one exists, Voronoi's algorithm does not always do this. For example, when $D = 850$, we find that $N(\alpha) = 150$ and $150 \mid S$ for $\alpha = 180 + 19\delta + 10\bar{\delta}$; however, the only Θ_r in the chain (3.1) such that $N(\theta_r) \mid S$ has $N(\theta_r) = 1$.

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