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A Block-by-Block Method for Volterra Integro-Differential Equations With Weakly-Singular Kernel

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Abstract. The theory of a block-by-block method for solving Volterra integro-differential equations with continuous kernels (see Makroglou [4], [5]) is adapted to Volterra integro-differential equations with weakly-singular kernels, and a rate of convergence is given.

1. Introduction. Consider the nonlinear Volterra integro-differential equation

$$y'(x) = G\left(x, y(x), \int_0^x K(x, t, y(t))\right) dt \qquad (x \geq 0), \tag{1.1}$$

given $y(0)$, written in the form,

$$y(x) = \int_0^x G(s, y(s), z(s))\, ds + y(0) \qquad (x \geq 0), \tag{1.2}$$

$$z(x) = \int_0^x K(x, t, y(t))\, dt \qquad (x \geq 0), \tag{1.3}$$

with

$$K(x, s, y(s)) \equiv K(x, s)y(s),$$
$$K(x, s) = 1/|x - s|^\alpha, \qquad 0 < \alpha < 1, 0 \leq s \leq x \leq X. \tag{1.4}$$

For the discretization of the equation (1.3), we shall use a product integration technique in such a way that when the method is used for solving examples with $K(x, s, y(s)) = H(x, s, y(s))/|x - s|^\alpha$ it will not require the evaluation of $H(x, s, y(s))$ for $s > x$, where it might, for example, not be defined (see Section 2). Product integration techniques have been used for the solution of weakly-singular integral equations; see for example Linz [3], Weiss [6], de Hoog and Weiss [2], Baker [1].

For the discretization of Eq. (1.2) we shall use Eqs. (2.3) in Makroglou [5] and produce a scheme which we called a generalized block-by-block method after Weiss, scheme GC, though it is a new method for integro-differential equations, see Section 3 below, originated in [4]. ('G' stands for 'Generalized' and 'C' is kept here in agreement with the notation used in [4] where it meant the third of the G schemes GA, GB, GC.)

A rate of convergence of the scheme is given in Section 4.

For use in the discussion to follow, we define $x_{m,j} = mh + u_j h$, $x_{m,j,k} = mh + u_j u_k h$, $j, k = 0, 1, \ldots, p$; $m = 0, 1, \ldots, N - 1$, where N, p integers, $h > 0$ so that $Nh = X$ and $0 \leq u_0 < u_1 < \cdots < u_p = 1$. We also assume the preliminaries and definitions given in Makroglou [5].

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2. Discretization of Eq. (1.3). Consider the equation (1.3) with $K(x, s, y(s))$ as in (1.4), that is the equation,

$$z(x) = \int_0^x K(x, t)y(t)\,dt, \tag{2.1}$$

where $K(x, t)$ is given by (1.4). Discretizing at the points $x_{m,j}$ we have

$$z(x_{m,j}) = \sum_{i=0}^{m-1} \int_{ih}^{(i+1)h} K(x_{m,j}, s)y(s)\,ds + \int_{mh}^{x_{m,j}} K(x_{m,j}, s)y(s)\,ds, \tag{2.2}$$

or

$$\begin{aligned} z(x_{m,j}) &= h \sum_{i=0}^{m-1} \int_0^1 K(x_{m,j}, ih + ht)y(ih + ht)\,dt \\ &\quad + hu_j \int_0^1 K(x_{m,j}, mh + u_j ht)y(mh + u_j ht)\,dt. \end{aligned} \tag{2.3}$$

We now use the approximations

$$y(ih + ht) \simeq \sum_{k=0}^{p} L_k(t)y(x_{i,k}), \tag{2.4}$$

$$\begin{aligned} y(mh + hu_j t) &\simeq \sum_{k=0}^{p} L_k(t)y(mh + u_j u_k h) \\ &\simeq \sum_{k=0}^{p} L_k(t) \sum_{r=0}^{p} L_r(u_j u_k)y(x_{m,r}), \end{aligned} \tag{2.5}$$

where $L_k(t)$ are the Lagrangian coefficients, giving

$$\begin{aligned} z_{m,j} &= hu_j \sum_{r=0}^{p} \sum_{k=0}^{p} V^{(m)}(m, j, k)L_r(u_j u_k)y_{m,r} \\ &\quad + h \sum_{i=0}^{m-1} \sum_{k=0}^{p} V^{(m)}(i, j, k)y_{i,k}, \end{aligned} \tag{2.6}$$

$m = 0, 1, \ldots, N - 1; j = 0, 1, \ldots, p$, ($j = 1, 2, \ldots, p$, if $u_0 = 0$), where we have put

$$V^{(m)}(i, j, k) = \int_0^1 K(x_{m,j}, ih + uht)L_k(t)\,dt, \tag{2.7}$$

with

$$\begin{aligned} u &= u_j \quad \text{if } i = m, \\ u &= 1 \quad \text{if } i = 0, 1, \ldots, m - 1. \end{aligned} \tag{2.8}$$

2.1. *Estimation of the Coefficients* $V^{(m)}(i, j, k)$. Using the kernel (1.4) in (2.7), we obtain

$$V^{(m)}(i, j, k) = \int_0^1 \frac{\prod_{q=0; q\neq k}^{p} (t - u_q)}{|l - t|^\alpha}\,dt \Big/ (u^\alpha h^\alpha D(k)), \tag{2.9}$$

where

$$D(k) = \prod_{q=0; q\neq k}^{p} (u_k - u_q), \tag{2.10}$$

and

$$(2.11)\qquad \begin{aligned} &l = m + u_j - i \quad && \text{for } i = 0, 1, \ldots, m-1,\\ &l = 1 && \text{for } i = m, \end{aligned}$$

or

$$(2.12)\qquad V^{(m)}(i, j, k) = (-1)^{p+1}\int_{l^\alpha}^{(l-1)^\alpha} \prod_{q=1}^{p}\left(t^{1/\alpha} - a_q\right) t^{1/\alpha - 2}\, dt/\left(\alpha u^\alpha h^\alpha D(k)\right),$$

where

$$(2.13)\qquad \begin{aligned} &a_{q+1} = l - u_q, \quad && q = 0, 1, \ldots, k-1,\\ &a_q = l - u_q, && q = k+1, \ldots, p. \end{aligned}$$

The product $\prod_{q=1}^{p}(t^{1/\alpha} - a_q)$ in (2.12) can be written as

$$(2.14)\qquad \prod_{q=1}^{p}\left(t^{1/\alpha} - a_q\right) = c_0(t^{1/\alpha})^p + c_1(t^{1/\alpha})^{p-1} + \cdots + c_p,$$

where, with $S_m = a_1^m + a_2^m + \cdots + a_p^m$, we have

$$(2.15)\qquad \begin{aligned} c_0 &= 1,\\ c_1 &= -S_1,\\ c_j &= -\left(S_j + c_1 S_{j-1} + c_2 S_{j-2} + \cdots + c_{j-1} S_1\right)/j, \qquad j = 2, 3, \ldots. \end{aligned}$$

Substituting (2.14) in (2.12) and integrating, we find

$$(2.16)\qquad V^{(m)}(i, j, k) = \frac{(-1)^{p+1}}{u^\alpha h^\alpha D(k)} \sum_{r=0}^{p} c_{p-r} \frac{\left\{(l-1)^{r-\alpha+1} - l^{r-\alpha+1}\right\}}{r - \alpha + 1},$$

$i = 0, 1, \ldots, m$; $k = 0, 1, \ldots, p$; $j = 1, \ldots, p$ if $u_0 = 0$, $j = 0, 1, \ldots, p$ if $u_0 \neq 0$.

3. Statement of the Method. According to the illustration given in the introduction, the approximate equations for scheme GC are

$$(3.1)\qquad \begin{aligned} y_{m,j} &= h \sum_{k=0}^{p} w_k^j G(x_{m,k}, y_{m,k}, z_{m,k})\\ &\quad + h \sum_{i=0}^{m-1} \sum_{k=0}^{p} w_k G(x_{i,k}, y_{i,k}, z_{i,k}) + y(0), \end{aligned}$$

$$(3.2)\qquad \begin{aligned} z_{m,j} &= h u_j \sum_{r=0}^{p} \sum_{k=0}^{p} V^{(m)}(m, j, k) L_r(u_j u_k) y_{m,r}\\ &\quad + h \sum_{i=0}^{m-1} \sum_{k=0}^{p} V^{(m)}(i, j, k) y_{i,k}, \end{aligned}$$

$m = 0, 1, \ldots, N-1$; $j = 0, 1, \ldots, p$, ($j = 1, 2, \ldots, p$ if $u_0 = 0$), where

$$(3.3)\qquad w_k^j = \int_0^{u_j} L_k(x)\, dx,$$

$$(3.4)\qquad w_k = w_k^p = \int_0^1 L_k(x)\, dx,$$

$$L_k(x) = \prod_{j=0;j\neq k}^{p} (x - u_j)/(u_k - u_j), \tag{3.5}$$

and $V^{(m)}(i, j, k)$ are given by (2.16).

Equations (3.1)–(3.2) constitute a system of $2p + 2$ ($2p$ if $u_0 = 0$) in general nonlinear equations for $y_{m,0}, y_{m,1}, \ldots, y_{m,p}$; $z_{m,0}, z_{m,1}, \ldots, z_{m,p}$.

4. Convergence. For the complete convergence proofs we refer to [4]. There, we started by obtaining an asymptotic expansion for the error $\varepsilon_m \equiv \max_{0<j<p}|\varepsilon_{m,j}|$, $\varepsilon_{m,j} \equiv z(x_{m,j}) - z_{m,j}$ in the approximations (3.2). In doing this, the work in [2] was of great help. Having obtained this expansion, one can then obtain a bound on $\mathbf{s}_m = [e_m, \varepsilon_m]^T$ along the lines of the convergence proof given in [5]. The convergence result obtained is given as Theorem 1 below.

THEOREM 1. *Let*

(i) $g(x) \in P_v$ *(see preliminaries in* [5]),

(ii) $y(x)$ *is* $p + 2$ *times continuously differentiable on* $0 \leqslant x \leqslant X$,

(iii) $G(x, y, z)$ *be* $p + v + 2$ *times continuously differentiable with respect to* x, y, z, *respectively, on* $0 \leqslant x \leqslant X$, $|y| \leqslant \bar{y}$, $|z| \leqslant \bar{z}$ *where* $\bar{y} = \max_{0<x<X}|y(x)|$ *and* $\bar{z} = \max_{0<x<X}|z(x)|$. *Then, there are constants* C_1, C_2, C_3, C_4, C_5 *such that*

$$\|\mathbf{s}_m\|_\infty \leqslant C_5 h^{p+1} \quad \text{if } v = 0,$$

$$\|\mathbf{s}_m\|_\infty \leqslant \begin{cases} C_1 h^{p+2} & (1) \\ C_2 h^{p+2-\alpha**} & (2) \end{cases} \quad \text{if } v > 0, \tag{4.1}$$

$m = 1, 2, \ldots, N - 1$, *and*

$$\|\mathbf{s}_0\|_\infty \leqslant \begin{cases} C_3 h^{p+2} & (1) \\ C_4 h^{p+2-\alpha} & (2) \end{cases} \tag{4.2}$$

and the inequalities occur with (1) *or* (2) *according to where the maximum occurs when considering* $\|\cdot\|_\infty$.

Some numerical results obtained by testing scheme GC on a linear and a nonlinear example for both $u_0 = 0$, $u_0 \neq 0$ are displayed in [4] (see [4, Examples 3, 4, p. 97; pp. 152, 153, 157, 158]). Order of convergence at least $O(h^{p+1})$ was verified.

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**The result (2) in (4.1) is changed here to $C_2 h^{p+2-\alpha}$ from $C_2 h^{p+1}$ in [4]. This because in [4, p. 201, Eq. III-1.108] we have $\int_0^1 g(t)P_0(t)\,dt = 0$ for $g \in P_{v(>0)}$.

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