On the Quasi-Optimality in $L_\infty$ of the $H^1$-Projection into Finite Element Spaces*

By A. H. Schatz and L. B. Wahlbin

Abstract. The $H^1$-projection into finite element spaces based on quasi-uniform partitions of a bounded smooth domain in $\mathbb{R}^N$, $N > 2$ arbitrary, is shown to be stable in the maximum norm (or, in the case of piecewise linear or bilinear functions, almost stable). It is not assumed that the mesh-domains coincide with the basic domain.

1. Introduction. Let $u$ be a function on a bounded closed domain $\Omega$ with smooth boundary in $\mathbb{R}^N$, $N > 2$. With $0 < h < \frac{1}{2}$ a parameter, let $\Omega_h = \bigcup_{i=1}^{N(h)} \tau_i^h$ be mesh-domains partitioned into finite elements $\tau_i^h$, and assume temporarily that $\Omega_h \subseteq \Omega$. (As will be seen in (1.6) et seq., the last restriction is easy to overcome when applying our result.) Denote by $W^1_\infty(\Omega_h)$ the class of functions with essentially bounded first derivatives (in the distribution sense), and let $S_h$, $0 < h < \frac{1}{2}$, be finite-dimensional subspaces of $W^1_\infty(\Omega_h)$, consisting of functions $\chi$ that vanish on $\partial \Omega_h$ and are such that $\chi|_{\tau_i^h} \in C^2(\tau_i^h)$.

Define $u_h \equiv Pu \in S_h$ as the $H^1$-projection of $u$; i.e.,

$$\int_{\Omega_h} \nabla u_h \cdot \nabla \chi = \int_{\Omega_h} \nabla u \cdot \nabla \chi - \sum_{i=1}^{N(h)} \left( - \int_{\tau_i^h} u \Delta \chi + \int_{\partial \tau_i^h} u \frac{\partial \chi}{\partial n} \right)$$

(1.1)

Note that $u_h$ is well defined for any continuous $u$. All integrals occurring are assumed to be exactly evaluated; hence, the influence of numerical quadrature is not considered, cf. Wahlbin [25].

Concerning the spaces $S_h$, certain further conditions, detailed in Section 3, are imposed. A brief summary of these is as follows: (i) The partitions of the $\Omega_h$'s are quasi-uniform; (ii) With

$$\delta \equiv \max_{x \in \partial \Omega_h} \text{dist}(x, \partial \Omega),$$

(1.2)

we have $\delta \leq C h^2$; (iii) For smooth functions $v$ that vanish on $\partial \Omega$, we can approximate $v$ by functions in the spaces $S_h$ to order $h^r + \delta$, $r > 2$ an integer. The exact conditions are easily verified in many concrete examples, including such with isoparametric modifications.

Received June 27, 1980; revised March 26, 1981.
1980 Mathematics Subject Classification. Primary 65N30, 65N15.
* This work was supported by the National Science Foundation.
Our main result, Theorem 5.1, is that

\[
\|u - u_h\|_{L^\infty(\mathcal{R}_h)} \leq C \left( \frac{1}{h} \right)^{\tilde{r}} \inf_{\chi \in \mathcal{S}_h} \|u - \chi\|_{L^\infty(\mathcal{R}_h)},
\]

where

\[
\tilde{r} = \begin{cases} 1, & r = 2, \\ 0, & r > 3. \end{cases}
\]

For \( r > 3 \), \( u_h \) is thus a quasi-optimal approximation to \( u \).

One would wish to apply the above result when \( u \) is the solution of a model Dirichlet problem

\[
-\Delta u = f \quad \text{in } \mathcal{R}, \quad u = 0 \quad \text{on } \partial \mathcal{R},
\]

so that

\[
\int_{\mathcal{R}_h} \nabla u_h \cdot \nabla \chi = \int_{\mathcal{R}_h} f \chi \quad \text{for all } \chi \in \mathcal{S}_h.
\]

In general, one has \( \mathcal{R}_h \subseteq \mathcal{R} \), unless: (i) \( \mathcal{R} \) is convex and the partitions of the \( \mathcal{R}_h \) are straight-edged, or: (ii) \( \partial \mathcal{R} \) is a polynomial curve and isoparametric modifications are used at the boundary. Hence, in general, \( f \) is not given on all of \( \mathcal{R}_h \), so that \( u_h \) is not well defined by (1.5) (this difficulty disappears with judicious choice of a numerical integration procedure). In the present analysis, it is assumed that \( f \) is suitably extended to \( \tilde{f} \) and that \( \tilde{f} \) is used in the definition (1.5) of \( u_h \). Then \( u_h \) can be regarded as the \( H^1 \)-projection of a function \( u^\delta \) which solves the problem

\[
-\Delta u^\delta = \tilde{f} \quad \text{in } \mathcal{R}^\delta, \quad u^\delta = 0 \quad \text{on } \partial \mathcal{R}^\delta,
\]

where \( \mathcal{R}^\delta \) is a domain with smooth boundary such that \( \mathcal{R}_h \cup \mathcal{R} \subseteq \mathcal{R}^\delta \). It is clearly possible, when \( h \) is small enough, to construct such domains with \( \max_{x \in \partial \mathcal{R}} \text{dist}(x, \partial \mathcal{R}^\delta) < Ch\delta \); compare (1.2) for notation.

By the maximum principle and (1.3), one has

\[
\|u - u_h\|_{L^\infty(\mathcal{R}_h \cap \mathcal{R})} \leq \|u - u^\delta\|_{L^\infty(\mathcal{R})} + \|u^\delta - u_h\|_{L^\infty(\mathcal{R}_h)} \leq \|u^\delta\|_{L^\infty(\mathcal{R})} + C \left( \frac{1}{h} \right)^{\tilde{r}} \inf_{\chi \in \mathcal{S}_h} \|u^\delta - \chi\|_{L^\infty(\mathcal{R}_h)},
\]

where \( C \) can be taken independent of \( \delta \) (see the proof of (1.3)).

From the above (1.3), or (1.6) when \( \mathcal{R}_h \not\subseteq \mathcal{R} \), it is possible to derive various convergence estimates for \( u - u_h \) in terms of data \( f \). Consider only the "isoparametric" situation; i.e., take \( \delta < Ch^r \). (In general, the highest order that can be obtained is \( \|u - u_h\|_{L^\infty(\mathcal{R}_h)} \leq C(f)(\ln 1/h)^r(h^r + \delta) \). Assume first that \( \mathcal{R}_h \subseteq \mathcal{R} \). Using approximation theory, Schauder estimates, and interpolation of function spaces, one may establish, for a large class of finite element spaces, that

\[
\|u - u_h\|_{L^\infty(\mathcal{R}_h)} \leq C_l h^{\min(l,r)} \left( \ln \frac{1}{h} \right)^{\tilde{r}} \|f\|_{C^{r-2}(\mathcal{R})},
\]

for \( 2 < l \neq r \). The method of analysis indicated gives constants \( C_l \) that tend to infinity as \( l \) tends to \( r \) from above or below.
For a sharper estimate when \( f \in W^{-2}_\infty \), one can proceed in many situations in the following way (which was pointed out to us by V. Thomée): Assume that for a suitable \( \chi \) in \( S_h \), typically an interpolant,
\[
\| u - \chi \|_{L_\infty(S_h)} \leq Ch^{-N/p} \| u \|_{W^r_\infty(S_\infty)}
\]
for any \( p < \infty \) large enough, where \( C \) does not depend on \( p \); cf. Ciarlet [6, Theorem 3.1.6]. Tracing constants in Agmon, Douglis, and Nirenberg [1], one finds that
\[
\| u \|_{W^r_\infty(S_\infty)} \leq Cp \| f \|_{W^r_\infty(S_\infty)}.
\]
Taking \( p = \ln 1/h \) and combining with (1.3), we obtain
\[
\| u - u_h \|_{L_\infty(S_h)} \leq Ch^r \left( \frac{1}{h} \right)^{r+1} \| f \|_{W^r_\infty(S_\infty)}.
\]
A similar result has been obtained in the piecewise linear case by Rannacher [17].

By (1.6), one has the corresponding estimates for \( \| u - u_h \|_{L_\infty(S_h \cap \Omega)} \) when \( \Omega_h \subset \Omega \), and the domains differ by at most \( Ch^r \); here the mean value theorem and elliptic regularity are used to handle the term \( |u|_{L_\infty(\partial \Omega)} \) of (1.6).

We have chosen to treat the \( H^1 \)-projection and the model problem (1.4) in this paper. This choice was made for notational simplicity. More general second-order elliptic Dirichlet problems, and the corresponding projections, can be analyzed by making appropriate modifications in our method.

Let us briefly list other work on quasi-optimal estimates for \( u - u_h \) in various norms.

The question is trivial in the \( H^1 \)-norm.

In the \( L_2 \)-norm, Babuška and Aziz [2, Theorem 6.3.8] showed that when \( S_h \subset H^2(\Omega) \) and \( \Omega_h = \Omega \), i.e., in practice when \( S_h \) consists of \( C^1 \) elements, then
\[
\| u - u_h \|_{L_2(\Omega)} \leq C \inf_{\chi \in S_h} \| u - \chi \|_{L_2(\Omega)}.
\]
The result is false when \( S_h \not\subset H^2(\Omega) \); see Babuška and Osborn [3, p. 58] for a simple counterexample. In the one-dimensional situation on an interval \( I \) for \( C^0 \) piecewise polynomials, the estimate (1.7) holds provided the infimum is taken only over functions \( \chi \) in \( S_h \) that interpolate \( u \) in \( C^0(I) \) at mesh-points \( x_j \); cf. Eisenstat, Schreiber, and Schultz [9]. In a similar vein, in [3] the \( L_2 \)-norm is replaced by a mesh-dependent norm,
\[
\| v \|_{L_p(I, (x_j))} = \left( \int_I |v|^p + \sum_j \left( \frac{x_j+1 - x_j-1}{2} \right)^p |v(x_j)|^p \right)^{1/p}, \quad 1 < p < \infty,
\]
and quasi-optimality in this norm is verified.

As noted also in [3], the estimate (1.3) in the maximum norm is true in one dimension, without the logarithm when \( r = 2 \); cf. Descloux [7], Douglas, Dupont, and Wahlbin [8], and Wheeler [26]. (It is also very easy to translate the methods of the present paper to the one-dimensional situation.)

Concerning estimates in the maximum norm in any number of space dimensions, much work has been devoted to showing quasi-optimality in the \( W^{1}_\infty \)-norm (or the
norm $\| \cdot \|_{L^\infty} + h \cdot \| \cdot \|_{H^1}$; cf. Natterer [14], Nitsche [15], Rannacher [17], and Scott [23]. A typical result is that (when $R_h = R$)

$$\| u - u_h \|_{L^\infty(R)} \leq C \inf_{\chi \in S_h} \| u - \chi \|_{L^\infty(R)},$$

Note that there is no logarithmic factor for $r = 2$; this is a recent result of Rannacher and Scott [18]. (An example by Fried [10] and Jespersen [12] indicates that the logarithmic factor in (1.3) might be necessary for $r = 2$.)

In the maximum norm itself, quasi-optimality (modulo logarithmic factors or factors $h^{-\varepsilon}$, $\varepsilon$ small) is previously known on plane polygonal domains, for meshes with or without refinements, and on convex polyhedral domains in $\mathbb{R}^3$; see Schatz [19] and Schatz and Wahlbin [21].

It is frequently of interest to localize stability estimates of the form above. As an example, one has results of the type

$$\| u - u_h \|_{L^\infty(R)} \leq C \left( \ln \frac{1}{h} \right)^\gamma \inf_{\chi \in S_h} \| u - \chi \|_{L^\infty(R)} + C \| u - u_h \|_{S_h},$$

where $\Omega \subset \Omega^1 \subset R_h$ and $\| \cdot \|_{S_h}$ denotes some weak norm measuring global effects; cf. Bramble, Nitsche, and Schatz [4], Bramble and Schatz [5], Nitsche and Schatz [16], and Schatz and Wahlbin [20], [22].

Our technique of analysis in the present paper does not distinguish between different dimensions $N$ and requires no relations between $r$ and $N$; for $r = N = 2$, however, a shorter proof is possible; see Remark 5.3. In a broad outline our argument is a simplification of that in [20], but additional and lengthy details are needed to take into account the discrepancy between $\mathbb{R}$ and $\mathbb{R}_h$.

We shall use standard notation for the Sobolev spaces $W^k_p(\Omega)$ and $H^k(\Omega) = W^k_2(\Omega)$, $k$ a non-negative integer, $1 < p < \infty$, and for the Hölder spaces $C^l(\Omega)$. We also set $\| v \|_{H^k(\Omega)} = \| \nabla v \|_{L^p(\Omega)}$ with a slight abuse of the norm notation. Generic constants $C$ and $c$ will be independent of $h$ and of essential variables and functions involved; these essential quantities are separately indicated. Two important constants which are not generic are $c'$ and $C_\ast$.

We thank K. Eriksson and V. Thomée for many valuable suggestions in connection with this paper.

2. Preliminaries. Consider the problem of finding $w$ such that, with $\eta$ given,

$$\begin{cases}
-\Delta w = \eta & \text{in } R, \\
w = 0 & \text{on } \partial R,
\end{cases}$$

where, for simplicity, the boundary $\partial R$ is infinitely differentiable. It is well known that $\| w \|_{H^1(R)} \leq C \| \eta \|_{L^2(R)}$, a result we shall use many times. Also,

$$w(x) = \int_{\operatorname{supp} \eta} G^x(y) \eta(y) \, dy,$$

where $G^x(y)$ is the Green’s function for (2.1). It is known (see, e.g., Krasovskiï [13]) that, for $x, y$ in $R$,

$$|D^\alpha_x G^x(y)| \leq \begin{cases}
C(1 + \|\ln|x - y|\|) & \text{for } |\alpha| = 0, N = 2, \\
C|\alpha||x - y|^{2-N-|\alpha|} & \text{otherwise}.
\end{cases}$$
Our most common use of this will be the following: Assume that dist(Ω, supp η) = d > 0. Then, for ℓ ≠ 0,

\[ \|w\|_{W_2^p(Ω)} \leq C d^{2-N-\ell} \int_{\text{supp } η} |\eta(y)| \, dy \]

\[ \leq C d^{2-N-\ell} (\text{diam(supp } η))^N/2 \|\eta\|_{L_2(Ω)}. \]

(2.3)

3. The Finite Element Spaces. In A.1–A.6 we collect the assumptions that we shall need on the finite element spaces. We phrase these assumptions so that they can be readily verified in many concrete situations.

Let 0 < h < \frac{1}{2} be a parameter and \( R_h \), with \( R_h \subseteq R \), mesh-domains made up of closures of disjoint open elements \( τ_i^h, i = 1, \ldots, I(h) \),

\[ \bigcup_{i=1}^{I(h)} τ_i^h. \]

Denote by \( δ = δ_h \) the maximal distance between \( \partial R_h \) and \( \partial R \),

\[ δ = \max_{x \in \partial R_h} \text{dist}(x, \partial R). \]

We let the notation \( W_k^p(R_h) \), for \( R \subseteq R_h \), stand for the piecewise norms relative to the partitions above.

We assume the following two properties of the partitions.

A.1. \( R_h \subseteq R \), where \( R_h \) is infinitely differentiable. The boundaries \( \partial R_h \) are sectionally smooth and uniformly Lipschitz for 0 < h < \frac{1}{2}, and there exists a constant C such that \( δ \leq Ch^2 \).

A.2. There exists a constant C such that, for any \( f \in W_1^1(τ_i^h), 0 < h < \frac{1}{2}, i = 1, \ldots, I(h) \),

\[ \int_{\partial τ_i^h} |f| \leq C \left\{ h^{-1} \|f\|_{L_1(τ_i^h)} + \|f\|_{W_1^1(τ_i^h)} \right\}. \]

The assumption A.2 is easy to verify for quasi-uniform partitions occurring in practice.

Let \( S_h = S_h(R_h) \) be a finite-dimensional subspace of \( W_{∞}^1(R_h) \cap W_{∞}^2(R_h) \), and let furthermore the functions in \( S_h \) vanish on \( \partial R_h \). Here \( W_p^{l,h}(R_h) \) is defined by the norm

\[ \|v\|_{W_p^{l,h}(R_h)} = \left( \sum_i \|v\|_{W_p^{l,h}(τ_i^h)}^p \right)^{1/p}, \]

with the appropriate modifications for \( p = \infty \). Also, \( H^{l,h} = W_2^{l,h} \).

After extension by zero, we can regard functions in \( S_h \) as being in \( W_{∞}^1(R) \).

For the spaces \( S_h \) we first assume an inverse property:

A.3 (Inverse Property). There exist constants C and \( c' > 0 \) such that, for any \( χ \) in \( S_h \) and \( τ = τ_i^h \),

\[ \left( \sum_{|α| = ℓ} \|D^α χ\|^p_{L_p(τ)} \right)^{1/p} \leq Ch^{m-l-N(1/q-1/p)} \|χ\|_{W_p^{l,h}(τ)}, \]

for 0 < m < l < 2, 1 < q < p < ∞, where \( τ' = \{x ∈ τ: \text{dist}(x, \partial τ) > c'h\} \).
This assumption is like a well-known one valid for quasi-uniform partitions, except for the smaller domain \( \tau' \) on the right. Its proof, however, would be the same in all concrete cases.

We shall finally list three different approximation hypotheses:

A.4 \textbf{(High Order Local Approximation).} There exist integers \( r > 2 \) and \( M \), and constants \( C \) and \( c > 0 \) such that the following holds.

For any \( v \in W^r_w(\mathbb{R}) \) with \( v \) vanishing on \( \partial \mathbb{R} \), there exists \( \chi \) in \( S_h \) with the following property.

Let \( B = B(y, d) \) and \( B' = B(y, 2d) \) be concentric balls of radii \( d \) and \( 2d \), respectively, where \( d > ch \), and set \( D_h = B \cap \mathbb{R}_h, D' = B' \cap \mathbb{R} \). Then

\[
\begin{align*}
&\frac{1}{h} \| \nu - \chi \|_{W^r_w(D_h)} + \| \nu - \chi \|_{W^r_w(D_h)} + h \| \nu - \chi \|_{W^r_w(D_h)} \\
&\leq C h^{r-1} \| \nu \|_{W^r_w(D')} + C h^{r-1} \sum_{m=1}^{M} d^{m-1} \| \nu \|_{W^r_w(D')}, \tag{3.1}
\end{align*}
\]

We have phrased this assumption in terms of certain concentric balls, but it is easily extended to more general domains.

The last term on the right of (3.1) merits some elucidation: For concreteness, consider a space \( S_h \) which comes from a larger finite element space \( S_{h}^0 \) by restricting functions to be zero on \( \partial \mathbb{R}_h \). Assume that \( S_h \) admits an interpolant \( \tilde{\chi} = \tilde{\chi}(\nu) \) such that

\[
\frac{1}{h} \| \nu - \tilde{\chi} \|_{W^r_w(D_h)} + \| \nu - \tilde{\chi} \|_{W^r_w(D_h)} + h \| \nu - \tilde{\chi} \|_{W^r_w(D_h)} \leq C h^{r-1} \| \nu \|_{W^r_w(D')}. \tag{3.1}
\]

Such an estimate can often be derived, e.g., by use of the Bramble-Hilbert lemma.

To obtain \( \chi \) in \( S_h \), \( \tilde{\chi} \) is cut down to be zero on \( \partial \mathbb{R}_h \). Often then \( \chi \) and \( \tilde{\chi} \) differ only in a boundary layer \( L_h \) of width approximately \( h \) and by the inverse property

\[
\begin{align*}
&\frac{1}{h} \| \chi - \tilde{\chi} \|_{W^r_w(L_h)} + \| \chi - \tilde{\chi} \|_{W^r_w(L_h)} + h \| \chi - \tilde{\chi} \|_{W^r_w(L_h)} \\
&\leq C h^{r-1} \| \chi - \tilde{\chi} \|_{W^r_w(L_h)} \leq C h^{r-1} \| \chi - \tilde{\chi} \|_{W^r_w(\partial \mathbb{R}_h \cap B)}.
\end{align*}
\]

The last inequality would often be true in practical situations. If the interpolation process uses only point values of \( \nu \), and not derivatives, then the above estimates can often be continued as

\[
\begin{align*}
&\leq C h^{r-1} \| \nu \|_{L_w(\partial \mathbb{R}_h \cap B)} \leq C h^{r-1} \| \nu \|_{W^{r,w}(\mathbb{R} \setminus \mathbb{R}_h \cap B')}.
\end{align*}
\]

where the last step used the mean value theorem. Therefore, (3.1) would obtain with \( M = 1 \) (and \( D' \) replaced by \((\mathbb{R} \setminus \mathbb{R}_h) \cap B' \) in the last term). Higher \( M \) are needed for interpolation processes that involve derivatives of \( v \), and where consequently tangential derivatives along \( \partial \mathbb{R}_h \) are cut down to zero. Most often, the last part of (3.1) could be improved to

\[
C h^{r-1} \delta \sum_{m=1}^{M} h^{m-1} \| \nu \|_{W^m_w((\mathbb{R} \setminus \mathbb{R}_h) \cap B')},
\]

but we shall have no use for such an improvement.

A.5 \textbf{(Low-Order Global Approximation).} There exists a constant \( C \) such that, for \( \nu \) in \( H^2(\mathbb{R}) \) and vanishing on \( \partial \mathbb{R} \), there exists \( \chi \) in \( S_h \) such that

\[
\begin{align*}
&\frac{1}{h} \| \nu - \chi \|_{L^2(\mathbb{R})} + \| \nu - \chi \|_{H^1(\mathbb{R})} + h \| \nu - \chi \|_{H^2_w(\mathbb{R})} \leq C h \| \nu \|_{H^2(\mathbb{R})}.
\end{align*}
\]
Let us briefly comment on how one would check A.5 in concrete cases. Since \( \|v\|_{L^2(\Omega_h \setminus \Omega)} < C\delta \|v\|_{H^1(\Omega)} \) and \( \|v\|_{H^1(\Omega_h \setminus \Omega_h)} < C\delta^{1/2} \|v\|_{H^2(\Omega)} \) by A.1 it suffices to consider the mesh-domain \( \Omega_h \) on the left. For \( N \) high one has to apply a preliminary smoothing argument since an interpolant, requiring point values, cannot immediately be used; see Hilbert [11] and Strang [24]. In our low-order case, this preliminary smoothing of \( v \) can be arranged to preserve the boundary condition \( v = 0 \) on \( \partial \Omega_h \). For, first flatten the boundary patchwise, then extend \( v \) oddly over the boundary, thus preserving \( H^2 \), and then employ an even smoothing kernel. The analysis of [11], [24], combined with ideas outlined in the comment after A.4, could then be carried through in many practical examples.

A.6 ("Superapproximation"). There exist constants \( C \) and \( c > 0 \), and an integer \( K \), such that the following holds:

Let \( B = B(y, d) \) with \( d > ch \), and set \( D_h = B \cap \Omega_h \). Let \( \omega \) be an infinitely differentiable function with support in \( B_1 \) and such that

\[
\left\| \omega^2 \right\|_{H^k(\Omega_h)} \leq Ld^{-k}, \quad k = 0, \ldots, K, \text{ and } \omega \equiv 1 \text{ on } B_2.
\]

Then for any \( v_h \) in \( S_h \) there exists \( \chi \) in \( S_h \) with support in \( D_h^4 \) and with \( \chi \equiv v_h \) on \( D_h^1 \). Further,

\[
\left\| \omega^2 v_h - \chi \right\|_{H^k(\Omega)} \leq CLh \left\{ d^{-2} \|v_h\|_{L_2(D_h \setminus B_1)} + d^{-1} \|v_h\|_{H^1(D_h \setminus B_1)} \right\}.
\]

Again the above is easily extended to more general domains.

For a discussion of superapproximation, see Nitsche and Schatz [16] and also Bramble, Nitsche, and Schatz [4]. The proofs there are easily adjusted to include, e.g., isoparametric modifications. Often, \( \chi \) can simply be taken as a local interpolant of \( \omega^2 v_h \).

4. Local \( \tilde{H}^1 \)-Estimates. This section is devoted to proving Theorem 4.1 below. It is assumed that \( \Omega_h \subseteq \Omega \).

The result and proof are similar to those in [16], but care needs to be exercised to account for the discrepancy between \( \Omega_h \) and \( \Omega \), and to trace constants depending on sizes of domains. Therefore we feel that a self-contained proof is in order.

Let \( B = B(y, d) \) and \( B' = B(y, 2d) \) be closed concentric balls centered at \( y \) and of radii \( d \) and \( 2d \), respectively. Set

\[
D_h = B \cap \Omega_h, \quad D_h' = B' \cap \Omega_h.
\]

For a domain \( \Omega \), let

\[
S_h^\#(\Omega) = \{ \chi \in S_h; \supp \chi \subseteq \Omega \cap \Omega_h \}.
\]

**Theorem 4.1.** Assume that \( \Omega_h \subseteq \Omega \) and that the assumptions of Section 3 hold. There exist constants \( C \) and \( c > 0 \), independent of \( y, d \) and \( h \), such that for \( d > ch \) the following holds: If \( v \in \tilde{H}^1(\Omega) \) and \( v_h \in S_h \) with

\[
\int \nabla (v - v_h) \cdot \nabla \chi = 0 \quad \text{for } \chi \in S_h^\#(D_h'),
\]

then

\[
\|v - v_h\|_{\tilde{H}_0^1(D_h)} \leq C \left( \|v\|_{\tilde{H}_0^1(D_h')} + d^{-1} \|v\|_{L_2(D_h')} + d^{-1} \|v - v_h\|_{L_2(D_h')} \right).
\]
Remark 4.1. Writing \( v - v_h = (v - \chi) - (v_h - \chi) \) for any \( \chi \in \{S_h} \), the first two terms on the right of (4.2) can be replaced by
\[
\inf_{\chi \in \{S_h}} \left( \|v - \chi\|_{L^2(D_h)} + d^{-1}\|v - \chi\|_{L^2(D_h^*)} \right)
\]

Proof. We shall need a few auxiliary domains “between” \( D_h \) and \( D_h^* \); for this let \( B^k = B(y, (1 + 1/k) \cdot d) \), \( k = 1, 2, \ldots \), and \( D_h^k = B^k \cap B_h \), \( k = 2, 3, 4 \). Then \( D_h \subseteq D_h^k \subseteq D_h^3 \subseteq D_h^2 \subseteq D_h^* \).

Consider first functions \( v_h \in \{S_h} \) which are “discrete harmonic” in \( D_h^2 \), i.e., such that
\[
(4.3) \quad \nabla v_h \cdot \nabla \chi = 0 \quad \text{for} \quad \chi \in \{S_h^2}(D_h^2).
\]

We shall show then that for \( d \geq ch \), \( c \) large enough,
\[
(4.4) \quad \|v_h\|_{L^2(D_h)} \leq Cd^{-1}\|v_h\|_{L^2(D_h^2)}.
\]

We introduce an infinitely differentiable cutoff function \( \omega, 0 \leq \omega \leq 1 \), such that
\[
\omega = 1 \quad \text{on} \quad B, \quad \text{supp} \ \omega \subseteq B^5,
\]
and with
\[
(4.5) \quad \|\omega\|_{W^{k,1}(R^3)} \leq C_k d^{-k}, \quad k = 1, 2, \ldots.
\]
Such a function is easily constructed by change of variables in one valid for \( d = 1 \). Now
\[
(4.6) \quad \|v_h\|_{L^2(D_h)} \leq \|\omega v_h\|_{L^2(B^5)}.
\]

Here
\[
\|\omega v_h\|_{L^2(B^5)}^2 = \int_{B^5} \nabla (\omega v_h) \cdot \nabla (\omega v_h)
\]
\[
= \int_{B^5} \nabla \omega \cdot v_h \nabla (\omega v_h) + \int_{B^5} \nabla v_h \cdot \omega \nabla (\omega v_h)
\]
\[
= \int_{B^5} \nabla \omega \cdot v_h \nabla (\omega v_h) + \int_{B^5} \nabla v_h \cdot \nabla (\omega^2 v_h) - \int_{B^5} \nabla v_h \cdot (\nabla \omega) \omega v_h.
\]

The last term on the right equals
\[
- \int_{B^5} \nabla (\omega v_h) \cdot (\nabla \omega) v_h + \int_{B^5} |\nabla \omega|^2 v_h^2
\]
and hence, cancelling terms and using the discrete harmonicity of \( v_h \), (4.3),
\[
\|\omega v_h\|^2_{L^2(B^5)} = \int_{B^5} |\nabla \omega|^2 v_h^2 + \int_{B^5} \nabla v_h \cdot \nabla (\omega^2 v_h - \chi) \quad \text{for any} \quad \chi \in \{S_h^2}(D_h^2).
\]

For the rest of the proof we drop the \( h \)'s in the notation for \( D_h, D_h^k \), and \( D_h^* \).

We next use Schwarz' inequality, the properties of \( \omega \), and, for choosing \( \chi \), the superapproximation hypothesis A.6. Note that, since \( \omega \) is supported in \( B^5 \), only the behavior of \( v_h \) on \( D^4 \) need influence \( \chi \), provided \( d \) is sufficiently large relative to \( h \). We obtain
\[
\|\omega v_h\|_{L^2(B^5)} \leq Cd^{-2}\|v_h\|_{L^2(D^4)}^2 + C\|v_h\|_{L^2(D^4)} \left\{ hd^{-2}\|v_h\|_{L^2(D^4)} + hd^{-1}\|v_h\|_{L^2(D^4)} \right\}.
\]
Via (4.6) we arrive at
\[ \|v_h\|_{H^1(D)}^2 < C d^{-2} \|v_h\|_{L^2(D)}^2 + Ch\|v_h\|_{H^1(D)} d^{-2} \|v_h\|_{L^2(D)} + Chd^{-1} \|v_h\|_{H^1(D)}^2 \]
\[ < C d^{-2} \|v_h\|_{L^2(D)}^2 + Chd^{-1} \|v_h\|_{H^1(D)}^2. \]

In the last step we used the fact that $hd^{-1} < C$.

Repeat the above procedure, with appropriate notational changes, on the last term on the right to obtain
\[ \|v_h\|_{H^1(D)}^2 < C d^{-2} \|v_h\|_{L^2(D)}^2 + Chd^{-1} \left( d^{-2} \|v_h\|_{L^2(D)}^2 + hd^{-1} \|v_h\|_{H^1(D)}^2 \right) \]
\[ < C d^{-2} \|v_h\|_{L^2(D)}^2 + C d^{-2} \|v_h\|_{H^1(D)}^2. \]

The inverse assumption A.3 is now applied to the last term to complete the proof of (4.4).

We proceed to prove (4.2). This time we employ a cutoff function, still denoted by $\omega$, such that
\[ \omega \equiv 1 \quad \text{on } B^2, \quad \text{supp } \omega \subseteq B', \]
and satisfying (4.5). Let $P$ be the $H^1(\mathcal{R}_h)$-projection to $S_h$. Note that since $\mathcal{R}_h \subseteq \mathcal{R}$, $P$ is also the $H^1(\mathcal{R})$-projection to $S_h$, if functions in $S_h$ are extended by zero. Now,
\[ \|v - v_h\|_{H^1(D)} = \|\omega v - v_h\|_{H^1(D)} \]
\[ < \|\omega v - P(\omega v)\|_{H^1(\mathcal{R}_h)} + \|P(\omega v) - v_h\|_{H^1(D)}. \]

Using (4.5), we have
\[ \|\omega v - P(\omega v)\|_{H^1(\mathcal{R}_h)} < C \|\omega v\|_{H^1(\mathcal{R}_h)} < C \|v\|_{H^1(\mathcal{R})} + C d^{-1} \|v\|_{L^2(\mathcal{R})}. \]

Since $\omega \equiv 1$ on $B^2$, using (4.1) it is easily seen that $P(\omega v) - v_h \in S_h$ is discrete harmonic on $D^2$, (4.3). Therefore, from (4.4),
\[ \|P(\omega v) - v_h\|_{H^1(D)} < C d^{-1} \|P(\omega v) - v_h\|_{L^2(\mathcal{R})} \]
\[ < C d^{-1} \|P(\omega v) - \omega v\|_{L^2(\mathcal{R})} + C d^{-1} \|v - v_h\|_{L^2(\mathcal{R})}. \]

By (4.7)–(4.9) we find that
\[ \|v - v_h\|_{H^1(D)} < \|v\|_{H^1(D)} + C d^{-1} \|v\|_{L^2(\mathcal{R})} \]
\[ + C d^{-1} \|v - v_h\|_{L^2(\mathcal{R})} + C d^{-1} \|P(\omega v) - \omega v\|_{L^2(\mathcal{R})}. \]

To handle the last term on the right, we utilize a duality argument over the domain $\mathcal{R}$, which has $H^2$-regularity for the Dirichlet problem. Thus,
\[ \|P(\omega v) - \omega v\|_{L^2(\mathcal{R})} = \sup_{\|\varphi\|_{H^2(\mathcal{R})} = 1} \int_{\mathcal{R}} (P(\omega v) - \omega v) \varphi. \]

For each fixed $\varphi$, let $\psi$ be the solution of the problem
\[ -\Delta \psi = \varphi \quad \text{in } \mathcal{R}, \quad \psi = 0 \quad \text{on } \partial \mathcal{R}. \]

Since $P(\omega v) = 0$ on $\partial \mathcal{R}_h$, we have, from Green's formula,
\[ \int_{\mathcal{R}_h} (P(\omega v) - \omega v) \varphi = \int_{\mathcal{R}_h} \nabla (P(\omega v) - \omega v) \cdot \nabla \psi - \int_{\partial \mathcal{R}_h} \omega v \frac{\partial \psi}{\partial n} \equiv I_1 + I_2. \]
Here, by the properties of the projection \( P \), by the low-order approximation assumption A.5, and by elliptic regularity,

\[
I_1 = - \int_{\Omega_h} \nabla (\omega v) \nabla (\psi - P\psi) \leq C \| \omega v \|_{H^1(\Omega_h)} \| \psi \|_{H^1(\Omega)} 
\]

\[
\leq Ch \left( \| v \|_{H^1(D')} + d^{-1} \| v \|_{L^2(D')} \right).
\]

For the term \( I_2 \) we note that it only enters if \( B' \cap \partial \Omega_h \) is not empty. We have

\[
|I_2| < \| \omega v \|_{L^2(\Omega_h)} \| \nabla \psi \|_{L^2(\Omega_h)}.
\]

Since \( \partial \Omega_h \) is uniformly Lipschitz, one knows (or easily deduces) that

\[
|\omega v|_{L^2(\Omega_h)} \leq C(\| \omega v \|_{L^2(\Omega_h)} \| \omega v \|_{H^1(\Omega_h)})^{1/2}
\]

\[
< C(d^{-1/2} \| v \|_{L^2(D')} + \| v \|_{L^2(D')} \| v \|_{H^1(D')})^{1/2}
\]

\[
< C(d^{-1/2} \| v \|_{L^2(D')} + d^{1/2} \| v \|_{H^1(D')}).
\]

Further,

\[
|\nabla \psi|_{L^2(\Omega_h)} \leq C(\| v \|_{H^1(\Omega_h)} \| \psi \|_{H^1(\Omega_h)})^{1/2}.
\]

Here, by elliptic regularity, \( \| \psi \|_{H^1(\Omega_h)} \leq C \). Also,

\[
\| \psi \|_{H^1(\Omega)}^2 = \int_D \psi \nabla \psi \leq \| \psi \|_{L^2(D')}.
\]

Since \( B(y, 2d) \cap \partial \Omega_h \) is not empty, \( \psi \) vanishes at some points on the boundary \( \partial \Omega \) that are within a distance \( O(\delta) \ll d \) of \( D' \). Considering the domain \( B(y, 4d) \cap \Omega \subset D' \), \( \psi \) vanishes on a part of its boundary which contains a fixed fraction of its total surface measure, and hence, by Poincaré's inequality,

\[
\| \psi \|_{L^2(D')} \leq Cd \| \psi \|_{H^1(\Omega)}.
\]

where it is not hard to see that the constant may be taken uniformly in \( d \) and \( y \). Therefore, \( \| \psi \|_{H^1(\Omega)} \leq Cd \), and hence, \( |\nabla \psi|_{L^2(\Omega_h)} \leq Cd^{1/2} \). Combining this with (4.14), (4.15),

\[
|I_2| \leq C(\| v \|_{L^2(D')} + d \| v \|_{H^1(D')}).
\]

So, by (4.10)-(4.13), since \( hd^{-1} \leq C \),

\[
\| v - v_h \|_{H^1(D')} \leq C \| v \|_{H^1(D')} + Cd^{-1} \| v \|_{L^2(D')} + Cd^{-1} \| v - v_h \|_{L^2(D')}.
\]

This completes the proof of Theorem 4.1.

5. The Main Result. This section contains the main result of the paper.

THEOREM 5.1. Let the assumptions of Section 3 hold. There exists a constant \( C \) such that if \( u \) in \( C^0(\Omega) \) and \( u_h \) in \( S_h \), \( u_h = Pu \), satisfy (1.1), then

\[
\| u - u_h \|_{L^\infty(\Omega_h)} \leq C \left( \ln \frac{1}{h} \right)^{r} \inf_{\chi \in S_h} \| u - \chi \|_{L^\infty(\Omega_h)},
\]

where \( \tilde{r} = 1 \) for \( r = 2 \), \( \tilde{r} = 0 \) for \( r \geq 3 \).
The rest of the section is devoted to a proof of Theorem 5.1. We first note, for simplicity in writing, that it suffices to establish the estimate

\[(5.1)' \quad \|u - u_h\|_{L^\infty(\Omega_h)} \leq C \left( \ln \frac{1}{h} \right)^r \|u\|_{L^\infty(\Omega_h)}; \]

for then (5.1) would follow upon writing \(u - u_h = (u - \chi) - (u_h - \chi)\) for \(\chi \in S_h\). We may also assume in the proof that \(u \in C^1(\Omega_h)\).

For further simplicity in writing, we shall often employ the convention that, in norms and integrals over the mesh-domain \(\Omega_h\), the domain is suppressed in the notation. Thus, \(\|u\|_{L^\infty} = \|u\|_{L^\infty(\Omega_h)}\). We remind the reader that \(\Omega_h \subseteq \Omega\) is assumed.

Let \(x_0\) be a point in \(\Omega_h\) where

\[(5.2) \quad |(u - u_h)(x_0)| = \|u - u_h\|_{L^\infty}. \]

We shall first show that we may assume that \(\text{dist}(x_0, \partial \Omega_h) > c'h\) for some \(c' > 0\); cf. Remark 5.1 below.

**Lemma 5.1.** There exists a constant \(c' > 0\) such that if \(\text{dist}(x_0, \partial \Omega_h) < c'h\), then

\[(5.3) \quad \|u - u_h\|_{L^\infty} \leq 2\|u\|_{L^\infty}. \]

**Proof.** Set \(\delta_0 = \text{dist}(x_0, \partial \Omega_h)\). Since \(u_h = 0\) on \(\partial \Omega_h\), we have, by the mean value theorem,

\[\|u - u_h\|_{L^\infty} \leq |u(x_0)| + |u_h(x_0)| < \|u\|_{L^\infty} + \delta_0 \|\nabla u_h\|_{L^\infty}. \]

Using the inverse property A.3,

\[\|u - u_h\|_{L^\infty} < \|u\|_{L^\infty} + c\delta_0 h^{-1} \|u_h\|_{L^\infty} \]

\[< (1 + c\delta_0 h^{-1})\|u\|_{L^\infty} + c\delta_0 h^{-1}\|u - u_h\|_{L^\infty}. \]

If \(c\delta_0 h^{-1} < 1/3\), we obtain (5.3). This proves the lemma.

Thus, in the remainder of this section we assume that \(\text{dist}(x_0, \partial \Omega_h) > c'h\), \(c' > 0\). We need some more notation. Let \(\tau\) be a finite element in the partition that has \(x_0\) in it, and let \(\tau'\) be the part of \(\tau\) with \(\text{dist}(x_0, \partial \Omega_h) > c'h\). Then \(x_0 \in \tau'\) is assumed. Assume also that \(c'\) is so small that the employment of the inverse property A.3 over \(\tau'\) is justified.

The notation just introduced will be fixed for the rest of the section.

We have, by A.3,

\[|u - u_h(x_0)| < \|u\|_{L^\infty} + |u_h(x_0)| < \|u\|_{L^\infty} + Ch^{-N/2} \|u_h\|_{L^2(\tau)} \]

\[< \|u\|_{L^\infty} + Ch^{-N/2} \|u\|_{L^2(\tau)} + Ch^{-N/2} \|u - u_h\|_{L^2(\tau)} \]

\[< C\|u\|_{L^\infty} + Ch^{-N/2} \|u - u_h\|_{L^2(\tau)}. \]

We proceed to estimate the last term on the right. We first use a duality argument:

\[(5.5) \quad \|u - u_h\|_{L^2(\tau)} = \sup_{\varphi \in C^0_c(\tau')} \|\varphi\|_{L^2} \int_{\tau'} (u - u_h) \varphi. \]

For each fixed \(\varphi\), let \(v\) be the solution of the Dirichlet problem

\[(5.6) \quad -\Delta v = \varphi \quad \text{in} \ \Omega, \quad v = 0 \quad \text{on} \ \partial \Omega. \]
A. H. SCHATZ AND L. B. WAHLBIN

Such a $v$ can be considered, loosely, as a scaled smooth "Green's function" with singularity at $x_0$. By Green's formula, and letting $v_h \in S_h$ be the $H^1$-projection of $v$,

$$
\int_{\partial \Omega_h} (u - u_h) \varphi = - \int_{\partial \Omega_h} u \frac{\partial v}{\partial n} + \int_{\partial \Omega_h} \nabla (u - u_h) \cdot \nabla v
$$

(5.7)

To estimate $I_1$, we have

$$
|I_1| \leq \|u\|_{L^\infty} \int_{\partial \Omega_h} |\nabla v|,
$$

and we appeal then to the following result.

**Lemma 5.2.** For $v$ as in (5.6) with $\varphi \in C^\infty_0(\tau')$ of unit $L_2$-norm,

$$
\int_{\partial \Omega_h} |\nabla v| \leq C h^{N/2},
$$

(5.8)

$$
\int_{\partial \Omega \setminus \partial \Omega_h} |\nabla v| \leq C h^{N/2}.
$$

(5.9)

Admitting this lemma for a moment, we have

(5.10) $|I_1| \leq C h^{N/2} \|u\|_{L^\infty}$.

To estimate $I_2$, use Green's formula over each element,

$$
I_2 = - \sum_i \int_{\eta_i^*} u \Delta (v - v_h) + \sum_i \int_{\eta_i^*} u \frac{\partial}{\partial n} (v - v_h).
$$

Then, from A.2,

$$
|I_2| \leq C \|u\|_{L^\infty} \left( \|\nabla (v - v_h)\|_{W^{1,\infty}} + h^{-1} \|\nabla (v - v_h)\|_{L_1} \right).
$$

We now record the crucial

**Lemma 5.3.** For $v$ as in (5.6) with $\varphi \in C^\infty_0(\tau')$ of unit $L_2$-norm, and $v_h$ its $H^1$-projection,

(5.11) $\|\nabla (v - v_h)\|_{W^{1,\infty}(\partial \Omega_h)} + h^{-1} \|\nabla (v - v_h)\|_{L_1(\partial \Omega_h)} \leq C h^{N/2} \left( \ln \frac{1}{h} \right)^{\frac{1}{2}}$.

The proof of this will be given later in this section. Using the lemma,

$$
|I_2| \leq C h^{N/2} \left( \ln \frac{1}{h} \right)^{\frac{1}{2}} \|u\|_{L^\infty}.
$$

Combining the above estimate with (5.10) into (5.7) and (5.5),

$$
\|u - u_h\|_{L_2(\tau')} \leq C h^{N/2} \left( \ln \frac{1}{h} \right)^{\frac{1}{2}} \|u\|_{L^\infty},
$$

so that by (5.4) the desired result (5.1)' obtains.

It remains now to prove Lemmas 5.2 and 5.3.
Proof of Lemma 5.2. Let us first consider
\[ \int_{\partial \Omega} |\nabla v| = \int_{\partial \Omega} \left| \frac{\partial v}{\partial n} \right|, \]
which equals
\[ \sup_{|\eta|_{L^2(\partial \Omega)} = 1} \int_{\partial \Omega} \left| \frac{\partial v}{\partial n} \right| \eta. \]
If \( w \) denotes the harmonic extension of \( \eta \) into \( \mathbb{R} \), then, since \( v = 0 \) on \( \partial \mathbb{R} \), Green’s second formula gives
\[ -\int_{\partial \mathbb{R}} \frac{\partial v}{\partial n} = \int_{\partial \mathbb{R}} \eta \]
where we used the maximum principle in the last step. Hence,
\[ (5.12) \int_{\partial \mathbb{R}} |\nabla v| < C h^{-N/2}. \]
We need to show the same estimate with \( \partial \mathbb{R} \) replaced by \( \partial \mathbb{R}_h \). To do so, let us work on a coordinate patch, where, after a smooth transformation,
\[ x = (x', x_N), \quad x' \in \Omega' \subset \subset \mathbb{R}^{N-1}, \]
\[ \partial \mathbb{R} = \{ x : x_N = 0, x' \in \Omega \}, \]
\[ \partial \mathbb{R}_h = \{ x : x_N = b(x'), x' \in \Omega \}, \]
with \( 0 < b(x') < C \delta < C h^2 \), and where \( b(x') \) is sectionally smooth and uniformly Lipschitz. Note that hence \( (1 + |\nabla b|^2)^{1/2} \) is uniformly bounded below and above so that we may freely go from integrals over \( \Omega' \) to surface integrals over the corresponding part of \( \partial \mathbb{R}_h \), and vice versa. With \( Dv \) a generic first derivative,
\[ Dv(x', b(x')) = Dv(x', 0) + \int_0^{b(x')} \frac{\partial}{\partial x_N} Dv(x', z) \, dz. \]
Here, \( v(x) = \int_{\Omega'} G^x(y) \varphi(y) \, dy \), so that, by the properties of the Green’s function, (2.2), (2.3), and since \( \text{dist}(\tau', \partial \mathbb{R}_h) > c'h \) and \( |z| < C h^2 \),
\[ \left| \frac{\partial}{\partial x_N} Dv(x', z) \right| < \int_{\Omega'} \frac{C}{|y - (x', z)|^N} |\varphi(y)| \, dy \leq \frac{C h^{N/2}}{|x' - x_0'|^N + h^N}, \]
with \( x_0 = (x_0', x_{0,N}) \).
Remark 5.1. To ensure the above estimate is the reason for our assumption that \( \text{dist}(\tau', \partial \mathbb{R}_h) > c'h \) and the ensuing additional work in Lemma 5.1.
Hence, using (5.12) and an elementary calculation,
\[ \int_{\Omega'} |Dv(x', b(x'))| \, dx' \]
\[ < \int_{\Omega'} |Dv(x', 0)| \, dx' + C h^{N/2} \int_0^{C \delta} dz \int_{\Omega'} \frac{dx'}{|x' - x_0'|^N + h^N} \]
\[ < C h^{N/2} + C h^{N/2 - 1} \delta < C h^{N/2}. \]
This proves (5.8).
For (5.9), in the transformed coordinates we have the estimate (5.8) over any level piece \( \{ x = (x', x_N), x' \in \Omega', x_N = k, k \leq C\delta \} \). An integration in the \( x_N \) direction then gives (5.9).

This completes the proof of Lemma 5.2.

We are now left with proving Lemma 5.3; this will occupy us for the rest of this section.

**Proof of Lemma 5.3.** Set \( e = v - v_h \). We shall first show that

\[
\|\nabla e\|_{L_1} \leq C h^{N/2+1} \left( \ln \frac{1}{h} \right)^{7/4}.
\]

It will be seen later that this is the hard step in proving (5.11). Recall our notational convention that a nondisplayed domain equals \( \mathcal{R}_h \).

We need some auxiliary notation. For this, recall our fixed notation \( x_0 \) and \( \tau' \), cf. (5.2) and the discussion immediately before (5.4). Set

\[
A_j = \{ x : 2^{-j} < |x - x_0| < 2^{-j+1} \}, \quad j \text{ integer},
\]
\[
\Omega_j = A_j \cap \mathcal{R}_h.
\]

Assume for simplicity that \( \mathcal{R}_h = \bigcup_{j=0}^{\infty} \Omega_j \). Next let \( C_* > 1 \) be a quantity to be chosen later (sufficiently large but independent of \( h \)) and let \( J = J(C_*, h) \) be the integer such that

\[
2^{-J} > C_* h \geq 2^{-J-1}.
\]

Further introduce

\[
B_* = \{ x : |x - x_0| < 2^{-J} \}, \quad \Omega_* = B_* \cap \mathcal{R}_h.
\]

For \( C_* \) large enough, \( \Omega_* \) contains \( \tau' \) which contains \( x_0 \). Also set

\[
d_j = 2^{-j},
\]

and

\[
\begin{align*}
A_j' &= A_{j-1} \cup A_j \cup A_{j+1}, \\
A_j'' &= A_{j-1} \cup A'_j \cup A'_{j+1}, \\
A_j^v &= A_{j-1}^v \cup A_j^v \cup A_{j+1}^v; \\
\Omega_j &= A_j' \cap \mathcal{R}_h ( = \Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1}), \\
\Omega_j^v &= A_j^v \cap \mathcal{R}_h.
\end{align*}
\]

Note that

\[
\mathcal{R}_h = \left( \bigcup_{j=0}^{J} \Omega_j \right) \cup \Omega_*;
\]

assume also that \( C_* \) is large enough so that with a positive constant \( c \),

\[
dist(\tau', A_j^v) \geq c d_j, \quad j = 0, \ldots, J + 1.
\]

A sketch of the situation might be helpful, Figure 1. (In the sketch we place \( x_0 \) quite close to \( \partial \mathcal{R}_h \), this being the harder case. Note also that the sketch is not to scale.)
We have now

\[
\|\nabla e\|_{L^1} = \|\nabla e\|_{L^1(\Omega)} + \sum_{j=0}^{J} \|\nabla e\|_{L^1(\Omega_j)}.
\]

Here, by the low-order approximation property A.5 and by elliptic regularity for (5.6),

\[
\|\nabla e\|_{L^1(\Omega)} \leq CC^{N/2}h^{N/2}\|e\|_{H^1(\Omega)}
\]

\[
\leq CC^{N/2}h^{N/2} \inf_{x \in S_h} \|v - x\|_{H^1(\Omega)}
\]

\[
\leq CC^{N/2}h^{N/2+1}\|v\|_{H^1(\Omega)} \leq CC^{N/2}h^{N/2+1}.
\]

Next,

\[
\|\nabla e\|_{L^1(\Omega_j)} \leq 2^N d_j^{N/2}\|e\|_{H^1(\Omega_j)},
\]

so that, with

\[
S = \sum_{j=0}^{J} d_j^{N/2}\|e\|_{H^1(\Omega_j)},
\]

we have, by (5.22), (5.23),

\[
\|\nabla e\|_{L^1} \leq CC^{N/2}h^{N/2+1} + 2^{N/2}S.
\]
Remark 5.2. Note that for the function $v$, which is harmonic away from the region $Q_m$, one has
\[ cd_{j/2}^N \| v \|_{H^i(A_j)} < \| v \|_{W^i(A_j)} < Cd_{j/2}^N \| v \|_{H^i(A_j)}, \]
with positive constants $c$ and $C$. A similar estimate can be derived for the "discrete harmonic" function $v_h$. Therefore, the bound in (5.25) appears sharp. Note further that the right-hand side of (5.25) can be bounded by a weighted $H^1$-norm, viz.,
\[ C \left( \ln \frac{1}{h} \right)^{1/2} \left( \int_{S_A} \left( \text{dist}(x, \tau') + C_s h \right)^N \| \nabla e(x) \|^2 \, dx \right)^{1/2}, \]
cf. [14], [15], [17].
To estimate each term in $S$ we use the local $H^1$-estimates of Theorem 4.1. Since $A_j$ can be covered by a bounded number of balls of radius $d_j/4$, Theorem 4.1 applies with $D_h = \Omega_j$, $D_{h'} = \Omega_j'$, and $d = d_j$. Heeding Remark 4.1, we thus obtain
\[ d_{j/2}^N \| e \|_{H^i(\Omega_j)} < d_{j/2}^N C \inf_{x \in S_A} \left( \| v - x \|_{H^i(\Omega_j)} + d_{j/2}^{-1} \| v - x \|_{L^q(\Omega_j)} \right) \]
\[ + C d_{j/2}^N \| e \|_{L^q(\Omega_j)} \]
\[ < C d_{j/2}^N \inf_{x \in S_A} \left( \| v - x \|_{W^i(\Omega_j)} + d_{j/2}^{-1} \| v - x \|_{L^q(\Omega_j)} \right) \]
\[ + C d_{j/2}^N \| e \|_{L^q(\Omega_j)}. \]
By the local approximation property A.4, and since $h d_{j/2} < C$, \[ \inf_{x \in S_A} \left( \| v - x \|_{W^i(\Omega_j)} + d_{j/2}^{-1} \| v - x \|_{L^q(\Omega_j)} \right) \]
\[ < C \delta^{-1} \| v \|_{W^i(\Omega_j \cap \mathfrak{R})} + C \delta^{-1} \sum_{m=1}^M d_{m-1}^{N/2} \| v \|_{W^i(A_j \cap \mathfrak{R})}. \]
Recall, (5.21), that $\text{dist}(\tau', A_j) > c d_j$, $c > 0$ may be assumed. Since $\phi$ is supported in $\tau'$, the properties of the Green’s function, (2.2), (2.3), give
\[ \| v \|_{W^i(A_j \cap \mathfrak{R})} < C d_{j/2}^{-N/2} h^{-N/2}, \quad l = 1, \ldots, \text{Max}(r, M). \]
Substituting now (5.28) into (5.27), and the result of that into (5.26), we obtain
\[ d_{j/2}^N \| e \|_{H^i(\Omega_j)} < C d_{j/2}^{-N/2 + r - 1} + C d_{j/2}^{-N/2 + r - 1} \]
\[ + C d_{j/2}^N \| e \|_{L^q(\Omega_j)}. \]
Inserting this into (5.25) and summing the geometric series and, for $r = 2$, noting that the sum involves approximately $\ln(1/h)$ terms, and also remembering that $\delta < Ch^2$, we find that
\[ \| \nabla e \|_{L^q(S_A)} < C C_{r/2}^N h^{N/2 + 1} + 2^{N/2} S \]
\[ < C C_{r/2}^N h^{N/2 + 1} + C \sum_{j=0}^J d_j h^{r-2} + C \sum_{j=0}^J d_j h^{r-2} \sum_{j=0}^j d_j \]
\[ + C \sum_{j=0}^J d_j^{N/2-1} \| e \|_{L^q(\Omega_j)} \]
\[ < C h^{N/2 + 1} \left( C_{r/2}^N + \left( \ln \frac{1}{h} \right)^r \right) + C \sum_{j=0}^{J+1} d_j^{N/2-1} \| e \|_{L^q(\Omega_j)}. \]
Remark 5.3. If \( r = 2, N = 2 \), we may now easily conclude the proof of (5.13). For then we estimate the last sum in (5.30) by
\[
\sum_{0}^{J+1} \|e\|_{L^2(\Omega)} \leq C \left( \ln \frac{1}{h} \right)^{1/2} \|e\|_{L^2} \leq Ch^2 \left( \ln \frac{1}{h} \right)^{1/2};
\]
the last estimate here is well known by the low-order approximation hypothesis A.5 and a duality argument.

In general, our argument is more involved; to estimate \( \|e\|_{L^2(\Omega)} \), we call on an additional local duality procedure. Write
\[
(5.31) \quad \|e\|_{L^2(\Omega)} = \sup_{\eta \in \mathcal{C}^0(\Omega)} \int_{\Omega} \eta.
\]
For each such fixed \( \eta \), let \( w \) be the solution of
\[-\Delta w = \eta \quad \text{in } \Omega, \quad w = 0 \quad \text{on } \partial \Omega.\]
Then, for any \( \chi \) in \( S_h \),
\[
(5.32) \quad \int_{\Omega} \eta \chi = \int_{\Omega} \nabla \cdot \nabla w = \int_{\Omega} \nabla \cdot (\nabla (w - \chi)).
\]

We shall now construct an approximation \( \chi \) to \( w \) that, roughly speaking, will be the low-order approximation of A.5 on \( \Omega_j \), and will be the high-order local approximation of A.4 outside of \( \Omega_j \). The blending of the two will be accomplished via "superapproximation", A.6. (We thank K. Eriksson for his help in this argument.)

Let \( \omega, \ 0 < \omega < 1 \), be a smooth function on \( \mathbb{R}^N \) such that (cf. (5.19) for notation)
\[
(5.33) \quad \omega^2 \equiv 1 \quad \text{on } A_j^{'''}, \quad \text{supp } \omega^2 \subseteq A_j^{iv},
\]
and
\[
(5.34) \quad \|
\omega
\|_{W^{k}_{\infty}(\mathbb{R}^N)} \leq C d_j^{-k}, \quad k = 0, \ldots , K \quad \text{(cf. A.6)},
\]
where \( C \) is independent of \( j \). (Construct such a function on unit size domains and then scale.)

Let \( \chi_H \) be the high-order local approximant to \( w \) of A.4, and let \( \chi_L \) denote the low-order global approximant to \( w \) of A.5. Set \( \psi = \omega^2 (\chi_L - \chi_H) \), and let \( \psi_S \in S_h \) be the "super"-approximation to \( \psi \) given in A.6. Then
\[
(5.35) \quad \psi_S \equiv 0 \quad \text{outside } \Omega_j^{''},
\]
and
\[
(5.36) \quad \psi_S \equiv \psi \quad \text{in } \Omega_j^{'''}.
\]
We now set \( \chi = \chi_H + \psi_S \); then, on \( \Omega_j^{'''} \), \( \chi = \chi_H + \psi = \chi_L \), and on \( \mathbb{R}_h \setminus \Omega_j^{''} \), \( \chi = \chi_H \).
We use the $\chi$ just constructed in (5.32). Then,

$$
\int \nabla e \cdot \nabla (w - \chi) = \int \nabla e \cdot \nabla (\omega^2 w + (1 - \omega^2)w - \chi_H - \psi_S)
$$

$$
= \int \nabla e \cdot \nabla (\omega^2 w - \chi_L)
$$

$$
+ \int \nabla e \cdot \nabla ((1 - \omega^2)(w - \chi_H)) + \int \nabla e \cdot \nabla (\psi - \psi_S)
$$

$$
= J_1 + J_2 + J_3.
$$

We proceed to estimate the three terms above.

For $J_1$: By (5.33), (5.34), and A.5,

$$
|J_1| \leq C\|e\|_{\dot{H}^1((\mathcal{R} \setminus A_j)^c)} (d_j^{-1} \|w - \chi_L\|_{L_2(\mathcal{R})} + \|w - \chi_L\|_{\dot{H}^1(\mathcal{R})})
$$

$$
\leq C (\|\nabla e\|_{L_2((\mathcal{R} \setminus A_j)^c)} + \|e\|_{\dot{H}^1((\mathcal{R} \setminus A_j)^c)}) h.
$$

By the Green's function representation, $v(x) = \int G^*(y) \varphi(y) \, dy$ (cf. (5.6)), and by (5.21),

$$
\|\nabla v\|_{L_2((\mathcal{R} \setminus A_j)^c)} \leq C (\delta_j N^{-1})^{1/2} \|\nabla v\|_{L_2((\mathcal{R} \setminus A_j)^c)}
$$

$$
\leq C (\delta_j N^{-1})^{1/2} d_j^{1 - N/2} h^{N/2} = C d_j^{1/2} d_j^{1/2} h^{N/2}.
$$

Thus,

$$
|J_1| \leq C h^{N/2} d_j^{1/2} d_j^{1/2} h^{N/2} + C \|e\|_{\dot{H}^1((\mathcal{R} \setminus A_j)^c)}.
$$

For $J_2$: Note that $1 - \omega^2$ is supported in $\mathcal{R} \setminus A_j''$. Since $\mathcal{R} \setminus A_j'' = (\mathcal{R}_h \setminus A_j'') \cup ((\mathcal{R} \setminus \mathcal{R}_h) \setminus A_j'')$,

$$
|J_2| = \left| \int \nabla e \cdot \nabla ((1 - \omega^2)(w - \chi_H)) \right|
$$

$$
\leq \|\nabla e\|_{L_2((\mathcal{R} \setminus A_j)^c)} C \{ d_j^{-1} \|w - \chi_H\|_{L_2(\mathcal{R}_h \setminus A_j''')} + \|\nabla (w - \chi_H)\|_{L_2((\mathcal{R} \setminus A_j)^c)} \}
$$

$$
+ \|\nabla v\|_{L_2((\mathcal{R} \setminus A_j)^c)} C \{ (d_j^{-1}) \|w - \chi_H\|_{L_2(\mathcal{R}_h \setminus A_j''')} + \|\nabla w\|_{L_2((\mathcal{R} \setminus A_j)^c)} \}.
$$

We note that for $k \neq j - 3, \ldots, j + 3, k > J + 5$ say, we have by A.4 and the Green's function representation $w(x) = \int \Omega_j G^*(y) \eta(y) \, dy$,

$$
(d_j^{-1}) \|w - \chi_H\|_{L_2(\mathcal{R}_h \setminus A_j''')} + \|\nabla (w - \chi_H)\|_{L_2(\mathcal{R}_h \setminus A_j''')} \leq C h^{-1} \|w\|_{W^2(\mathcal{R}_h \setminus A_j)} + C h^{-1} \delta \sum_{m=1}^{M} d_k^{m-1} \|w\|_{W^2(\mathcal{R}_h \setminus A_k)}
$$

$$
\leq C h^{-1} (\max(d_k, d_j))^{2 - N - r} d_j^{N/2} + C h^{-1} \|w\|_{L_2(\mathcal{R}_h \setminus A_j'')} d_j^{N/2}.
$$

Since $\mathcal{R}_h \setminus A_j''$ is the union of such $\Omega_k$ and a small inner "core" domain, for which a similar estimate is easily derived (for $C_\star$ large enough), we find that

$$
|J_2| \leq C h^{-1} d_j^{N/2} d_j^{N/2} + C h^{-1} \delta d_j^{N/2}.
$$
By Lemma 5.2,
\[ \| \nabla v \|_{L_1(\Omega_\varepsilon \setminus \Omega_h)} < Ch^{N/2}\delta \]  
and, again by the Green's function representation,
\[ d_j^{-1} \| w \|_{L_w((\Omega_\varepsilon \setminus \Omega_h) \setminus \partial \Omega)} + \| \nabla w \|_{L_w((\Omega_\varepsilon \setminus \Omega_h) \setminus \partial \Omega)} < C d_j^{1-N/2}. \]

Using (5.40), (5.41) and (5.42) in (5.39), we see that
\[ |J_2| < C \| \nabla e \|_{L_1(\Omega_\varepsilon)} \left( h^{r-1} d_j^{2-N/2-r} + h^{-1} d_j^{1-N/2} \right) + Ch^{N/2}\delta d_j^{1-N/2}. \]

For \( J_3 \): By (5.35) and (5.36) and A.6,
\[ |J_3| = \left| \int \nabla e \cdot \nabla (\psi - \psi_\varepsilon) \right| \leq \| e \|_{H^1(\Omega_\varepsilon \cap \Omega_h)} \| \psi \|_{H^1(\Omega_\varepsilon \cap \Omega_h)} \| \psi - \psi_\varepsilon \|_{H^1(\Omega_\varepsilon \cap \Omega_h)} \]
\[ \leq C \| e \|_{H^1(\Omega_\varepsilon \cap \Omega_h)} h \left( d_j^{-2} \| x_L - x_H \|_{L_2((\Omega_\varepsilon \cap \Omega_h) \setminus \partial \Omega)} + d_j^{-1} \| x_L - x_H \|_{H^1(\Omega_\varepsilon \cap \Omega_h)} \right) \]
\[ + d_j^{-1} \| x_L - x_H \|_{L_2((\Omega_\varepsilon \cap \Omega_h) \setminus \partial \Omega)} + d_j^{-1} \| x_H - w \|_{L_2((\Omega_\varepsilon \cap \Omega_h) \setminus \partial \Omega)} \]
\[ \leq C \| e \|_{H^1(\Omega_\varepsilon \cap \Omega_h)} h \left( d_j^{-2} \| x_L - x_H \|_{L_2} + d_j^{-1} \| x_L - x_H \|_{H^1} \right) \]
\[ + d_j^{-2} \| x_H - w \|_{L_2((\Omega_\varepsilon \cap \Omega_h) \setminus \partial \Omega)} + d_j^{-1} \| x_H - w \|_{H^1((\Omega_\varepsilon \cap \Omega_h) \setminus \partial \Omega)} \].

Here, by A.5,
\[ d_j^{-2} \| x_L - w \|_{L_2} + d_j^{-1} \| x_L - x_H \|_{H^1} < C \| w \|_{H^2(\Omega_\varepsilon)} < C. \]

Further, by A.4 and the Green's function representation,
\[ d_j^{-2} \| x_H - w \|_{L_2((\Omega_\varepsilon \cap \Omega_h) \setminus \partial \Omega)} < C d_j^{-2} d_j^{N/2} \| x_H - w \|_{L_w((\Omega_\varepsilon \cap \Omega_h) \setminus \partial \Omega)} \]
\[ \leq C d_j^{N/2-2} \left( h^r \| w \|_{H_w((\Omega_\varepsilon \cap \Omega_h) \setminus \partial \Omega)} + C \delta \sum_{m=1}^M d_j^{m-1} \| w \|_{H_w((\Omega_\varepsilon \cap \Omega_h) \setminus \partial \Omega)} \right) \]
\[ \leq C d_j^{N/2-2} \left( h^r d_j^{2-N} + C \delta \sum_{m=1}^M d_j^{m-1} d_j^{2-N-m} d_j^{N/2} \right) < C, \]

and, similarly,
\[ d_j^{-1} \| x_H - w \|_{H^1((\Omega_\varepsilon \cap \Omega_h) \setminus \partial \Omega)} < C. \]

Thus,
\[ |J_3| < Ch \| e \|_{H^1(\Omega_\varepsilon \cap \Omega_h)}. \]

Using (5.44), (5.43) and (5.38) in (5.37), and the result in (5.32) and (5.31),
\[ \| e \|_{L_2(\Omega_\varepsilon)} < Ch \| e \|_{H^1(\Omega_\varepsilon \cap \Omega_h)} + Ch^{N/2+1/2} d_j^{1-N/2} \]
\[ + C \| \nabla e \|_{L_2(\Omega_\varepsilon)} \left( h^{r-1} d_j^{2-N/2-r} + h^{-1} d_j^{1-N/2} \right) \]
\[ + Ch^{N/2}\delta d_j^{1-N/2}. \]
Hence, from (5.30),

\[
\|\nabla e\|_{L^1} < C C_{*}^{N/2} h^{N/2+1} + 2^{N/2} S < C h^{N/2+1} \left( C_{*}^{N/2} + \left( \ln \frac{1}{h} \right)^{2} \right)
\]

\[
+ C \|\nabla e\|_{L^1} \sum_{j=0}^{J+1} \left( h^{r-1} d_{j}^{1-r} + h^{-1} \delta \right)
\]

(5.45)

\[
+ C \sum_{j=0}^{J+1} h d_{j}^{N/2-1} \|e\|_{\dot{H}^1(\Omega_{j} \cap \Omega_{j}^{'})}
\]

\[
+ C \sum_{j=0}^{J+1} \left( h^{N/2} + h^{N/2+1} \delta^{1/2} d_{j}^{-1/2} \right).
\]

Here, remembering that \(\delta < C h^{2}\),

\[
\sum_{j=0}^{J+1} \left( h^{r-1} d_{j}^{1-r} + h^{-1} \delta \right) < C h^{r-1} (C_{*} h)^{1-r} + C h \ln \frac{1}{h} < \frac{C}{(C_{*})^{r-1}}.
\]

Further, cf. (5.24) for notation,

\[
\sum_{j=0}^{J+1} h d_{j}^{N/2-1} \|e\|_{\dot{H}^1(\Omega_{j} \cap \Omega_{j}^{'})}
\]

\[
< C \sum_{j=0}^{J} h d_{j}^{N/2-1} \|e\|_{\dot{H}^1(\Omega_{j})} + C h (C_{*} h)^{N/2-1} \|e\|_{\dot{H}^1(\Omega_{j})}
\]

\[
< C \frac{h}{d_{j}} S + C C_{*}^{N/2-1} h^{N/2+1} < \frac{C}{C_{*}} S + C C_{*}^{N/2-1} h^{N/2+1}.
\]

Also,

\[
\sum_{j=0}^{J+1} \left( h^{N/2} + h^{N/2+1} \delta^{1/2} d_{j}^{-1/2} \right) < C h^{N/2+1} \left( h \ln \frac{1}{h} + h^{1/2} \right) < C h^{N/2+1}.
\]

Inserting the above three estimates in (5.45),

\[
\|\nabla e\|_{L^1} < C C_{*}^{N/2} h^{N/2+1} + 2^{N/2} S
\]

\[
< C h^{N/2+1} \left( C_{*}^{N/2} + \left( \ln \frac{1}{h} \right)^{2} \right) + \|\nabla e\|_{L^1} \frac{C}{(C_{*})^{r-1}} + S \frac{C}{C_{*}}.
\]

Taking now \(C_{*}\) large enough, we deduce in succession that

\[
S < C h^{N/2+1} \left( C_{*}^{N/2} + \left( \ln \frac{1}{h} \right)^{2} \right) + \|\nabla e\|_{L^1} \frac{C}{C_{*}^{r-1}}
\]

and that

\[
\|\nabla e\|_{L^1} < C h^{N/2+1} \left( C_{*}^{N/2} + \left( \ln \frac{1}{h} \right)^{2} \right).
\]

This proves the desired estimate (5.13).
It remains now to show (5.11). In the notation of (5.14)–(5.21),

$$\| \nabla e \|_{W^1_h(\Omega_h)} = \| \nabla e \|_{W^1_h(A_j)} + \sum_{j=0}^J \| \nabla e \|_{W^1_h(A_j)}.$$ 

Here, for any $\chi_j \in S_h$, by the inverse property A.3 (where, by subtracting constants over each element, it is seen that it suffices to include the pure gradient term),

$$\| \nabla e \|_{W^1_h(A_j)} < \| \nabla (v - \chi_j) \|_{W^1_h(A_j)} + Ch^{-1}\| \nabla (\chi_j - v_h) \|_{L_1(A_j)}$$ 

$$< CI(v - \chi_j, \Omega_j, 1) + Ch^{-1}\| \nabla e \|_{L_1(A_j)},$$ 

where we have used the shorter notation

$$I(g, \Omega, p) = \| \nabla g \|_{W^1_h(\Omega)} + h^{-1}\| \nabla g \|_{L_1(\Omega)}.$$ 

Similarly,

$$\| \nabla e \|_{W^1_h(\Omega_h)} < CI(v, \Omega_h, 1) + Ch^{-1}\| \nabla e \|_{L_1(\Omega_h)}.$$ 

Hence,

$$\| \nabla e \|_{W^1_h(A_j)} < CI(v - \chi_j, \Omega_j, 1) + C \sum_{j=0}^J I(v - \chi_j, \Omega_j, 1)$$ 

$$+ Ch^{-1}\| \nabla e \|_{L_1(A_j)}.$$ 

(5.46)

Here, by low-order approximation A.5,

$$I(v - \chi_j, \Omega_j, 1) < (8C_h h)^{N/2} I(v - \chi_j, \Omega_h, 2)$$ 

$$< C(C_h h)^{N/2}\| v \|_{H^2(\Omega)} < Ch^{N/2}. (5.47)$$ 

By local approximation A.4 and the Green’s function representation of Section 2, using (5.21),

$$I(v - \chi_j, \Omega_j, 1) < 4^N d^N I(v - \chi_j, \Omega_j, \infty)$$ 

$$< Cd^N \left( h^{-2}\| v \|_{W^2(\Omega \cap A_j)} + Ch^{-2} \sum_{m=1}^M d_j^{m-1} \| v \|_{W^2(\Omega \cap A_j)} \right)$$ 

$$< Cd^N \left( h^{-2}d_j^{2-N} + Ch^{-2}d_j^{1-N}h^{N/2} \right)$$ 

$$< Ch^{N/2}(h^{-2}d_j^{2-r} + d_j),$$ 

(5.48)

where the last step used that $\delta < Ch^2$.

Inserting (5.47) and (5.48) in (5.46) and using (5.13) for the last term of (5.46),

$$\| \nabla e \|_{W^1_h(\Omega_h)} < Ch^{N/2}\left( \frac{1}{h} \right)^r + h^{N/2} \sum_{j=0}^J (h^{-2}d_j^{2-r} + d_j) < Ch^{N/2}\left( \frac{1}{h} \right)^r.$$ 

This completes the proof of Lemma 5.3.

Theorem 5.1 is now completely verified.

Department of Mathematics
Cornell University
Ithaca, New York 14853


