Spline Interpolation at Knot Averages on a Two-Sided Geometric Mesh

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Abstract. For splines of degree $k > 1$ with knots $-t_i = t_{2m+1-i} = 1 + q + q^2 + \cdots + q^{m-1}, i = 1, \ldots, m$, where $0 < q < 1$, it is shown that spline interpolation to continuous functions at nodes $\tau_i = \Sigma_k^{j=0} w_{j+i}, i = 1, \ldots, n = 2m - k - 1$, has operator norm $\|P\|$ which is bounded independently of $q$ and $m$ as $q$ tends to zero if and only if $(1 - w_j)^k < \frac{1}{2}$, $(1 - w_k)^k < \frac{1}{2}$, and $w_j > 0$, $j = 1, \ldots, k$. The choice of nodes $\tau_i = \Sigma_k^{j=0} w_{j+i}$ and the growth rate of $\|P\|$ with $k$ are also discussed.

1. Two-Sided $q$-Splines. To integers $n > 0$, $k > 0$, and a nondecreasing sequence $t = (t_i)_{i=0}^{n+k+1}$ with $t_i < t_{i+k+1}$, $i = 1, \ldots, n$, is associated $S_{k+1,1}$, the space of polynomial splines of order $k + 1$ with knot sequence $t$, defined by $S_{k+1,1} = \text{span}\{N_1, \ldots, N_n\}$, where each $N_i = N_i_{k+1}$ is an appropriate normalized $B$-spline. See [1] for specific details.

With $q > 0$, $m$ a positive integer, $n = 2m - k - 1$, and

$$t_i = - (1 + q + \cdots + q^{m-1}), \quad i = 1, \ldots, m,$$

$$= 1 + q + \cdots + q^{i-m-1}, \quad i = m + 1, \ldots, 2m,$$

$S_{k+1,1}$ is the space of two-sided $q$-splines. Each two-sided $q$-spline can be represented as

$$s(t) = \sum_{i=1}^{m-1} A_j \left[q^{-m}(t_{j+1} - t_i)\right]^k + \sum_{j=0}^k A_{m+j} t_j^j$$

$$+ \sum_{j=1}^{m-1} A_{m+k+j} \left[q^{-j}(t - t_{m+j})\right]^j,$$

where $u_+ = \max\{u, 0\}$, with the endpoint conditions

$$s^{(i)}(t_i) = s^{(i)}(t_{2m}) = 0, \quad i = 0, \ldots, k - 1.$$

Conversely, each function of the form (1.2) which satisfies (1.3) is a two-sided $q$-spline.

With the notation

$$[i] = 1 + q + \cdots + q^{i-1}, \quad i = 0, 1, \ldots,$$

relations such as

$$t_{j+1} - t_i = q^{m-j}[j + 1 - i], \quad 0 < i < j < m,$$

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and
\[ t_{i+1} - t_j = q^{l-m}[i + 1 - j], \quad m < j < i < 2m, \]
can be stated in a compact form. The notation
\[ [i]! = [i][i-1] \cdots [2][1] \quad \text{and} \quad [j] = \frac{[j]!}{[i]![j-i]!} \]
will also be useful.

The clause “as \( q \) tends to zero” appears throughout this paper. It will always mean “for all \( q \) satisfying \( 0 < q < q_0 \)”. The specific choice of \( q_0 \) will vary from instance to instance. However, \( q_0 \) will never depend on \( m \).

**Lemma 1.1.** With \( k \) and \( m \) fixed, let \( \{ s \} \) be a set of two-sided \( q \)-splines with \( \{(A_1, \ldots, A_{2m+k-1})\} \) the corresponding set of coefficient vectors in (1.2). Then \( \{ s \} \) is uniformly bounded as \( q \) tends to zero if and only if \( \{(A_j)\} \) is uniformly bounded as \( q \) tends to zero. Moreover, if the bound on \( \{ s \} \) is independent of \( m \), then so is the bound on \( \{(A_j)\} \).

**Proof.** Let \( 1 > q_0 > 0 \) and \( C \) be such that
\[ |A_j| < C, \quad \text{all } j \text{ and } 0 < q < q_0. \]
Then, for each real \( t \) and \( 0 < q < q_0 \),
\[ |s(t)| < C \left( \sum_{j=0}^{m-1} q^{-j}(t_{j+1} - t_j)^k + \sum_{j=0}^{m-1} q^{-j}(t_{2m} - t_{m+j})^k \right) \]
\[ = C \left( \sum_{j=0}^{m-1} [j]^k + \sum_{j=0}^{m-1} [m]^j + \sum_{j=0}^{m-1} [m-j]^k \right) \leq (2m + k - 1)C[m]^k. \]

Conversely, let \( 1 > q_0 > 0 \) and \( B \) be such that
\[ |s(t)| < B, \quad \text{all real } t \text{ and } 0 < q < q_0. \]
Since
\[ \sum_{j=0}^{k} A_{m+j}(i/k)^j = s(i/k), \quad i = 0, \ldots, k, \]
is a matrix equation with nonsingular coefficient matrix \( V = ((i/k)^j) \) depending only on \( k \),
\[ |A_{m+j}| < (k + 1)B_kB, \quad j = 0, \ldots, k, \]
where \( B_k \) is a bound on the entries of \( V^{-1} \). Set \( C_0 = (k + 1)B_kB \) and assume inductively that \( q_1 \) is such that \( |A_{m-j}| < C_j \) for \( j = 0, 1, \ldots, i - 1 \) for \( q < q_1 \).

From (1.2)
\[ s(t_{m-i}) - s(t_{m-i+1}) = A_{m-i} + \sum_{j=0}^{i-1} A_{m-j}([i-j+1]^k - [i-j]^k) \]
\[ + \sum_{j=0}^{k} A_{m+j}(-1)^j([i+1]^j - [i]^j), \]
so that

$$|A_{m-i}| < 2B + \sum_{i=0}^{i-1} C_j \left( [i - j + 1]^k - [i - j]^k \right) + C_0 \sum_{i=1}^{k} \left( [i + 1]^i - [i]^i \right)$$

$$< 2B + \sum_{i=0}^{i-1} C_j q^{-j} R_k (1 - q_0)^{-k} + C_0 \sum_{i=1}^{k} q^j (1 - q_0)^{-j}$$

$$< 2B + \sum_{i=0}^{i-1} C_j q^{-j} R_k \quad \text{with} \quad R_k = k^2 (1 - q_0)^{-k}.$$ 

Setting $C_i = 2B + \sum_{i=0}^{i-1} C_j q^{-j} R_k$ allows the induction to proceed. Then $C_1 = 2B + C_0 q_1 R_k$, and $C_{i+1} = q_1 (1 + R_k) C_i + 2B (1 - q_1), i = 1, \ldots, m - 2$. This recurrence solves as

$$C_i = \frac{2B (1 - q_1)}{1 - q_1 - q_1 R_k} \left[ 1 - (q_1 + q_1 R_k)^{i-1} \right] + C_1 (q_1 + q_1 R_k)^{i-1},$$

$$i = 1, \ldots, m - 1,$$

if $q_1 + q_1 R_k \neq 1$. Imposing the added restriction $q_1 + q_1 R_k < \frac{1}{2}$ and noting that a symmetric argument will yield $|A_{m+k+j}| < C_j, j = 1, \ldots, m - 1$, establishes that

$$\max_j |A_j| < \max_i C_i < 4B + C_1 + C_0.$$

This bound is independent of $m$ if $B$ is independent of $m$. □

**Lemma 1.2.** Let $k$ and $m$ be fixed. As $q$ tends to zero, the coefficients $(A_j)$ satisfy

$$A_i + \sum_{i+1}^{m-1} A_j q^{i-j} (k-i) [j] + \sum_{k-i}^{k} A_{m+j} O(q^{m-i}(k-i)) = 0, \quad i = 1, \ldots, k - 1,$$

and

$$A_k + \sum_{k+1}^{m-1} A_j [j] + \sum_{0}^{k} A_{m+j} \left( \frac{j!}{[k]!} + O(q^{m-k+1}) \right) = 0.$$

**Proof.** This follows from (1.3). Let functionals $\Lambda_{1s}$, $1 < i < \nu < k$, be defined by

$$\Lambda_{1s} = q^{m-i}(k-\nu) \frac{\nu!}{k!} (-1)^{k-\nu} s^{(k-\nu)}(t_1)$$

and, recursively,

$$\Lambda_{\nu s} = q^{s-k}(\Lambda_{i-1,\nu - s} - [i - 1] \Lambda_{i-1,\nu - 1 s}) / [i].$$

From (1.2)

$$s^{(k-\nu)}(t_1) = \sum_{i=0}^{m-1} A_j q^{i-m}(k-\nu) \frac{k!}{\nu!} (-1)^{k-\nu} [j]^\nu$$

$$+ \sum_{k-\nu}^{k} A_{m+j} \frac{j!}{(j - k + \nu)!} t_1^{j-k+\nu},$$

whence

$$\Lambda_{1s} = \sum_{i=0}^{m-1} A_j q^{i-m}(k-\nu) [j]^\nu + \sum_{k-\nu}^{k} A_{m+j} q^{m-i}(k-\nu) C_{1s}. $$
where

\[ C_{ijr} = \frac{\nu j!}{k! (j - k + 1)!} (-1)^{k-n} t_i^{j-k+r}. \]

The recursion formula gives

\[ A_{i} = \sum_{r} A_{i+j} q^{(m-r)(k-r)} C_{ijr}, \]

where

\[ C_{ijr} = \binom{C_{i-1,j,r}}{i-1} q^{m-i+1} C_{i-1,j,r-1}. \]

From (1.3) each \( A_{i} = 0 \) and, in particular, \( A_{i} = 0 \). This fact, along with the observation that \( C_{ijk} = C_{ijr} / [k! + O(q^{m-k+1}) \] completes the proof. □

Combining Lemmas 1.1, 1.2, and a symmetric counterpart of Lemma 1.2 yields

**Lemma 1.3.** Let \( k \) and \( m \) be fixed and let \{s\} be a set of two-sided q-splines which is bounded as \( q \) tends to zero. Then the corresponding set of coefficient vectors \([^A_i]\) satisfies

\[
A_i = O(q^{k-i}), \quad i = 1, \ldots, k - 1, \\
A_i = O(1), \quad i = k, \ldots, 2m, \\
A_{2m+i} = O(q^i), \quad i = 1, \ldots, k - 1,
\]

as \( q \) tends to zero. If the bound on \{s\} is independent of \( m \), then so are the bounds on the \( A_i \).

The independence of \( m \) in the \( O(q^{k-i}) \) and \( O(q^i) \) bounds follows from the exponential decay of the coefficients in the first \( k - 1 \) equations of Lemma 1.2.

2. Spline Interpolation. Let \( \tau = (\tau_i) \) be a strictly increasing sequence. It is known [1] that: For each function \( f \) defined on \( \tau \) there is exactly one \( s \in S_{k+1,t} \) such that \( s(\tau_i) = f(\tau_i), i = 1, \ldots, n, \) if and only if \( N_i(\tau_i) > 0, i = 1, \ldots, n, \) or, equivalently, if and only if

\[
t_i < \tau_i < t_{i+k+1}, \quad i = 1, \ldots, n.
\]

When \( \tau \) satisfies (2.1) a linear map \( P \) into \( S_{k+1,t} \) which reproduces \( S_{k+1,t} \) may be defined by: For each function \( f \) defined on \( \tau \), \( Pf \in S_{k+1,t} \) and \( (Pf)(\tau_i) = f(\tau_i), i = 1, \ldots, n \). In fact, \( Pf = \sum f(\tau_j)L_j \) where \( L_j(\tau_i) = \delta_{ij}, i, j = 1, \ldots, n \). The operator norm of \( P \) is

\[ ||P|| = \sup_{f} \frac{||Pf||}{||f||}, \]

where the sup is taken over all \( f \in C[t_1, t_{n+k+1}] \) and

\[ ||f|| = \sup\{|f(t)|: t_1 < t < t_{n+k+1}\}. \]

It is well known that

\[ ||P|| = \max \binom{\sum_{1}^{n} |L_j(t)| = \max \binom{\max_{0 < \mu < n} \binom{\tau_{\mu} < i < \tau_{\mu+1}}{s_{\mu}(t)}}{1}. \]
where \( \tau_0 = t_1, \tau_{n+1} = t_{n+k+1} \) and \((s_{\mu})^n_0\) is defined by

\[
\begin{align*}
  s_{\mu}(t_i) &= (-1)^{i+\mu}, & i &= 1, \ldots, \mu, \\
  &= (-1)^{i+\mu}, & i &= \mu + 1, \ldots, n.
\end{align*}
\]

For each \( \mu \), the so-called Lebesgue function \( \Sigma |L_{\mu}(t)| \) coincides with \( s_{\mu}(t) \) on the interval \([\tau_{\mu}, \tau_{\mu+1}]\).

One way of specifying \( \tau \) is to require that the nodes be knot averages, i.e.,

\[
(2.3) \quad \tau_i = \sum_{j=0}^{k+1} w_j t_{i+j}, \quad i = 1, \ldots, n,
\]

where the \( w_j \) are fixed nonnegative numbers which sum to one.

**Theorem 1.** Let \( k \geq 2, m, \) and \((w_j)_{k+1}^0\) be fixed. Let \( t \) be given by (1.1) and \( \tau \) be given by (2.3). If \( \|P\| \) is bounded as \( q \) tends to zero, then

\[
(2.4) \quad w_i > 0, \quad i = 1, \ldots, k.
\]

If the bound on \( \|P\| \) is also independent of \( m \), then either

\[
(2.5) a \quad w_0 = 0 \quad \text{and} \quad (1 - w_1)^k < \frac{1}{2}
\]

or

\[
(2.5) b \quad w_0 > 0 \quad \text{and} \quad \frac{1}{2} < (1 - w_0)^k
\]

and, either

\[
(2.6) a \quad w_{k+1} = 0 \quad \text{and} \quad (1 - w_k)^k < \frac{1}{2}
\]

or

\[
(2.6) b \quad w_{k+1} > 0 \quad \text{and} \quad \frac{1}{2} < (1 - w_{k+1})^k.
\]

Conversely, if (2.4), (2.5), (2.6) hold, then \( \|P\| \) is bounded independently of \( m \) as \( q \) tends to zero.

**Proof.** Let \( w_a \) be the first positive weight and \( w_b \) be the last positive weight, so that \( \tau_i = \sum_{a}^{b} w_j t_{i+j} \), and set

\[
\begin{align*}
  \theta_1 &= (1 - w_a) + (1 - w_a - w_{a+1})q + \cdots + w_b q^{b-a-1}, \\
  \theta_2 &= (1 - w_b) + (1 - w_b - w_{b-1})q + \cdots + w_a q^{b-a-1}.
\end{align*}
\]

If \( a = b \), then \( \theta_1 = \theta_2 = 0 \). If \( a < b \), then \( 0 < \theta_1 < 1 \) and \( 0 < \theta_2 < 1 \) as \( q \) tends to zero. Therefore,

\[
(2.7) \quad t_{i+b} - \theta_2 q^{m+1-b-i} < t_{i+b}, \quad i = 1, \ldots, m - b,
\]

\[
(2.8) \quad t_{i+a} + \theta_1 q^{i+a-m} < t_{i+a+1}, \quad i = m - a + 1, \ldots, n,
\]

for all sufficiently small \( q > 0 \). Since

\[
(2.9) \quad \tau_i = 1 - 2 \sum_{a}^{m-i} w_j + O(q), \quad i = m - b + 1, \ldots, m - a,
\]

as \( q \) tends to zero, it follows that also

\[
(2.9) \quad -1 < \tau_{m-b+1} < \tau_{m-b+2} < \cdots < \tau_{m-a} < +1
\]

for all sufficiently small \( q > 0 \).
Henceforth, we require that \( q \) be such that the inequalities in (2.7) and (2.9) hold. This requirement is independent of \( m \).

Now let \( \|P\| \) be bounded independently of \( m \) as \( q \) tends to zero. We shall prove that (2.4) and (2.6) must hold. A symmetric argument, which we omit, will give (2.5).

Let \( s = s_\mu \) be defined by (2.2) with \( \mu < m - b + 1 \) or \( \mu > m - a - 1 \). There is a constant \( C \) which bounds \( \|P\| \) so that \( \|s\| \leq C \) as \( q \) tends to zero. Since the restriction of \( s \) to \([-1, +1]\) is a polynomial of degree \( k \), it follows from a theorem of A. A. Markov (see [7]) that

\[
\max\{|s'(t)|: -1 < t < 1\} \leq C k^2.
\]

Thus, (2.8), (2.9), and the mean-value theorem imply that

\[
2 = |s(\tau_i) - s(\tau_{i+1})| \leq C k^2 (\tau_{i+1} - \tau_i) \leq 2 C k^2 w_{m-i} + O(q)
\]

for \( i = m - b + 1, \ldots, m - a - 1 \) as \( q \) tends to zero. Thus, \( w_i \geq 1/Ck^2 > 0 \), \( i = a + 1, \ldots, b - 1 \).

Suppose that \( b < k \). Then, on the one hand, (1.2) gives

\[
\pm 1 = s(\tau_1) = \sum_{b}^{m-1} A_j ([j - b] + \theta_2 q^{j-b})^k + \sum_{0}^{k} A_{m+j} (-[m - b] - \theta_2 q^{m-b})
\]

\[
= A_b \theta_2^k + \sum_{b+1}^{m-1} A_j ([j - b] + O(q^{j-b}))
\]

\[
+ \sum_{0}^{k} A_{m+j} (-[m - b]) + O(q^{m-b})
\]

whereas, on the other hand, with \( \Lambda_{i} s \) as in the proof of Lemma 1.2,

\[
0 = \theta_2^k \Lambda_{bb} s + \sum_{b+1}^{k-1} [i - b]^k \Lambda_{i}s + [k]! \Lambda_{kk} s
\]

\[
= A_b \theta_2^k + \sum_{b+1}^{m-1} A_j ([j - b] + O(q^{j-b}))
\]

\[
+ \sum_{0}^{k} A_{m+j} (-[m - b]) + O(q^{m-b})
\]

Subtraction yields

\[
\pm 1 = \sum_{b+1}^{m-1} A_j O(q^{j-b}) + \sum_{0}^{k} A_{m+j} O(q^{m-b}),
\]

so that \( A_j \) cannot be bounded as \( q \) tends to zero. This contradiction to Lemma 1.3 shows that \( b > k \).

A similar argument with \( s(\tau_n) \) shows that \( a < 1 \), so that (2.4) is proved.

To prove (2.6), we first suppose that \( w_{k+1} = 0 \). We must show that \( (1 - w_k)^k < \frac{1}{2} \)
or, equivalently, that

\[
r_2 = \frac{\theta_2^k}{(1 - \theta_2^k)} < 1 \quad \text{as } q \text{ tends to zero}.
\]

Again, let \( s = s_\mu \) be defined by (2.2). Then Lemma 1.2 and (1.2) give

\[
-s(\tau_1) = [k]! \Lambda_{kk} s - s(\tau_1) = \sum_{0}^{m-k-1} M_{0j} A_{k+j} + \sum_{0}^{k} R_{0j} A_{m+j}
\]

\[
r_2 = \frac{\theta_2^k}{(1 - \theta_2^k)} < 1 \quad \text{as } q \text{ tends to zero}.
\]

(2.11) \[-s(\tau_1) = [k]! \Lambda_{kk} s - s(\tau_1) = \sum_{0}^{m-k-1} M_{0j} A_{k+j} + \sum_{0}^{k} R_{0j} A_{m+j}\]
and

\[(2.12) \quad s(\tau_i) - s(\tau_{i+1}) = \sum_{i=1}^{m-k-1} M_{ij} A_{k+j} + \sum_{j=0}^{k} R_{ij} A_{m+j}, \quad i = 1, \ldots, m - k - 1,\]

where

\[M_{ij} = \frac{[k + j]!/[j]! - ([j] + \theta_2 q^j)^k}{j = 0, \ldots, m - k - 1,}\]

\[M_{i,i-1} = \theta_2^k, \quad i = 1, \ldots, m - k - 1,\]

\[M_{ij} = \left([j - i + 1] + \theta_2 q^{j-i+1}\right)^k - \left([j - i] + \theta_2 q^{j-i}\right)^k, \quad i = 1, \ldots, m - k - 1; j = i, \ldots, m - k - 1,\]

\[R_{ij} = \frac{1}{i!} C_{ij} - \tau_i^j, \quad j = 0, \ldots, k,\]

\[R_{ij} = \tau_i - \tau_{i+1}, \quad i = 1, \ldots, m - k - 1; j = 0, \ldots, k,\]

with \(C_{ij} = \frac{t_i^k}{k!} + O(q^{m-k+1})\) as in the proof of Lemma 1.2.

Since the \(A_j\) are bounded and

\[M_{ii} = 1 - \theta_2^k + O(q), \quad i = 0, \ldots, m - k - 1,\]

\[M_{ij} < [k + j]^k - [j]^k < q^k k^k [k + j]^{-1} < q^k (1 - q)^{-k},\]

\[M_{ij} < [j - i + 2]^k - [j - i]^k < q^i i^q (1 - q)^{-k},\]

\[j = i + 1, \ldots, m - k - 1,\]

\[|R_{ij}| < q^{m-k}(j + 1)(1 - q)^j,\]

\[|R_{ij}| < q^{m-k-j}(1 - q)^j,\]

the system (2.11) and (2.12) has the form

\[(1 - \theta_2^k) A_k = -s(\tau_i) + O(q),\]

\[\theta_2^k A_{k+i-1} + (1 - \theta_2^k) A_{k+i} = s(\tau_i) - s(\tau_{i+1}) + O(q), \quad i = 1, \ldots, m - k - 1,\]

which solves as

\[A_{k+i} = \frac{2(-1)^{i+\mu}}{1 - 2\theta_2} \left[1 - \frac{2^{r_{i+1}}}{\theta_2^k} + O(q),\right] \quad i = 0, \ldots, \min(\mu - 1, m - k - 1),\]

\[A_{k+\mu+i} = \frac{2(-1)^{i+1}}{1 - 2\theta_2} \left[1 - \frac{2^{r_{i+1}}}{\theta_2^k} + \frac{2^{r_{i+1}}}{\theta_2^k}\right] + O(q), \quad i = 0, \ldots, m - k - 1 - \mu,\]

if \(r_2 \neq 1\) and as

\[A_{k+1} = (-1)^{i+1}(2 + 4i) + O(q), \quad i = 0, \ldots, \min(\mu - 1, m - k - 1),\]

\[A_{k+\mu+i} = (-1)^{i+1}(2 + 4i - 4\mu) + O(q), \quad i = 0, \ldots, m - k - 1 - \mu,\]

if \(r_2 = 1\) provided that the buildup of \(O(q)\) terms is bounded independently of \(m\). This will be the case if \(qr_2 < 1\) as \(q\) tends to zero, a condition that can be met independently of \(m\). By Lemma 1.3 these \(A_j\) are bounded independently of \(m\). Therefore, (2.10) must hold and \((1 - \theta_2^k)^k < \frac{1}{2}^k\).
To complete the proof of (2.6) we now suppose that $w_{k+1} > 0$. Since $\theta_2 = 1 - w_{k+1} + O(q)$, we must now show that $\theta_2^k > 1/2$ as $q$ tends to zero, that is,

\[ r_2 = \theta_2^k / (1 - \theta_2^k) > 1 \quad \text{as } q \text{ tends to zero.} \]

Since the $\tau_i$ have “moved over one interval”, Eqs. (2.11) and (2.12) are replaced by

\[ -s(\tau_i) = [k]! A_k + \sum_{j=1}^{m-k-1} \left( [k+j]!/j! - \left( [j-1] + \theta_2 q^{j-1}\right)^k \right) A_{k+j} \]

(2.15)

and

\[ s(\tau_i) - s(\tau_{i+1}) = \sum_{j=1}^{m-k-1} M_{i,j-1} A_{k+j} + \sum_{j=0}^{k} R_{ij} A_{m+j}, \]

(2.16)

and the bounds on $M_{ij}$ and $R_{ij}$ are replaced by

\[ \left| [k+j]!/j! - \left( [j-1] + \theta_2 q^{j-1}\right)^k \right| < q^{j-1} k (1 - q)^k, \]

\[ |R_{ij}| < q^{m-k-1} (j + 1)(1 - q)^j, \]

\[ |R_{ij}| < q^{m-k-1} j (1 - q)^j. \]

This incomplete system now has the form

\[ A_k + (1 - \theta_2^k) A_{k+1} = -s(\tau_1) + O(q), \]

\[ \theta_2^k A_{k+1} + (1 - \theta_2^k) A_{k+1} = s(\tau_k) - s(\tau_{k+1}) + O(q), \quad i = 1, \ldots, m - k - 2. \]

Adding the equation

\[ s(\tau_{m-k-1}) = A_{m-k} \theta_2^k + \sum_{j=0}^{k} A_{m+j} \tau_{m-k-1} = A_{m-k} \theta_2^k + s(-1) + O(q) \]

and imposing the restriction $qr_2^{-1} < 1$ permits us to solve this system backwards in terms of $s(-1)$ as

\[ A_{m-i} = \frac{(-1)^{m-1-k-\mu-i}}{2\theta_2^k - 1} \left[ 2 - (1 + r_2) r_2^{-i} \right] + (1 + r_2) (-r_2)^{-i} s(-1) \]

\[ + O(q), \quad i = 1, \ldots, m - k - 1 - \mu, \]

(2.18)

\[ A_{k+\mu+1-i} = \frac{(-1)^{i-1}}{2\theta_2^k - 1} \left[ 2 - (1 + r_2) r_2^{-i} \left( 2 - r_2^{-m+k+\mu+1} \right) \right] \]

\[ + (1 + r_2) (-r_2)^{-m+k+\mu+1-i} s(-1) + O(q), \quad i = 1, \ldots, \mu, \]

if $0 < \mu < m - k - 2$ and as

\[ A_{m-i} = \frac{(-1)^{m-1-k-i}}{2\theta_2^k - 1} \left[ 2 - (1 + r_2) r_2^{-i} \right] \]

(2.19)

\[ + (1 + r_2) (-r_2)^{-i} s(-1) + O(q), \quad i = 1, \ldots, m - k - 1, \]

if $\mu > m - k - 1$. Since the $A_j$ are bounded independently of $m$, (2.14) must hold and $(1 - w_{k+1})^k > \frac{1}{2}$. 


The proof that (2.4), (2.5), (2.6) are necessary conditions for $\|P\|$ to be bounded independently of $m$ as $q$ tends to zero is complete.

To prove that (2.4), (2.5), (2.6) are sufficient that $\|P\|$ be bounded independently of $m$ as $q$ tends to zero, we will use the approach outlined in the proof of Lemma 1.1. That is, we will first show that, for each $s = s_\mu$, the block $A_m, \ldots, A_{m+k}$ is bounded and then argue recursively from bounds on $s(\tau_i)$ (replacing $s(\tau_i)$ in the proof of Lemma 1.1) that $A_{m-i}$ (and $A_{m+k+i}$), $i = 1, \ldots, m-1$, are bounded independently of $m$. Finally, we will use (1.2) and (2.13) or (2.18) or (2.19) to bound $s_\mu(\tau)$ for all $\tau$ and all $\mu$.

If $a = 0$ and $b = k + 1$, the first step, bounding the block $A_m, \ldots, A_{m+k}$ is easy since (2.9) implies that

$$\sum_{i=k+1-b}^{k} A_{m+j} \sigma_{m-k+i} = \pm 1, \quad i = k + 1 - b, \ldots, k - a,$$

and (2.4), (2.8) give a bounded inverse for the Vandermonde matrix $(\sigma_{m-k+i})$. However, if $b = k$ then the $i = 0$ equation of (2.20) is replaced by

$$\theta_k A_{m-1} + \sum_{i=0}^{k} A_{m+j} \sigma_{m-k} = \pm 1.$$

If $a = 1$, there is a similar replacement of

$$\sum_{i=0}^{k} A_{m+j} \sigma_{m} + \theta_k A_{m+k+1} = \pm 1$$

for the $i = k$ equation of (2.20).

Therefore, if $b = k$ (and/or $a = 1$), a preliminary step to eliminate $\theta_k A_{m-1}$ from (2.21), at the expense of adding a bounded quantity to the right member, is necessary. While eliminating $\theta_k A_{m-1}$ through a sequence of upper triangulation steps on (2.11), (2.12), (2.21) is straightforward, there must be an argument that $\theta_k A_{m-1}$ is bounded independently of $m$ as $q$ tends to zero independently of $m$. The following lines supply this argument.

Let $b = k$ and let $s$ be any of the $s_\mu$ given by (2.2). Using the bounds on $M = (M_{ij})$, we see that this matrix is diagonally dominant if $q$ is such that $1 - 02^* > 02^* + kq(1 - q)^{-k-1}$. But (2.5) is equivalent to $1 - 02^* > 02^*$ for sufficiently small $q$, so that this condition can be met by imposing a further restriction on $q$.

Let $q_0 > 0$ and $\delta > 0$ be such that $\delta = 1 - 2\theta_k^* - kq_0(1 - q_0)^{-k-1}$. Then the solutions of a system

$$Mx = b$$

satisfy $\max |x_i| < \delta^{-1} \max |b_i|$ by the usual diagonal dominance argument. Applying this fact with

$$b_0 = \begin{bmatrix} k \\ \Lambda_{kk} s - s(\tau_1) \end{bmatrix} = -s(\tau_1),$$

$$b_i = s(\tau_i) - s(\tau_{i+1}), \quad i = 1, \ldots, m - k - 1,$$

as well as with

$$b_i = -R_{iy}, \quad i = 0, \ldots, m - k - 1,$$
for each \( j = 0, \ldots, k \), yields

\[
A_{m-1} = C + \sum_{j=0}^{k} C_j A_{m+j}
\]

with

\[
|C| < \delta^{-1} \max \{|s(\tau_i)|, |s(\tau_i) - s(\tau_{i+1})|: i = 1, \ldots, m - k - 1\} = 2/\delta,
\]

\[
|C_0| < \delta^{-1} \max_i |R_{00}| = |R_{00}|/\delta = O(q^{m-k}),
\]

\[
|C_j| < \delta^{-1} \max_i |R_{ij}| = |\tau_{m-k-1}^j - \tau_{m-k}^j|/\delta
\]

\[
= |(1 + q + \theta_2 q^2) - (1 + \theta_2 q)|/\delta < q[2] j[3]^{j-1}/\delta
\]

\[
< qj[3]^j/\delta = O(q), \quad j = 1, \ldots, k.
\]

Combining these deductions with (2.21) gives the equation

\[
(2.23) \quad \sum_{j=0}^{k} A_{m+j}(\tau_{m-k}^j + C_j \theta_2^j) = s(\tau_{m-k}) - C \theta_2^k,
\]

which can be adjoined to (2.20). Since \( C_j = O(q) \) and \( \tau_{m-k+1} - \tau_{m-k} = 2w_k + O(q) \), the resulting system has a bounded solution as \( q \) tends to zero. We have assumed that \( a = 0 \). If \( a = 1 \), a similar argument at \( \tau_{m} \) is needed.

We have completed the first step in the proof of sufficiency, i.e., we have shown that the set \( A_m, \ldots, A_{m+k} \) is bounded. But now (2.12) or (2.16) and their symmetric counterparts imply immediately that the set \( A_k, \ldots, A_{2m} \) is bounded. An argument similar to the proof of Lemma 1.2 gives \( O(q^i) \) bounds on \( A_{k-i} \), and \( A_{2m+i}, i = 1, \ldots, k - 1 \). The second step in the proof is completed.

Now we must bound \( s_\mu(t) \) for all \( t \) and all \( \mu \). For \(-1 < t < +1\), the boundedness of \( A_m, \ldots, A_{m+k} \) and (2.4) give a uniform bound on \( s_\mu(t) \). If \( t_1 < t < t_m \), there is a \( \theta_i \) in \( [0, 1] \) and an \( i > 0 \) such that \( t_{m-i} < t = t_{m-i+1} - \theta_i q^i \). Then

\[
s(t) = \sum_{m-i} A_j([i + j - m] + \theta_i q^{i+j-m})^k + \sum_{j=0}^{k} A_{m+j} t^j.
\]

If \( i < m - b \), then \( \tau_{m+1-b-i} = [-i] - \theta_2 q^i \) and

\[
|s(t)| < |s(\tau_{m+1-b-i})| + |A_{m-i}| + O(q) = 1 + |A_{m-i}| + O(q)
\]

can be easily shown. If \( i > m - b \), a modified argument gives

\[
|s(t)| < |s(\tau_i)| + \sum_{j=1}^{k} |A_j| + O(q) = 1 + |A_k| + O(q).
\]

Thus, the \( s_\mu(t) \) are uniformly bounded for all \( \mu \) and all \( t \) so that \( ||P|| \) is bounded independently of \( m \) as \( q \) tends to zero. \( \Box \)

3. Two Special Cases. Theorem 1 provides counterexamples when (2.4), (2.5), (2.6) are not satisfied, e.g., interpolation at the knots with \( k > 2 \) or interpolation at weighted two-knot averages with \( k > 3 \). The condition that \( q \) tend to zero compares (contrasts?) with the often-used condition that the local mesh ratios \((t_{j+1} - t_j)/(t_{i+1} - t_i), |i - j| = 1 \) be bounded.
For \( k > 3 \) and \( q = 1 \), it is easy to select weights \( w_j \) satisfying (2.4), (2.5), (2.6) which still produce unbounded spline interpolation. Thus, even for two-sided \( q \)-splines, these conditions are not sufficient to guarantee bounded interpolation. Indeed, the method of their derivation suggests that they are linked quite closely to the tendency of \( q \) to zero.

For the two special cases which follow it is not clear that we need \( q \) to tend to zero. Computational evidence with small \( k \) suggests, in fact, that \( q \) tending to zero gives "worst-case" results. Thus, Theorems 2 and 3 are imperfect in that the condition that \( q \) tend to zero may be superfluous.

**Theorem 2.** Let \( t \) be given by (1.1) and, for each \( k > 1 \) and \( m > k \), let \( \tau \) be given by
\[
\tau_i = (t_{i+1} + t_{i+2} + \cdots + t_{i+k})/k, \quad i = 1, \ldots, n.
\]
Then, \( \|P\| \) is bounded as \( q \) tends to zero. Moreover, there exist absolute constants \( 1 < C_1 < C_2 \) such that, for each \( k > 2 \),
\[
C_1^k < \|P\| < C_2^k \quad \text{as } q \text{ tends to zero.}
\]

**Theorem 3.** Let \( t \) be given by (1.1) and, for each \( k > 1 \) and \( m > k \), let \( \tau \) be given by (2.3) with
\[
w_0 = w_{k+1} = \sin^2(\alpha_k/2),
\]
\[
w_j = \sin(\alpha_k)\sin(2j\alpha_k), \quad j = 1, \ldots, k,
\]
where \( \alpha_k = \pi/(2k + 2) \). Then, \( \|P\| \) is bounded as \( q \) tends to zero. Moreover, there exist absolute constants \( 0 < C_3 < C_4 \) such that, for each \( k > 2 \),
\[
C_3 \log k < \|P\| < C_4 \log k \quad \text{as } q \text{ tends to zero.}
\]

**Proof of Theorems 2 and 3.** The assertions that \( \|P\| \) is bounded as \( q \) tends to zero are proved by showing that (2.4), (2.5), (2.6) hold. These follow readily, since, in Theorem 2,
\[
(1 - w_k)^k = (1 - w_l)^k = (k - 1)^k/k < 1/e < 3/8
\]
while, in Theorem 3,
\[
(1 - w_{k+1})^k = (1 - w_0)^k = \cos^{2k}(\alpha_k/2) > (1 - \alpha_k^2/8)^{2k} > 1 - \pi\alpha_k/8 > (8k + 3)/(8k + 8) > 3/4.
\]

In Theorem 2, the lower bound on \( \|P\| \) follows from the fact that, as \( q \) tends to zero, the nodes
\[
\tau_{m-k+1}, \tau_{m-k+2}, \ldots, \tau_{m-k+j}, \ldots, \tau_{m-1}
\]
tend to
\[
(2 - k)/k, (4 - k)/k, \ldots, (2j - k)/k, \ldots, (k - 2)/k,
\]
and that, for \( s = s_\mu \) with \( m - k \leq \mu \leq m - 1 \),
\[
|s(\pm 1)| = (1 - r_{2m-k}^{-1})/ (1 - 2\theta_k^m) + \text{O}(q) \geq 1/ (1 - \theta_k^m) + \text{O}(q) > 1,
\]
so that \( \|P\| \) is bounded below by any lower bound for polynomial interpolation on \([-1, +1]\) at the equally-spaced nodes
\[
-1, (2 - k)/k, (4 - k)/k, \ldots, (2j - k)/k, \ldots, (k - 2)/k, +1.
\]
See Rivlin [7, pp. 96–99] for a proof that such polynomial interpolation grows exponentially.

Similarly, in Theorem 3, the lower bound on \( \|P\| \) follows from the fact that \( \tau_{m-k}, \ldots, \tau_m \) approach the Chebyshev nodes \( -\cos(2j\alpha_k - \alpha_k), j = 1, \ldots, k+1 \), as \( q \) tends to zero and the fact that polynomial interpolation on these nodes has logarithmic growth. See [7, pp. 93–96].

To complete the proof that \( \|P\| \) grows exponentially or logarithmically in Theorem 2 or Theorem 3, respectively, it is necessary only to show that, for each \( \mu \), \( s_\mu(t) \) is "controlled" outside \((-1, +1)\). This fact follows from the closing lines of the proof of Theorem 1, where it was noted that, for \( t_1 < t < t_m \), there is a \( j < m \) such that \( |s(t)| < 1 + |A_j| + O(q) \). For Theorem 2, (3.1) and (2.13) imply that

\[
\max_{j<m} |A_j| < \frac{2}{1 - 2\theta_2^k} + O(q) < \frac{2e}{e - 2} + O(q) < 8,
\]

so that \( |s(t)| < 10 \) for \( t < -1 \) as \( q \) tends to zero. For Theorem 3, (3.2) and (2.18), (2.19) imply that

\[
\max_{j<m} |A_j| < \frac{2}{2\theta_2^k - 1} + 2|s(-1)| + O(q) < \frac{2}{2\cos^{2k}(\alpha_k/2) - 1} + 2|s(-1)| + O(q) < 4 + 2|s(-1)| + O(q),
\]

so that \( |s(t)| < 6 + 2|s(-1)| \) for \( t < -1 \) as \( q \) tends to zero. Symmetry considerations give like bounds for \( |s(t)| \) on \(+1 = t_{m+1} < t < t_{2m}\).

The proof of Theorem 2 and Theorem 3 is complete. □

If \( q = 1 \) (not covered by these theorems), two-sided \( q \)-spline interpolation is essentially the same as cardinal spline interpolation, for which logarithmic growth of \( \|P\| \) with \( k \) has been demonstrated; see [6]. This fact supports the conjecture that \( q \) tending to zero gives "worst-case" results for the nodes (3.1).

For cubic spline interpolation with arbitrary knot spacing and the nodes (3.1), de Boor [2] has shown that \( \|P\| < 27 \). He conjectures that \( \|P\| < 3 \) or 4 may be true. The following supplies a lower bound on \( \lim \sup \|P\| \), where the lim sup is taken over all ordered knot spacings.

**Theorem 4.** Let \( k = 3 \) and let \( t \) and \( \tau \) be given by (1.1) and (3.1), respectively. Then

\[
\lim \|P\| = \frac{(222\sqrt{1111} + 999)}{1331} = 2.507825 \ldots,
\]

where \( \lim \|P\| \) denotes the limiting value of \( \|P\| \) as \( q \) tends to zero and \( m \) tends to infinity.

**Proof.** Let \( s = s_\mu \) with \( \mu = m - 1 \). From (2.21) and (2.13)

\[
|s(-1)| = \frac{(-1)^{\mu - m + k}}{1 - 2\theta_2^k} (1 - r_m^{m-k}) + O(q)
\]

(3.3)
for each \( k > 2 \) and \( \mu > m - k \). Similarly,

\[
(3.4) \quad s(t) = \frac{(-1)^{m-1-\mu}}{1 - 2\theta_k^k} (1 - r_k^{m-k}) + O(q)
\]

for \( k > 2 \) and \( \mu < m - 1 \). Thus, for the case presently under consideration, \( s(t) \) tends, on \([-1, +1]\), to the cubic \( p(t) \) satisfying \( p(\pm 1) = 27/11 \) and \( p(\pm 1/3) = \pm 1 \). This cubic is

\[
p(t) = \left( -297t^3 + 243t^2 + 297t - 27 \right)/88.
\]

It has a maximum on \([-1, +1]\) of \( (222\sqrt{111} + 999)/1331 \) at \( t = (9 + 2\sqrt{111})/33 \). Showing that \( \lim ||P|| \) exists and is equal to this maximum requires a discussion (which we omit) similar to the last paragraph in the proof of Theorem 1 above.

For arbitrary \( k \) it is easy to find \( p(t) \), the polynomial which \( s_m(t) \) approaches as \( q \) tends to zero and \( m \) tends to infinity. From (3.1) and (3.4)

\[
\lim s(t) = \frac{1}{1 - 2((k - 1)/k)^k}.
\]

From (3.3), \( \lim s(-1) = (-1)^{k-1}z_k \). Then standard combinatorial formulas give (see Gould [4, p. 59])

\[
p(t) = -(-1)^l \sum_{0}^{l} \frac{(-4)^j T}{T + j} \left( T + j \right) + \frac{2T + k z_k}{k} \left( T + l \right),
\]

if \( k \) is even with \( l = k/2 \) and \( T = lt \), and

\[
p(t) = -(-1)^l \sum_{0}^{l} \frac{(-4)^j 2T}{2j + 1} \left( T + j - 1/2 \right) + \frac{2T + k z_k}{k} \left( T + l - 1/2 \right),
\]

if \( k \) is odd with \( l = (k - 1)/2 \) and \( T = kt/2 \). The maximum of \( p(t) \) on \((k - 2)/k < t < +1\) is a good lower bound on \( ||P|| \) as \( q \) tends to zero and \( m \) tends to infinity.

The following table was computed via double-precision arithmetic in FORTRAN on an Amdahl 470/V7 computer. All entries are rounded down.

<p>| Lower bounds on ( \lim \sup ||P|| ) |
|---|---|---|---|---|---|</p>
<table>
<thead>
<tr>
<th>( k )</th>
<th>( \max p(t) )</th>
<th>( k )</th>
<th>( \max p(t) )</th>
<th>( k )</th>
<th>( \max p(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2.0000</td>
<td>7</td>
<td>7.7939</td>
<td>12</td>
<td>9.02 \times 10^5</td>
</tr>
<tr>
<td>3</td>
<td>2.5078</td>
<td>8</td>
<td>11.8194</td>
<td>15</td>
<td>5.13 \times 10^5</td>
</tr>
<tr>
<td>4</td>
<td>3.0814</td>
<td>9</td>
<td>18.7344</td>
<td>18</td>
<td>3.17 \times 10^6</td>
</tr>
<tr>
<td>5</td>
<td>3.9686</td>
<td>10</td>
<td>30.7986</td>
<td>21</td>
<td>2.05 \times 10^6</td>
</tr>
<tr>
<td>6</td>
<td>5.4087</td>
<td>11</td>
<td>52.1254</td>
<td>24</td>
<td>1.37 \times 10^8</td>
</tr>
</tbody>
</table>

This table, in which the exponential growth is clear, is associated with Theorem 2 above. A corresponding table of lower bounds on \( \lim \sup ||P|| \) for the node assignment of Theorem 3 can be computed from the fact that the Lebesgue function for polynomial interpolation on the Chebyshev nodes attains its maximum.
at $t = 1$; see [7, Eq. (4.2.19)]. The first few entries of such a table are:

\[(1,1.414)\quad (2,1.666)\quad (3,1.847)\quad (4,1.988)\quad (5,2.104)\quad (6,2.202).\]

A later entry is (35,3.243). The logarithmic growth is clear. For $k = 1$ with arbitrary knots it can be shown that $\|P\| \leq \sqrt{2}$ when nodes are specified by (3.2) above. Whether the other bounds are "good" bounds for the arbitrary knot case is problematical.

4. Remarks. For one-sided $q$-splines with spline knots $t_i = (1 - q^i)/(1 - q)$, $i = \ldots, -1, 0, 1, 2, \ldots$, and interpolation nodes $\tau_i = t_i + \theta q^i$, where $\theta$ is fixed, $0 < \theta < 1$, S. L. Lee [5] has considered eigensplines, i.e., nontrivial splines $s(t)$ satisfying $s(t) = \lambda s(1 + qt)$ for some fixed eigenvalue $\lambda$. Setting $\lambda = -1$ yields, for each $k \geq 2$, a certain equation $F_k(q, \theta) = 0$. If $q$ and either $\theta_1$ or $\theta_2$ defined above satisfy this equation, then two-sided $q$-spline interpolation is unbounded. Lee [5] has shown that $F_k(0 + \theta, \theta) = C[2\theta^k - 1][2(1 - \theta)^k - 1]$.

For quadratic splines with arbitrary knots $t_i$, Demko [3] has shown that interpolation is bounded independently of $t_i$ and $\tau_i$ if the nodes $\tau_i$ satisfy $\tau_i = t_{i+2} - \lambda_i(t_{i+2} - t_{i+1})$ with $\lambda^2 \leq \gamma < \frac{1}{2}$ and $(1 - \lambda)^2 \leq \gamma < \frac{1}{2}$. Consequently, for $k = 2$, the results of Theorem 1 above with (2.5)a and (2.6)a are valid for all $q$ and not just as $q$ tends to zero.

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5. S. L. Lee, private communication.