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Inequalities for Certain Hypergeometric Functions

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Abstract. Theorems on two-sided inequalities for Gauss and Kummer's hypergeometric functions as given by Buschman have been improved. Complex analogues of the said inequalities have been developed and it is pointed out that a similar analysis gives extensions of Luke's, Flett's, and Carlson's theorems.

1. Let $F(\alpha)$ denote the hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; x)$. A contiguous relation for $F(\alpha)$, see, for instance, [6, 2.8(28)] wherein α is replaced by $\alpha + n$ can be rewritten as

$$F(\alpha + n + 1) = A_n F(\alpha + n) + B_n F(\alpha + n - 1),$$

where

$$(1.1)\quad A_n = \left(\frac{\beta x - \gamma}{\alpha + n} + 2 - x\right) \Big/ (1 - x), \qquad B_n = \left(\frac{\gamma}{\alpha + n} - 1\right) \Big/ (1 - x).$$

The recursion formula (1.1) enables us to represent $F(\alpha + n + 1)$ by an $(n + 2) \times (n + 2)$ determinant

$$(1.2)\qquad F(\alpha + n + 1) = \det \begin{bmatrix} F(\alpha) & -F(\alpha - 1) & & & & & \\ B_0 & A_0 & -1 & & & & \\ & B_1 & A_1 & -1 & & & \\ & & & \ddots & & & \\ & & & B_{n-1} & A_{n-1} & -1 \\ & & & & B_n & A_n \end{bmatrix}.$$

The determinant (1.2) will have a strictly dominant diagonal provided that

$$(1.3)\qquad \begin{array}{ll} \text{(i)} & |F(\alpha)| > |F(\alpha - 1)|, \\ \text{(ii)} & |A_k| > |B_k| + 1, \qquad \text{for } 0 < k < n, \\ \text{(iii)} & |A_n| > |B_n|. \end{array}$$

Assuming that α, β, γ and x all are positive real numbers, an examination of power series representations of $F(\alpha)$ and $F(\alpha - 1)$ with respect to x shows that

$$|F(\alpha)| > |F(\alpha - 1)| \quad \text{with } 0 < x < 1,$$

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provided that

$$(1.4)\qquad \left|\frac{\alpha-1}{\alpha+n-1}\right| \not> 1 \quad \text{for } n \geqslant 1 \quad \text{and} \qquad \left|\frac{\alpha-1}{\alpha+n-1}\right| < 1 \quad \text{for at least one value of } n.$$

It therefore follows that 1.3(i) holds for $\alpha \geqslant \frac{1}{2}$. Also, obviously 1.3(iii) will be valid under the same sets of conditions for which 1.3(ii) is valid. Now in order that 1.3(ii) may hold, in the first place, for $\gamma < \alpha$, it is sufficient that $\beta x > 0$, which is obvious since β and x are both positive real numbers.

In the next place, consider the situation $\gamma > \alpha$. Let $\gamma > \alpha + k$ for some positive integer k. The inequality

$$(1.5)\qquad |A_k| > |B_k| + 1$$

will be satisfied for

$$(1.6)\qquad \beta > \beta x > \max\{\gamma, 2(\gamma - \alpha)\} > 0.$$

Indeed this is so since in this case

$$|A_k| = \left[\frac{\beta x - \gamma}{\alpha + k} + 2 - x\right] \Big/ (1 - x), \qquad |B_k| = \left(\frac{\gamma}{\alpha + k} - 1\right) \Big/ (1 - x).$$

If $\gamma > \alpha + n$, nothing remains to say, but if $\alpha < \gamma \leqslant \alpha + n$, there exists a nonnegative integer k_0 such that $\alpha + k_0 < \gamma \leqslant \alpha + k_0 + 1$. Thus when $k > k_0$, (1.5) holds for $\beta x > 0$, and when $k \leqslant k_0$, (1.5) holds under the conditions (1.6). Thus, the sufficient conditions under which 1.3(i)–1.3(iii) hold may be summarized as

$$(1.7)\qquad \alpha \geqslant \tfrac{1}{2}, \quad \gamma \leqslant \alpha \quad \text{or} \quad \beta > \beta x > \max\{\gamma, 2(\gamma - \alpha)\} > 0.$$

Buschman [3] has claimed that (1.3) holds if all α, β, γ and x are real and positive and satisfy the set of conditions $\alpha > 1$, $\beta > \beta x > 2\gamma > 0$. A closer examination clearly reveals that our conditions are much weaker than those given by Buschman and hence one can expect to get estimates in a wider range.

Thus under the conditions (1.7), by the theorem of G. B. Price [9] we have

$$(1.8)\qquad \begin{aligned} A_n[F(\alpha) - |F(\alpha-1)|] \prod_{k=0}^{n-1} (A_k - 1) &< {}_2F_1(\alpha + n + 1, \beta; \gamma; x) \\ &< A_n[F(\alpha) + |F(\alpha-1)|] \prod_{k=0}^{n-1} (A_k + 1), \end{aligned}$$

where the absolute value symbols on $F(\alpha)$ and A_k's, $k = 0, \ldots, n$, have been dropped because of our assumptions. Further, the absolute value symbol on $F(\alpha - 1)$ can also be dropped by recourse to Erber's formula [5, (11)], which for real parameters and variables can be rewritten as

$$(1.9)\qquad |{}_2F_1(a, b; c; z)| \leqslant {}_2F_1(|a|, |b|; |c|; |z|); \qquad |z| < 1.$$

Consequently

$$(1.10)\qquad |F(\alpha - 1)| = |{}_2F_1(\alpha - 1, \beta; \gamma; x)| \leqslant {}_2F_1(|\alpha - 1|, \beta; \gamma; x).$$

Hence converting products into gamma-functions, the result (1.8) along with (1.10) enables us to write the modified version of Buschman's Theorem 1 in the following form.

THEOREM 1. *If* $\alpha \geqslant \frac{1}{2}$, $\gamma \leqslant \alpha$ *or* $\beta > \beta x > \max\{\gamma, 2(\gamma - \alpha)\} > 0$, *then*

$$g(x)L < {}_2F_1(\alpha + n + 1, \beta; \gamma; x) < g(x)U,$$

where

$$g(x) = (1 - x)^{-n-1}(\beta x - \gamma + (2 - x)(\alpha + n))\Gamma(\alpha)/\Gamma(\alpha + n + 1),$$

$$(1.11)\quad U = [F(\alpha) + F(|\alpha - 1|)](3 - 2x)^n \Gamma\left(\frac{\beta x - \gamma}{3 - 2x} + \alpha + n\right)\Big/\Gamma\left(\frac{\beta x - \gamma}{3 - 2x} + \alpha\right),$$

$$L = [F(\alpha) - F(|\alpha - 1|)]\Gamma(\beta x - \gamma + \alpha + n)/\Gamma(\beta x - \gamma + \alpha).$$

It is observed here that for $\alpha > 1$, $F(|\alpha - 1|) = F(\alpha - 1)$, and therefore U and L of (1.11) correspond to those of Theorem 1 of Buschman [3]. Further, by using the bounds for the determinant (1.2) as given by Brenner [1], we have

THEOREM 2. *If* $\alpha \geqslant \frac{1}{2}$, $\gamma \leqslant \alpha$ *or* $\beta > \beta x > \max\{\gamma, 2(\gamma - \alpha)\} > 0$, *then*

$$L' < {}_2F_1(\alpha + n + 1, \beta; \gamma; x) < U',$$

where

$$(1.12)\quad \begin{aligned} L' &= \left(\frac{F^2(\alpha) - F^2(\alpha - 1)}{F(\alpha)}\right)\prod_{k=0}^{n-1}\left(A_k - \frac{|B_k| + 1}{A_k}\right), \\ U' &= A_n\left(\frac{F^2(\alpha) + F^2(\alpha - 1)}{F(\alpha)}\right)\prod_{k=0}^{n-1}\left(A_k + \frac{|B_k| + 1}{A_k}\right). \end{aligned}$$

It should be noted that the absolute value symbol on B_k's can be dropped when $\gamma > \alpha + n$. Note also that whereas Theorem 2 gives improved lower and upper bounds for ${}_2F_1(\alpha + n + 1, \beta; \gamma; x)$ over Theorem 1, Theorem 1 is more suitable in applications because of its simplicity.

Improvements over Theorems 1 and 2, though in a restrictive domain, may further be obtained in the light of the suggestions made by Srivastava and Brenner [10] by writing the determinant (1.2) in the alternate form

$$(1.13)\quad F(\alpha + n + 1) = \det\begin{bmatrix} F(\alpha) & -F(\alpha - 1)\sqrt{B_0} & & & \\ \sqrt{B} & A_0 & & -\sqrt{B_1} & \\ & & \ddots & & \\ & & \sqrt{B_{n-1}} & A_{n-1} & -\sqrt{B_n} \\ & & & \sqrt{B_n} & A_n \end{bmatrix}.$$

Thus, for example, the inequality corresponding to Theorem 1 may be stated as follows:

THEOREM 3. *If* $\alpha \geqslant \frac{1}{2}$, *and either* $\alpha\beta > \beta\gamma > \alpha\beta x > \gamma > 0$, *or* $\alpha(1 - x) > \gamma - \alpha > 0$, $\beta x > \max\{\gamma, 2(\gamma - \alpha)\} > 0$, *then*

$$L'' < {}_2F_1(\alpha + n + 1; \beta; \gamma; x) < U'',$$

where

$$L'' = A_n[F(\alpha) - F(|\alpha - 1|)|\sqrt{B_0}|] \prod_{k=0}^{n-1} (A_k - |\sqrt{B_{k+1}}|), \tag{1.14}$$
$$U'' = A_n[F(\alpha) + F(|\alpha - 1|)|\sqrt{B_0}|] \prod_{k=0}^{n-1} (A_k + |\sqrt{B_{k+1}}|).$$

If $1 < \alpha < \gamma$, the ${}_2F_1$'s in the bounds of the above listed theorems can further be approximated by application of Luke's [8, 4.21, 4.23], Carlson's [4] or Flett's [7] theorems to obtain inequalities in terms of parameters and variables.

Proceeding as before, an improved version of Theorem 2 of Buschman [3] can be stated as

THEOREM 4. *If* $\alpha \geqslant \frac{1}{2}$, $\alpha \geqslant \gamma > 0$ *or* $x > \max\{\gamma, 2(\gamma - \alpha)\} > 0$, *then*

$$h(x)B < {}_1F_1(\alpha + n + 1; \gamma; x) < h(x)A,$$

where

$$h(x) = (x - \gamma + 2(\alpha + n))\Gamma(\alpha)/\Gamma(\alpha + n + 1),$$

$$A = [{}_1F_1(\alpha; \gamma; x) + {}_1F_1(|\alpha - 1|; \gamma; x)]3^n\Gamma\left(\frac{x-\gamma}{3} + \alpha + n\right)\Big/\Gamma\left(\frac{x-\gamma}{3} + \alpha\right),$$

$$B = [{}_1F_1(\alpha; \gamma; x) - {}_1F_1(|\alpha - 1|; \gamma; x)]\Gamma(x - \gamma + \alpha + n)/\Gamma(x - \gamma + \alpha).$$

Also, by the same analysis, it is found that Theorem 3 of Buschman, which gives bounds for the confluent hypergeometric function Ψ, is valid in a larger domain $2c - 1 > a > 0$, $x > 0$.

2. The Case of Complex Parameters and Variables. Erber [5] observed that for $n > 0$,

$$|(\alpha)_n| \leqslant (|\alpha|)_n, \quad |(\alpha)_n| \geqslant (\cos(\theta/2))^{n-1}(|\alpha|)_n, \quad \theta = \arg \alpha, |\theta| < \pi, \tag{2.1}$$

and used these to obtain

$$|{}_2F_1(\alpha, \beta; \gamma; z)| \leqslant \cos(\theta/2){}_2F_1(|\alpha|, |\beta|; |\gamma|; |z|\sec \theta/2), \tag{2.2}$$

where $\theta = \arg \gamma$, $|\theta| < \pi$, and $|z| < \cos(\theta/2)$. From (2.1) we can also have

$$|{}_pF_q(\alpha_p; \beta_q; z)| \leqslant \Pi \cos(\theta_q/2){}_pF_q(|\alpha_p|; |\beta_q|; |z|\Pi \sec(\theta_q/2)), \tag{2.3}$$

where $\theta_q = \arg(\beta_q)$, $|\theta_q| < \pi$, $|z| < \Pi \cos(\theta_q/2)$, and as usual Π stands for the product symbol. If $p \leqslant q$, the condition $|z| < \Pi \cos(\theta_q/2)$ in (2.3) can be dropped.

With the help of (2.2) and the triangle inequality $|\alpha + n| < |\alpha| + n$, n being any nonnegative integer, extensions of Theorems 1, 3, and 4 for complex parameters and arguments can be obtained. For reasons of brevity we shall however state only the extension of Theorem 1.

THEOREM 5. *If* a, b, c, *and* z *are complex numbers and* $\theta = \arg c$, $|\theta| < \pi$, $|z| < \cos(\theta/2)$, *then*

$$|{}_2F_1(a + n + 1, b; c; z)| < \cos(\theta/2)U \cdot g(z),$$

where

$$g(z) = \frac{\left[1 - |z|\sec(\theta/2)\right]^{-n-1}}{(|a|)_{n+1}} \left[|bz|\sec(\theta/2) - |c| + (2 - |z|\sec(\theta/2))(|a| + n)\right],$$

$$U = \left\{{}_2F_1(|a|, |b|; |c|; |z|\sec(\theta/2)) + {}_2F_1(|\,|a| - 1|, |b|; |c|; |z|\sec(\theta/2))\right\}$$
$$\cdot (3 - 2|z|\sec(\theta/2))^n((|bz|\sec(\theta/2) - |c|)/(3 - 2|z|\sec(\theta/2)) + |a|)_n,$$

provided

$$(2.4) \qquad |a| \geqslant \tfrac{1}{2}, \quad |c| \leqslant |a| \quad \text{or} \quad |b| > |bz|\sec(\theta/2) > \max\{|c|, 2(|c| - |a|)\} > 0.$$

In the sequel, complex analogues of inequalities of Luke [8, 4.21, 4.23, 5.6, 5.8] and those of Flett [7] and Carlson [4] could also be given similarly.

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