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A Note on the Stable Decomposition of Skew-Symmetric Matrices*

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Abstract. Computationally stable decompositions for skew-symmetric matrices, which take advantage of the skew-symmetry in order to halve the work and storage, are presented for solving linear systems of equations.

1. Introduction. We shall consider here the problem of solving $Ax = b$ on the computer, where A is either skew-symmetric ($A^T = -A$) or skew-Hermitian ($\overline{A}^T = -A$). We seek a generalization of the LU decomposition in order to obtain a stable decomposition which takes advantage of $A^T = -A$ (or $\overline{A}^T = -A$) so that the work and storage are halved. Although skew matrices do not occur as frequently as symmetric matrices, they are occasionally of interest [7], [9], [10], [12].

If A is $n \times n$ (real or complex) skew-symmetric, then the diagonal of A is null. Since $\det A = \det A^T = \det(-A) = (-1)^n \det A$, we have $\det A = 0$ if n is odd. If A^{-1} exists, then A^{-1} is also skew-symmetric.

If A is $n \times n$ skew-Hermitian, then the diagonal of A is purely imaginary but need not be null, e.g.,

$$A = \begin{bmatrix} i & -1+i \\ 1+i & 2i \end{bmatrix},$$

where $i = \sqrt{-1}$. Since $\overline{\det A} = \det(\overline{A}^T) = \det(-A) = (-1)^n \det A$, we have $\operatorname{Re}(\det A) = 0$ if n is odd and $\operatorname{Im}(\det A) = 0$ if n is even. If A^{-1} exists, then A^{-1} is skew-Hermitian.

2. Decomposition of Skew-Symmetric Matrices. Let A be a real or complex skew-symmetric matrix. We may generalize the diagonal pivoting method for symmetric matrices [2], [4], [5], [6], [8] as follows. First, partition A as

$$\left[\begin{array}{c|c} S & -C^T \\ \hline C & B \end{array}\right],$$

where S is $k \times k$, C is $(n-k) \times k$, and B is $(n-k) \times (n-k)$; clearly, S and B are skew-symmetric. If S and C are null, then we go on to B. If S is nonsingular, then

$$A = \left[\begin{array}{c|c} S & -C^T \\ \hline C & B \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline CS^{-1} & I \end{array}\right]\left[\begin{array}{c|c} S & 0 \\ \hline 0 & B + CS^{-1}C^T \end{array}\right]\left[\begin{array}{c|c} I & -S^{-1}C^T \\ \hline 0 & I \end{array}\right].$$

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But $B + CS^{-1}C^T$ is once again skew-symmetric. Hence, we need store only the strictly lower (or upper) triangular part of A and can overwrite those elements with the multipliers in CS^{-1} (or $-S^{-1}C^T$) and the strictly lower (or upper) triangular part of $B + CS^{-1}C^T$. Note that $(CS^{-1})^T = -S^{-1}C^T$ since $S^{-T} = -S^{-1}$.

Since $\operatorname{diag}(A) = 0$, we cannot take $k = 1$ unless the first column of A (and hence the first row) is null. Otherwise, we have $k = 2$ and

$$S = \begin{bmatrix} 0 & -a_{21} \\ a_{21} & 0 \end{bmatrix};$$

if $a_{21} \neq 0$, then S is nonsingular. If $a_{21} = 0$ but $a_{i1} \neq 0$ for some i, $2 \leqslant i \leqslant n$, then we can interchange the ith and second row and column of A, so that

$$A = P_1 \left[\begin{array}{c|c} S & -C^T \\ \hline C & B \end{array}\right] P_1$$

and S is nonsingular; $P_1 = P_1^T$ is obtained by interchanging the ith and first column of the identity matrix.

Thus, if the first column of A is null, take $P_1 = I$, $S = 0$ is 1×1, $C = 0$ is an $(n-1)$-vector, and we go directly to B. If the first column of A is not null, then A is 2×2 and nonsingular, C is $(n-2) \times 2$, and the reduced matrix is $B + CS^{-1}C^T$. Then we repeat this procedure for $B = -B^T$ of order $n - 1$ in the former case and for $B + CS^{-1}C^T = -(B + CS^{-1}C^T)^T$ of order $n - 2$ in the latter case.

In conclusion, we have

$$A = P_1 M_1 P_2 M_2 \cdots P_{n-1} M_{n-1} D \tilde{M}_{n-1} P_{n-1} \cdots \tilde{M}_2 P_2 \tilde{M}_1 P_1,$$

where P_j is the identity matrix or a permutation matrix, M_j is the identity matrix or a block unit lower triangular matrix containing two columns of multipliers in its jth and $(j+1)$st columns and $(j+2)$nd through nth rows, $\tilde{M}_j = M_j^T$, and D is skew-symmetric block diagonal with 1×1 and 2×2 diagonal blocks—all 1×1 blocks are zero and all 2×2 blocks are nonsingular. (If n is odd, then there is at least one 1×1 block.) Thus, we have reduced the skew matrix A to a block diagonal skew matrix D by a sequence of permutations and congruence transformations. Of course, all relevant elements of the M_j (or $\tilde{M}_j$) and D could be stored in the corresponding strictly lower (or upper) triangular part of A. One n-vector could store the relevant information in the permutations P_j.

Counting divisions as multiplications, the decomposition requires $\frac{1}{6}n^3 - \frac{1}{4}n^2 - \frac{1}{6}n$ multiplications and $\frac{1}{6}n^3 - \frac{3}{4}n^2 + \frac{5}{6}n$ additions if n is even, and $\frac{1}{6}n^3 - \frac{1}{2}n^2 + \frac{5}{6}n - \frac{1}{2}$ multiplications and $\frac{1}{6}n^3 - n^2 + \frac{11}{6}n - 1$ if n is odd. The number of comparisons is at most $\frac{1}{2}n^2 - \frac{1}{2}n$. Given the decomposition of A, we can now solve $Ax = b$ with $n^2 + \mathcal{O}(n)$ multiplications and additions.

3. Stability of the Decomposition. In order to have a stable decomposition, we need to ensure that catastrophic element growth in the reduced matrices does not occur from one step to the next [2], [13], [14]. No element growth occurred whenever S was 1×1. Let us now consider the case when S is 2×2.

Let

$$S = \begin{bmatrix} 0 & -a_{21} \\ a_{21} & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} S & -C^T \\ C & B \end{bmatrix}.$$

Then a row of CS^{-1} is

$$[a_{i1}, a_{i2}]\begin{bmatrix} 0 & \dfrac{1}{a_{21}} \\ -\dfrac{1}{a_{21}} & 0 \end{bmatrix} = [-a_{i2}/a_{21}, a_{i1}/a_{21}],$$

and an element of $A^{(3)} \equiv B + CS^{-1}C^T$ is of the form

$$a_{ij}^{(3)} = a_{ij} - \left(\frac{a_{i2}}{a_{21}}\right)a_{j1} + \left(\frac{a_{i1}}{a_{21}}\right)a_{j2}.$$

Thus, if $|a_{21}| = \max_{2\leq i\leq n}\{|a_{i1}|, |a_{i2}|\}$, then $|a_{ij}^{(3)}| \leq 3\max_{r,s}|a_{rs}|$ and $\max_{i,j}|(B + CS^{-1}C^T)_{ij}| \leq 3\max_{r,s}|a_{rs}|$. We can ensure this by interchanging the kth and second row and column if $|a_{k1}| = \max_{2\leq i\leq n}\{|a_{i1}|, |a_{i2}|\}$ or by interchanging the second and first row and column and then the kth and second row and column if $|a_{k2}| = \max_{2\leq i\leq n}\{|a_{i1}|, |a_{i2}|\}$.

If we do this at each step, then the element growth factor, the largest element (in modulus) in all the reduced matrices divided by $\max_{r,s}|a_{rs}|$, is bounded by

$$\left\{\begin{matrix} 3^{n/2-1} & \text{if } n \text{ is even} \\ 3^{(n-1)/2-1} & \text{if } n \text{ is odd} \end{matrix}\right\} \leq \left(\sqrt{3}\right)^{n-2} < (1.7321)^{n-2}.$$

This requires $\frac{1}{2}n^2 - \frac{1}{2}n$ comparisons, and is a partial pivoting strategy; cf. [4], [5], [13], [14]. The partial pivoting strategy for the diagonal pivoting method in the symmetric case gives a bound of $(2.57)^{n-1}$ [4], [5].

We can obtain a smaller bound on the element growth factor by employing a complete pivoting strategy. If $|a_{pq}| = \max_{r>s}\{|a_{rs}|\}$, then we can move the (p, q) element to the (2, 1) position symmetrically by interchanging the qth and first row and column and then the pth and second row and column. This requires at most $\frac{1}{12}n^3 + \frac{1}{8}n^2 - \frac{1}{12}n$ comparisons. By an analysis identical to Wilkinson's for Gaussian elimination with complete pivoting [13], we obtain the same bound as his on the element growth factor: $< \sqrt{n}f(n)$, where

$$f(n) = \left(\prod_{k=2}^{n} k^{1/(k-1)}\right)^{1/2} < 1.8n^{(\ln n)/4},$$

$f(100) \approx 330$. This compares with the bound of $3nf(n)$ for the complete pivoting strategy for the diagonal pivoting method in the symmetric case [2], [6].

4. Another Stable Decomposition for Skew-Symmetric Matrices. The other well-known stable decomposition for symmetric matrices is the tridiagonal decomposition developed by Aasen [1] and Parlett and Reid [11]. It decomposes $A = A^T$ as

$$A = P_2L_2 \cdots P_nL_nTL_n^TP_n \cdots L_2^TP_2,$$

where the P_j are permutation matrices, the L_j are unit lower triangular, and T is symmetric tridiagonal. It requires $\frac{1}{6}n^3 + \mathcal{O}(n^2)$ multiplications and additions, and $\frac{1}{2}n^2 + \mathcal{O}(n)$ comparisons; the bound on element growth is 4^{n-2} [3, p. 525].

If A is skew-symmetric, then, by modifying Aasen's algorithm in a manner similar to Section 2, we obtain

$$A = P_2L_2 \cdots P_nL_nT\tilde{L}_nP_n \cdots \tilde{L}_2P_2,$$

where the P_j and L_j are as above, $\tilde{L}_j = L_j^T$, but T is now skew-symmetric tridiagonal (with a null diagonal). It requires $\frac{1}{6}n^3 + \mathcal{O}(n^2)$ multiplications and additions, and $\frac{1}{2}n^2 + \mathcal{O}(n)$ comparisons; but now the bound on element growth is 3^{n-2} (this follows from [3, p. 525], since the diagonal of A is null).

5. Stable Decomposition of Skew-Hermitian Matrices. If A is skew-Hermitian ($\bar{A}^T = -A$), Aasen's algorithm gives

$$A = P_2 L_2 \cdots P_n L_n T \tilde{L}_n P_n \cdots \tilde{L}_2 P_2,$$

where the P_j and L_j are as above, $\tilde{L}_j = \bar{L}_j^T$, but T is now skew-Hermitian. Since the diagonal of A is not necessarily null, element growth is bounded by 4^{n-2}.

However, when A is skew-Hermitian, we cannot use the techniques of Sections 2 and 3 since the diagonal of A is now not necessarily null. But, if A is skew-Hermitian, then $B = iA$ is Hermitian since $\bar{B}^T = -i\bar{A}^T = -i(-A) = iA = B$. Since $B = iA$ is Hermitian, we can use the stable decomposition for Hermitian matrices [4], [5], [6] and the subroutines in LINPACK [8], obtaining a stable decomposition with $\frac{1}{6}n^3 + \mathcal{O}(n^2)$ multiplications and additions, and $\geqslant \frac{1}{2}n^2$ but $\leqslant n^2$ comparisons with a partial pivoting strategy as implemented in LINPACK [8], or $\geqslant \frac{1}{12}n^3$ but $\leqslant \frac{1}{6}n^3$ comparisons with a complete pivoting strategy [2], [6]. The element growth factor is bounded by $(2.57)^{n-1}$ for the partial pivoting strategy and $3nf(n)$ for the complete pivoting strategy. The decomposition can now be used to solve $Ax = b$ with $n^2 = \mathcal{O}(n)$ multiplications and additions (by solving $Bx = ib$).

6. Remarks. We could do the same thing when A is real skew-symmetric, but $B = iA$ is then *complex* (Hermitian). The algorithms in Sections 2–4 show how we may stay in real arithmetic with stable decompositions based on congruence transformations. Since the nonzero eigenvalues of a real skew-symmetric matrix occur in purely imaginary complex conjugate pairs ($\pm i\mu_j$ where the μ_j are positive), the "inertia" (π, ν, ζ) of A (defined to be the number of positive, negative, and zero imaginary parts of the eigenvalues of A) is $((n - \zeta)/2, (n - \zeta)/2, \zeta)$. If A is also nonsingular then its "inertia" is $(n/2, n/2, 0)$. This fixed inertia property is why skew-symmetric matrices are easier to decompose than symmetric indefinite matrices. We have an immediate modification of Sylvester's Inertia Theorem to skew-symmetric matrices: if A is skew-symmetric, then $B = MAM^T$ is skew-symmetric and B has the same "inertia" as A, where M is nonsingular.

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