A Note on the Stable Decomposition of Skew-Symmetric Matrices*

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Abstract. Computationally stable decompositions for skew-symmetric matrices, which take advantage of the skew-symmetry in order to halve the work and storage, are presented for solving linear systems of equations.

1. Introduction. We shall consider here the problem of solving $Ax = b$ on the computer, where $A$ is either skew-symmetric ($A^T = -A$) or skew-Hermitian ($A^T = -A$). We seek a generalization of the LU decomposition in order to obtain a stable decomposition which takes advantage of $A^T = -A$ (or $A^T = -A$) so that the work and storage are halved. Although skew matrices do not occur as frequently as symmetric matrices, they are occasionally of interest [7], [9], [10], [12].

If $A$ is $n \times n$ (real or complex) skew-symmetric, then the diagonal of $A$ is null. Since $\det A = \det A^T = \det(-A) = (-1)^n \det A$, we have $\det A = 0$ if $n$ is odd. If $A^{-1}$ exists, then $A^{-1}$ is also skew-symmetric.

If $A$ is $n \times n$ skew-Hermitian, then the diagonal of $A$ is purely imaginary but need not be null, e.g.,

$$A = \begin{bmatrix} i & -1 + i \\ 1 + i & 2i \end{bmatrix},$$

where $i = \sqrt{-1}$. Since $\det A = \det(\overline{A}^T) = \det(-A) = (-1)^n \det A$, we have $\text{Re}(\det A) = 0$ if $n$ is odd and $\text{Im}(\det A) = 0$ if $n$ is even. If $A^{-1}$ exists, then $A^{-1}$ is skew-Hermitian.

2. Decomposition of Skew-Symmetric Matrices. Let $A$ be a real or complex skew-symmetric matrix. We may generalize the diagonal pivoting method for symmetric matrices [2], [4], [5], [6], [8] as follows. First, partition $A$ as

$$A = \begin{bmatrix} S & -C^T \\ C & B \end{bmatrix},$$

where $S$ is $k \times k$, $C$ is $(n-k) \times k$, and $B$ is $(n-k) \times (n-k)$; clearly, $S$ and $B$ are skew-symmetric. If $S$ and $C$ are null, then we go on to $B$. If $S$ is nonsingular, then

$$A = \begin{bmatrix} S & -C^T \\ C & B \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & B + CS^{-1}C^T \end{bmatrix} \begin{bmatrix} I & -S^{-1}C^T \\ 0 & I \end{bmatrix}.$$
But $B + CS^{-1}C^T$ is once again skew-symmetric. Hence, we need store only the strictly lower (or upper) triangular part of $A$ and can overwrite those elements with the multipliers in $CS^{-1}$ (or $-S^{-1}C^T$) and the strictly lower (or upper) triangular part of $B + CS^{-1}C^T$. Note that $(CS^{-1})^T = -S^{-1}C^T$ since $S^{-T} = -S^{-1}$.

Since $\text{diag}(A) = 0$, we cannot take $k = 1$ unless the first column of $A$ (and hence the first row) is null. Otherwise, we have $k = 2$ and

$$S = \begin{bmatrix} 0 & -a_{21} \\ a_{21} & 0 \end{bmatrix},$$

if $a_{21} \neq 0$, then $S$ is nonsingular. If $a_{21} = 0$ but $a_{ij} \neq 0$ for some $i, 2 \leq i \leq n$, then we can interchange the $i$th and second row and column of $A$, so that

$$A = P_1 \begin{bmatrix} S & -C^T \\ C & B \end{bmatrix} P_1$$

and $S$ is nonsingular; $P_1 = P_1^T$ is obtained by interchanging the $i$th and first column of the identity matrix.

Thus, if the first column of $A$ is null, take $P_1 = I$, $S = 0$ is $1 \times 1$, $C = 0$ is an $(n - 1)$-vector, and we go directly to $B$. If the first column of $A$ is not null, then $A$ is $2 \times 2$ and nonsingular, $C$ is $(n - 2) \times 2$, and the reduced matrix is $B + CS^{-1}C^T$. Then we repeat this procedure for $B = -B^T$ of order $n - 1$ in the former case and for $B + CS^{-1}C^T = -(B + CS^{-1}C^T)^T$ of order $n - 2$ in the latter case.

In conclusion, we have

$$A = P_1 M_1 P_2 M_2 \cdots P_{n-1} M_{n-1} D \tilde{M}_{n-1} P_{n-1} \cdots \tilde{M}_2 P_2 \tilde{M}_1 P_1,$$

where $P_j$ is the identity matrix or a permutation matrix, $M_j$ is the identity matrix or a block unit lower triangular matrix containing two columns of multipliers in its $j$th and $(j + 1)$st columns and $(j + 2)$nd through $n$th rows, $\tilde{M}_j = M_j^T$, and $D$ is skew-symmetric block diagonal with $1 \times 1$ and $2 \times 2$ diagonal blocks—all $1 \times 1$ blocks are zero and all $2 \times 2$ blocks are nonsingular. (If $n$ is odd, then there is at least one $1 \times 1$ block.) Thus, we have reduced the skew matrix $A$ to a block diagonal skew matrix $D$ by a sequence of permutations and congruence transformations. Of course, all relevant elements of the $M_j$ (or $\tilde{M}_j$) and $D$ could be stored in the corresponding strictly lower (or upper) triangular part of $A$. One $n$-vector could store the relevant information in the permutations $P_j$.

Counting divisions as multiplications, the decomposition requires $\frac{1}{6}n^3 - \frac{1}{4}n^2 - \frac{1}{6}n$ multiplications and $\frac{1}{6}n^3 - \frac{1}{4}n^2 + \frac{5}{8}n$ additions if $n$ is even, and $\frac{1}{6}n^3 - \frac{1}{4}n^2 + \frac{3}{8}n - \frac{1}{4}$ multiplications and $\frac{1}{6}n^3 - n^2 + \frac{1}{6}n - 1$ if $n$ is odd. The number of comparisons is at most $\frac{1}{4}n^2 - \frac{1}{4}n$. Given the decomposition of $A$, we can now solve $Ax = b$ with $n^2 + O(n)$ multiplications and additions.

**3. Stability of the Decomposition.** In order to have a stable decomposition, we need to ensure that catastrophic element growth in the reduced matrices does not occur from one step to the next [2], [13], [14]. No element growth occurred whenever $S$ was $1 \times 1$. Let us now consider the case when $S$ is $2 \times 2$.

Let

$$S = \begin{bmatrix} 0 & -a_{21} \\ a_{21} & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} S & -C^T \\ C & B \end{bmatrix}.$$
Then a row of $CS^{-1}$ is
\[
\begin{bmatrix}
0 & \frac{1}{a_{21}} \\
-\frac{1}{a_{21}} & 0
\end{bmatrix}
\begin{bmatrix}
a_{i1} \\
a_{i2}
\end{bmatrix}
= \begin{bmatrix}
a_{i2}/a_{21}, \\
a_{i1}/a_{21}
\end{bmatrix},
\]
and an element of $A^{(3)} = B + CS^{-1}CT$ is of the form
\[
a_{ij}^{(3)} = a_{ij} - \begin{pmatrix}
a_{i2} \\
a_{i1}
\end{pmatrix} a_{j1} + \begin{pmatrix}
a_{i1} \\
a_{i2}
\end{pmatrix} a_{j2}.
\]

Thus, if $|a_{21}| = \max_{2 \leq i \leq n} \{|a_{i1}|, |a_{i2}|\}$, then $|a_{ij}^{(3)}| \leq 3 \max_{r,s} |a_{rs}|$ and
\[
\max_{i,j} |(B + CS^{-1}CT)_{ij}| \leq 3 \max_{r,s} |a_{rs}|.
\]
We can ensure this by interchanging the $k$th and second row and column if $|a_{k1}| = \max_{2 \leq i \leq n} \{|a_{i1}|, |a_{i2}|\}$ or by interchanging the second and first row and column and then the $k$th and second row and column if $|a_{k2}| = \max_{2 \leq i \leq n} \{|a_{i1}|, |a_{i2}|\}$.

If we do this at each step, then the element growth factor, the largest element (in modulus) in all the reduced matrices divided by $\max_{r,s} |a_{rs}|$, is bounded by
\[
\begin{cases}
3^{n/2-1} & \text{if } n \text{ is even} \\
3(n-1)/2-1 & \text{if } n \text{ is odd}
\end{cases}
\leq \left(\sqrt{3}\right)^{n-2} < 1.7321^{n-2}.
\]

This requires $\frac{1}{2}n^2 - \frac{1}{2}n$ comparisons, and is a partial pivoting strategy; cf. [4], [5], [13], [14]. The partial pivoting strategy for the diagonal pivoting method in the symmetric case gives a bound of $(2.57)^{n-1}$ [4], [5].

We can obtain a smaller bound on the element growth factor by employing a complete pivoting strategy. If $|a_{pq}| = \max_{r \geq s} \{|a_{rs}|\}$, then we can move the $(p, q)$ element to the $(2, 1)$ position symmetrically by interchanging the $q$th and first row and column and then the $p$th and second row and column. This requires at most $\frac{1}{2}n^3 + \frac{1}{2}n^2 - \frac{1}{2}n$ comparisons. By an analysis identical to Wilkinson's for Gaussian elimination with complete pivoting [13], we obtain the same bound as his on the element growth factor: $< \sqrt{n}f(n)$, where
\[
f(n) = \left(\prod_{k=2}^{n} k^{1/(k-1)}\right)^{1/2} < 1.8n(1/n)^{4},
\]
$f(100) \approx 330$. This compares with the bound of $3nf(n)$ for the complete pivoting strategy for the diagonal pivoting method in the symmetric case [2], [6].

4. Another Stable Decomposition for Skew-Symmetric Matrices. The other well-known stable decomposition for symmetric matrices is the tridiagonal decomposition developed by Aasen [1] and Parlett and Reid [11]. It decomposes $A = A^T$ as
\[
A = P_2L_2 \cdots P_nL_nTL_n^TP_n \cdots L_2^TP_2,
\]
where the $P_j$ are permutation matrices, the $L_j$ are unit lower triangular, and $T$ is symmetric tridiagonal. It requires $\frac{1}{6}n^3 + \Theta(n^2)$ multiplications and additions, and $\frac{1}{2}n^2 + \Theta(n)$ comparisons; the bound on element growth is $4^{n-2}$ [3, p. 525].

If $A$ is skew-symmetric, then, by modifying Aasen's algorithm in a manner similar to Section 2, we obtain
\[
A = P_2L_2 \cdots P_nL_nT\bar{L}_nP_n \cdots \bar{L}_2P_2,
\]
where the \( P_j \) and \( L_j \) are as above, \( \tilde{L}_j = L_j^T \), but \( T \) is now skew-symmetric tridiagonal (with a null diagonal). It requires \( \frac{1}{3} n^3 + \mathcal{O}(n^2) \) multiplications and additions, and \( \frac{1}{3} n^2 + \mathcal{O}(n) \) comparisons; but now the bound on element growth is \( 3^{n-2} \) (this follows from [3, p. 525], since the diagonal of \( A \) is null).

5. Stable Decomposition of Skew-Hermitian Matrices. If \( A \) is skew-Hermitian \((A^T = -A)\), Aasen’s algorithm gives

\[
A = P_2L_2 \cdots P_nL_nT\tilde{L}_n \cdots \tilde{L}_2P_2,
\]

where the \( P_j \) and \( L_j \) are as above, \( \tilde{L}_j = L_j^T \), but \( T \) is now skew-Hermitian. Since the diagonal of \( A \) is not necessarily null, element growth is bounded by \( 4^{n-2} \).

However, when \( A \) is skew-Hermitian, we cannot use the techniques of Sections 2 and 3 since the diagonal of \( A \) is now not necessarily null. But, if \( A \) is skew-Hermitian, then \( B = iA \) is Hermitian since \( B^T = -iA^T = -i(-A) = iA = B \). Since \( B = iA \) is Hermitian, we can use the stable decomposition for Hermitian matrices [4], [5], [6] and the subroutines in LINPACK [8], obtaining a stable decomposition with \( \frac{1}{6} n^3 + \mathcal{O}(n^2) \) multiplications and additions, and \( \geq \frac{1}{2} n^2 \) but \( \ll n^2 \) comparisons with a partial pivoting strategy as implemented in LINPACK [8], or \( \geq \frac{1}{5} n^3 \) but \( \ll \frac{1}{6} n^3 \) comparisons with a complete pivoting strategy [2], [6]. The element growth factor is bounded by \((2.57)^{n-1}\) for the partial pivoting strategy and \(3n \sqrt{n(n)}\) for the complete pivoting strategy. The decomposition can now be used to solve \( Ax = b \) with \( n^2 = \mathcal{O}(n) \) multiplications and additions (by solving \( Bx = ib \)).

6. Remarks. We could do the same thing when \( A \) is real skew-symmetric, but \( B = iA \) is then complex (Hermitian). The algorithms in Sections 2–4 show how we may stay in real arithmetic with stable decompositions based on congruence transformations. Since the nonzero eigenvalues of a real skew-symmetric matrix occur in purely imaginary complex conjugate pairs \((\pm i \mu_j \text{ where the } \mu_j \text{ are positive})\), the “inertia” \((\pi, \nu, \xi)\) of \( A \) (defined to be the number of positive, negative, and zero imaginary parts of the eigenvalues of \( A \)) is \((n - \xi)/2, (n - \xi)/2, \xi)\). If \( A \) is also nonsingular then its “inertia” is \((n/2, n/2, 0)\). This fixed inertia property is why skew-symmetric matrices are easier to decompose than symmetric indefinite matrices. We have an immediate modification of Sylvester’s Inertia Theorem to skew-symmetric matrices: if \( A \) is skew-symmetric, then \( B = MAM^T \) is skew-symmetric and \( B \) has the same “inertia” as \( A \), where \( M \) is nonsingular.

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