

Recurrence Relations for the Indefinite Integrals of the Associated Legendre Functions

By A. R. DiDonato

Abstract. Two recurrence relations are derived for the computation of the integral of the associated Legendre functions of real argument and integer order and degree.

The objective of this note is to develop recurrence relations for the integral

$$(1) \quad S_n^m(x) \equiv \int_a^x P_n^m(t) dt, \quad |x| \leq 1, -1 \leq a < 1,$$

on the indices $n \geq m \geq 0$, where $P_n^m(x)$ denotes the classical associated Legendre function of degree n and order m with real argument x . We use the definition

$$(2) \quad P_n^m(x) \equiv (1-x^2)^{m/2} \frac{d^m}{dx^m} [P_n(x)], \quad [4, \text{p. 323}],$$

with $P_n^0(x) \equiv P_n(x)$, the Legendre polynomial of degree n , i.e.,

$$(3) \quad P_n^x(x) \equiv \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n, \quad [4, \text{p. 303}].$$

The indices n and m are treated as nonnegative integers throughout, and a is a constant usually taken as zero or minus one.

The final expressions are given below by (12) and (17) and in a particular normalized form in (12) and (17). It is unlikely these relations are new, however we have not been able to find them in the literature, and their absence is conspicuous in such classical texts as [1], [4].* The need for such relations occurs in geodesy studies, [3], [5].

We first list for easy reference some well-known relations, [1, p. 1325]:

$$(4) \quad P_{n+1} = \frac{2n+1}{n+1} x P_n - \frac{n}{n+1} P_{n-1}, \quad n \geq 0, (P_{-1} \equiv 0),$$

$$(5) \quad P_{n+1}^m = \frac{2n+1}{n-m+1} x P_n^m - \frac{n+m}{n-m+1} P_{n-1}^m, \quad n \geq m \geq 0,$$

$$(6) \quad P_n^n = \frac{(2n)!}{2^n n!} (1-x^2)^{n/2}, \quad P_{n+1}^n = (2n+1)x P_n^n,$$

$$(7) \quad (1-x^2) \frac{d}{dx} (P_n^m) = (n+1)x P_n^m - (n-m+1) P_{n+1}^m,$$

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*After this work was completed, it was brought to our attention that (12) had recently appeared in the literature, [2]. It was decided to include our own derivation since it is different and more concise than the one in [2].

$$(8) \quad P_{n+1}^{m+1} = P_{n-1}^{m+1} + (n+m)(n+m-1)P_{n-1}^{m-1} \\ - (n-m+1)(n-m+2)P_{n+1}^{m-1},$$

$$(9) \quad \sqrt{1-x^2} P_n^m \\ = \frac{1}{2n+1} [(n+m)(n+m-1)P_{n-1}^{m-1} - (n-m+1)(n-m+2)P_{n+1}^{m-1}],$$

$$(10) \quad \frac{d}{dx} [(1-x^2)^{m/2} P_n^m] = -(n-m+1)(n+m)(1-x^2)^{(m-1)/2} P_n^{m-1}.$$

The desired recurrence relation on n is obtained without much difficulty. By first integrating (7) and then carrying out an integration by parts on the left we get

$$(11) \quad (1-t^2)P_n^m(t) \Big|_a^x + 2 \int_a^x t P_n^m(t) dt = (n+1) \int_a^x t P_n^m(t) dt - (n-m+1)S_{n+1}^m.$$

Then integrating (5) and using it in (11) gives the result we want, namely,

$$(12) \quad S_{n+1}^m = \frac{(n-1)(n+m)}{(n+2)(n-m+1)} S_{n-1}^m - \frac{2n+1}{(n+2)(n-m+1)} (1-t^2)P_n^m(t) \Big|_a^x, \\ 0 \leq m \leq n.$$

The steps leading to a recurrence relation on m for (1) begin with the integration of (8). The result is

$$(13) \quad S_{n+1}^{m+1} = S_{n-1}^{m+1} + (n+m)(n+m-1)S_{n-1}^{m-1} \\ - (n-m+1)(n-m+2)S_{n+1}^{m-1}.$$

Now replacing m by $(m-1)$ in (12) and using some obvious modifications gives

$$(14) \quad (n+m-1)(n+m)S_{n-1}^{m-1} \\ = \frac{(2n+1)(n+m)}{n-1} \left[\frac{(n-m+2)(n+2)}{2n+1} S_{n+1}^{m-1} + (1-t^2)P_n^{m-1}(t) \Big|_a^x \right].$$

Inserting this result in (13), we have

$$(15) \quad S_{n+1}^{m+1} = S_{n-1}^{m+1} + \frac{(2n+1)(m+1)(n-m+2)}{n-1} S_{n+1}^{m-1} \\ + \frac{(2n+1)(n+m)}{n-1} (1-t^2)P_n^{m-1}(t) \Big|_a^x.$$

In a similar way, we replace m by $(m+1)$ in (12) to get

$$\frac{(n-1)(n+m+1)}{2n+1} S_{n-1}^{m+1} = \frac{(n-m)(n+2)}{2n+1} S_{n+1}^{m+1} + (1-t^2)P_n^{m+1}(t) \Big|_a^x.$$

Then using this result in (15) yields

$$(16) \quad S_{n+1}^{m+1} = \frac{1}{m-1} \left\{ (m+1)(n-m+2)(n+m+1)S_{n+1}^{m-1} \right. \\ \left. + (1-t^2)[P_n^{m+1}(t) + (n+m)(n+m+1)P_n^{m-1}(t)] \Big|_a^x \right\}.$$

Replacing $n + 1$ by n in (16) gives the desired result, namely

$$(17) \quad S_n^{m+1} = \frac{1}{m-1} \left\{ (m+1)(n-m+1)(n+m)S_n^{m-1} + (1-t^2) [P_{n-1}^{m+1}(t) + (n+m-1)(n+m)P_{n-1}^{m-1}(t)] \Big|_a^x \right\},$$

$2 \leq m \leq n.$

To start the recurrence relations (12) and (17) we can use

$$(18) \quad P_0^0 = 1, \quad P_1^0 = x, \quad P_2^0 = \frac{1}{2}(3x^2 - 1), \quad P_1^1 = \sqrt{1-x^2},$$

$$(19) \quad P_{n+1} = \frac{2n+1}{n+1} x P_n - \frac{n}{n+1} P_{n-1}, \quad n \geq 0, \quad P_{-1} \equiv 0, \quad (\text{from (4)}),$$

$$(20) \quad S_0^0 = x - a, \quad S_1^1 = \frac{1}{2} [t\sqrt{1-t^2} + \sin^{-1} t] \Big|_a^x,$$

$$(21) \quad S_1^0 = \frac{x^2 - a^2}{2},$$

and, in general,

$$(22) \quad S_n^0 = \frac{1}{2n+1} [P_{n+1}(t) - P_{n-1}(t)] \Big|_a^x, \quad **$$

$$(23) \quad S_n^1 = \frac{(n-2)n}{(n-1)(n+1)} S_{n-2}^1 - \frac{2n-1}{(n-1)(n+1)} (1-t^2) P_{n-1}^1(t) \Big|_a^x,$$

$n \geq 2, \quad (\text{from (12)}),$

$$(24) \quad S_n^2 = -2S_n^0 + [(n+3)tP_n(t) - (n+1)P_{n+1}(t)] \Big|_a^x \quad (\text{from (2), (9), (4)}),$$

$$(25) \quad \begin{cases} P_{n+1}^m = \frac{2n+1}{n-m+1} x P_n^m - \frac{n+m}{n-m+1} P_{n-1}^m, & n \geq 0, n \geq m, \\ P_{n+1}^m = (2n+1)\sqrt{1-x^2} P_n^m & (\text{from (9) and (8)}). \end{cases}$$

$$(26) \quad S_n^n = \frac{1}{n+1} [n(2n-3)(2n-1)S_{n-2}^{n-2} + tP_n^n(t)] \Big|_a^x \left. \vphantom{S_n^n} \right\} \quad (\text{from (6) with integration by parts).}$$

$$(27) \quad S_{n+1}^{n+1} = -\frac{1}{n+2} \sqrt{1-t^2} P_{n+1}^{n+1}(t) \Big|_a^x$$

Normalizations of $P_n^m(x)$ and $S_n^m(x)$ may be needed in order to keep the results of the recurrence relations from becoming excessively large. Since

$$(28) \quad \int_{-1}^1 [P_n^m(t)]^2 dt = \left(\frac{2}{2n+1} \right) \frac{(n+m)!}{(n-m)!},$$

we define normalized Legendre functions $\bar{P}_n^m(x)$ by

$$(29) \quad \bar{P}_n^m \equiv \left[\frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} \right]^{1/2} P_n^m(x).$$

**The result in (22) follows from integrating (10), with $m = 1$, and then using (9).

Accordingly, for (5), (12), and (17), respectively, we have their normalized forms, where we number a normalized equation the same as the corresponding unnormalized equation except for the addition of an overbar.

$$\begin{aligned}
 \bar{P}_{n+1}^m(x) &= \left[\frac{(2n+3)(2n+1)}{(n+m+1)(n-m+1)} \right]^{1/2} x \bar{P}_n^m(x) \\
 (5) \quad &- \left[\frac{(2n+3)(n+m)(n-m)}{(2n-1)(n+m+1)(n-m+1)} \right]^{1/2} \bar{P}_{n-1}^m(x), \\
 &0 \leq m < n+1,
 \end{aligned}$$

$$\begin{aligned}
 \bar{S}_{n+1}^m(x) &= \frac{n-1}{n+2} \left[\frac{(2n+3)(n+m)(n-m)}{(2n-1)(n+m+1)(n-m+1)} \right]^{1/2} \bar{S}_{n-1}^m(x) \\
 (12) \quad &- \frac{(1-t^2)}{n+2} \left[\frac{(2n+3)(2n+1)}{(n+m+1)(n-m+1)} \right]^{1/2} \bar{P}_n^m(t) \Big|_a^x, \\
 &1 \leq m < n+1,
 \end{aligned}$$

$$\begin{aligned}
 \bar{S}_n^{m+1}(x) &= \frac{1}{m-1} \left\{ (m+1) \left[\frac{(n+m)(n-m+1)}{(n+m+1)(n-m)} \right]^{1/2} \bar{S}_n^{m-1} \right. \\
 (17) \quad &+ (1-t^2) \left[\frac{(2n+1)(n-m-1)}{(2n-1)(n+m+1)} \right]^{1/2} \bar{P}_{n-1}^{m+1}(t) \Big|_a^x \\
 &+ \left. (1-t^2) \left[\frac{(2n+1)(n+m)(n+m-1)}{(2n-1)(n+m+1)(n-m)} \right]^{1/2} \bar{P}_{n-1}^{m-1}(t) \Big|_a^x \right\}, \\
 &2 \leq m < n,
 \end{aligned}$$

where

$$\bar{S}_n^m(x) \equiv \left[\frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} \right]^{1/2} S_n^m(x).$$

The starting relations (18)–(27) in normalized form are given in the same order.

$$\begin{aligned}
 (18) \quad \bar{P}_0 &= 1/\sqrt{2}, \quad \bar{P}_1(x) = (3/2)^{1/2} x, \quad \bar{P}_2(x) = (5/2)^{1/2} \frac{1}{2} (3x^2 - 1) \\
 \bar{P}_1^1(x) &= [3(1-x^2)]^{1/2} / 2,
 \end{aligned}$$

$$\begin{aligned}
 (19) \quad \bar{P}_{n+1}(x) &= \frac{[(2n+3)(2n+1)]^{1/2}}{n+1} x \bar{P}_n(x) \\
 &- \frac{n}{n+1} \left(\frac{2n+3}{2n-1} \right)^{1/2} \bar{P}_{n-1}(x), \quad n \geq 0,
 \end{aligned}$$

$$(20) \quad \bar{S}_0^0 = \frac{1}{\sqrt{2}} (x-a), \quad \bar{S}_1^1 = \frac{\sqrt{3}}{4} \left[t \sqrt{1-t^2} + \sin^{-1} t \right] \Big|_a^x,$$

$$(21) \quad \bar{S}_1^0 = \frac{(3/2)^{1/2}}{2} (x^2 - a^2),$$

$$(22) \quad \bar{S}_n^0 = \left\{ \left[\frac{1}{(2n+1)(2n+3)} \right]^{1/2} \bar{P}_{n+1}(t) - \left[\frac{1}{(2n+1)(2n-1)} \right]^{1/2} \bar{P}_{n-1}(t) \right\} \Big|_a^x,$$

$$(23) \quad \bar{S}_n^1 = \frac{n-2}{n+1} \left[\frac{(2n+1)n(n-2)}{(2n-3)(n+1)(n-1)} \right]^{1/2} \bar{S}_{n-2}^1 - \frac{(1-t^2)}{n+1} \left[\frac{(2n+1)(2n-1)}{(n+1)(n-1)} \right]^{1/2} \bar{P}_{n-1}^1(t) \Big|_a^x,$$

$$(24) \quad \bar{S}_n^2 = \frac{1}{[(n+2)(n+1)n(n-1)]^{1/2}} \times \left\{ -2\bar{S}_n^0 + \left[(n+3)t\bar{P}_n(t) - (n+1)\left(\frac{2n+1}{2n+3}\right)^{1/2} \bar{P}_{n+1}(t) \right] \Big|_a^x \right\},$$

$$(25) \quad \bar{P}_{n+1}^{n+1}(x) = \left[\frac{2n+3}{2n+2} \right]^{1/2} (1-x^2)^{1/2} \bar{P}_n^n(x),$$

$$(26) \quad \bar{S}_n^n = \frac{1}{n+1} \left\{ \left[\frac{n(2n-1)(2n+1)}{4(n-1)} \right]^{1/2} \bar{S}_{n-2}^{n-2} + t\bar{P}_n^n \right\} \Big|_a^x,$$

$$(27) \quad \bar{S}_{n+1}^n = -\frac{1}{n+2} [2(n+1)]^{1/2} (1-t^2)^{1/2} \bar{P}_{n+1}^{n+1}(t) \Big|_a^x.$$

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Naval Surface Weapons Center
 Dahlgren Laboratory
 Dahlgren, Virginia 22448

1. P. M. MORSE & H. FESHBACH, *Methods of Theoretical Physics*, Vol. II, McGraw-Hill, New York, 1953.
2. M. K. PAUL, "Recurrence relations for integrals of associated Legendre functions," *Bull. Géodésique*, v. 52, 1978, pp. 177-190.
3. W. M. ROBERTSON, *Spherical Geodetic Transformations*, Vol. I of II, Report No. R-1181, The Charles Stark Draper Laboratories, Inc., Cambridge, Mass. 02139, September 1978.
4. E. T. WHITTAKER & G. N. WATSON, *A Course of Modern Analysis*, Cambridge Univ. Press, Oxford, 1952.
5. B. ZONDEK, *Aggregation Errors of Cell-Averaged Geoid Heights*, Naval Surface Weapons Center, Dahlgren Laboratory, TR-3608, April 1977.