

An Upper Bound for the First Zero of Bessel Functions

By LI. G. CHAMBERS

Abstract. It is shown, using the Rayleigh-Ritz method of the calculus of variations, that an upper bound for the first zero j_ν , of $z^{-\nu} J_\nu(z)$, $\nu > -1$, is given by

$$(\nu + 1)^{1/2} \{(\nu + 2)^{1/2} + 1\},$$

and that for large ν , $j_\nu = \nu + O(\nu^{1/2})$.

1. The following upper bound is given by Watson [4] for the first zero j_ν of $J_\nu(x)$ ($\nu > 0$)

$$(1) \quad j_\nu < \left\{ \frac{4}{3} (\nu + 1)(\nu + 5) \right\}^{1/2}.$$

It may be shown that a better bound may be obtained, valid for $\nu > -1$, namely

$$(2) \quad (\nu + 1)^{1/2} \{(\nu + 2)^{1/2} + 1\}.$$

2. Consider the function

$$(3) \quad u(z) = \Gamma(\nu + 1) (2/(\gamma z))^\nu J_\nu(\gamma z).$$

The differential equation satisfied by $u(z)$ is given by Watson [3] to be

$$(4) \quad z^2 u'' + (2\nu + 1) z u' + \gamma^2 z^2 u = 0$$

with the boundary condition $u(0) = 1$, and if γ is a zero of J_ν , $u(1) = 0$.

Equation (4) can be written in Sturm-Liouville form

$$(5) \quad \frac{d}{dz} \left(z^{2\nu+1} \frac{du}{dz} \right) + \gamma^2 z^{2\nu+1} u = 0.$$

Multiplying Eq. (5) by u and integrating over $0 \leq z \leq 1$, it follows that

$$(6) \quad \gamma^2 = \frac{\int_0^1 z^{2\nu+1} u'^2 dz}{\int_0^1 z^{2\nu+1} u^2 dz}.$$

On integration by parts, $uu'z^{2\nu+1}$ will vanish at $z = 0$, if $\nu > -\frac{1}{2}$, and $u(1)$ vanishes. Thus the relation (6) provides a variational formulation, as indicated by Irving and Mullineux [1], for γ^2 which is an eigenvalue for the differential equation (5). The first eigenvalue will be j_ν^2 . The functional

$$(7) \quad \Lambda(\omega) = \frac{\int_0^1 z^{2\nu+1} \omega'^2 dz}{\int_0^1 z^{2\nu+1} \omega^2 dz},$$

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as indicated by Irving and Mullineux [2], obeys the following relations

$$(8a) \quad \Lambda(u) = j_\nu^2,$$

$$(8b) \quad \Lambda(\omega) > \Lambda(u), \quad u \neq \omega.$$

Thus $\Lambda(\omega)$ provides an upper bound to j_ν^2 when $\omega(1) = 0$, by the Rayleigh-Ritz procedure.

3. Consider the approximating function

$$(9) \quad \omega = 1 - z^p,$$

where p is as yet unspecified.

$$(10) \quad \Lambda(\omega) = \frac{\int_0^1 z^{2\nu+1} p^2 z^{2p-2} dz}{\int_0^1 (1 - 2z^p + z^{2p}) z^{2\nu+1} dz}$$

$$(11) \quad = p^2 \left\{ \frac{1/(2\nu + 2p)}{1/(2\nu + 2) - 2/(2\nu + p + 2) + 1/(2\nu + 2p + 2)} \right\}$$

$$(12) \quad = \frac{(\nu + 1)(2\nu + p + 2)(\nu + p + 1)}{\nu + p}$$

on simplification.

Up till now p has not been specified. $\Lambda(\omega)$ may be regarded as a function of p , and the best upper bound will follow by minimizing $\Lambda(\omega)$ with respect to p . It can be verified by differentiation that this happens when

$$(13) \quad p + \nu = (\nu + 2)^{1/2}.$$

Although p is negative outside of $-1 < \nu < 2$, the process is still valid because all of the denominators in the expression (11) remain positive, and it can easily be seen that

$$\lim_{z \rightarrow 0} z^{2\nu+1} \omega \omega' = 0,$$

so that the endpoint condition at $z = 0$ remains satisfied. It follows that

$$j_\nu^2 < \frac{(\nu + 1)[(\nu + 2) + (\nu + 2)^{1/2}][(\nu + 2)^{1/2} + 1]}{(\nu + 2)^{1/2}},$$

which gives

$$(14) \quad j_\nu < (\nu + 1)^{1/2}((\nu + 2)^{1/2} + 1).$$

A straightforward reduction shows that, if $\nu + 1 > 0$,

$$(\nu + 1)^{1/2}((\nu + 2)^{1/2} + 1) < \left\{ \frac{4}{3}(\nu + 1)(\nu + 5) \right\}^{1/2}.$$

(For $\nu = 7$ there is in fact equality.)

It thus follows that [4]

$$(15) \quad \{\nu(\nu + 2)\}^{1/2} < j_\nu < (\nu + 1)^{1/2}((\nu + 2)^{1/2} + 1), \quad \nu > 1.$$

It follows from (15) that

$$(16) \quad j_\nu = \nu + O(\nu^{1/2}) \quad \text{for large } \nu.$$

As an example, the bound for $\nu = 0$ is given by $\sqrt{2} + 1 = 2.4142$ in comparison with the true value 2.4048.

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School of Mathematics and Computer Science
University College of North Wales
Bangor, Gwynedd LL57 2UW, Wales

1. J. IRVING & N. MULLINEUX, *Mathematics in Physics and Engineering*, Academic Press, New York, 1959, p. 388.
2. J. IRVING & N. MULLINEUX, *Loc. cit.*, p. 39.
3. G. N. WATSON, *A Treatise on the Theory of Bessel Functions*, The University Press, Cambridge, 1944, p. 98.
4. G. N. WATSON, *Loc. cit.*, p. 486.