MATHEMATICS OF COMPUTATION
VOLUME 38, NUMBER 158
APRIL 1982, PAGES 593-599

Evaluation of Generalized Howland Integrals

By Chih-Bing Ling

Abstract. This paper presents a method of evaluation of the generalized Howland integrals. The values are tabulated to 10D.

The generalized Howland integrals are defined by

$$\text{(1)}\qquad \begin{matrix} I_{k,s} \\ I^*_{k,s} \end{matrix} = \frac{2^k}{k!}\int_0^\infty \frac{x^k e^{-sx}\,dx}{\sinh 2x \pm 2x} = \frac{1}{2(k!)}\int_0^\infty \frac{x^k e^{-sx/2}\,dx}{\sinh x \pm x}, \quad \begin{matrix} (k \geqslant 1) \\ (k \geqslant 3) \end{matrix}$$

where k and s are integers. For the sake of convergence, k is restricted as indicated above and s is restricted to be not less than -1. Owing to their frequent occurrence in mathematical sciences, it is thought that they deserve a special consideration.

The four integrals for $s = 0$ and 2 are the ordinary Howland integrals. The two integrals for $s = 0$ have been evaluated to 25D by Ling and Lin [3] when k is odd and by Ling [4] when k is even. Those for $s = 2$ have recently been evaluated to 20D by Ling and Wu [5]. It is the endeavor of the present paper to evaluate the remaining integrals to 10D.

The following recurrence relations for $s \geqslant 1$ are readily verified:

$$\text{(2)}\qquad \begin{aligned} I_{k,s-2} + 2(k+1)I_{k+1,s} - I_{k,s+2} &= \left(\frac{2}{s}\right)^{k+1}, \\ I^*_{k,s-2} - 2(k+1)I^*_{k+1,s} - I^*_{k,s+2} &= \left(\frac{2}{s}\right)^{k+1}. \end{aligned}$$

By using these relations, the integrals $I_{k+1,s}$ and $I^*_{k+1,s}$ can be evaluated by recurrence in terms of $I_{k,s-2}$, $I_{k,s+2}$ and $I^*_{k,s-2}$, $I^*_{k,s+2}$ from the values of the leading integrals $I_{k,-1}$, $I_{k,0}$, $I_{1,s}$ and $I^*_{k,-1}$, $I^*_{k,0}$, $I^*_{3,s}$, respectively. Such a process of computation has the distinct advantage that no accuracy is lost in successive steps, except perhaps when $s = 1$. To avoid this possibility, we take $I_{k,1}$ and $I^*_{k,1}$ as the leading integrals intead of $I_{k,-1}$ and $I^*_{k,-1}$.

As mentioned before, the integrals $I_{k,0}$ and $I^*_{k,0}$ have been evaluated to high precision of 25D. Plana's method was used for their evaluation. This method, however, is no longer applicable if the value of s in the integrals is other than zero. By expanding $e^{-x/2}$ in the second form of the integrands in (1) into a series in x,

Received March 24, 1981; revised July 7, 1981.
1980 *Mathematics Subject Classification.* Primary 65A05, 65D20, 65D30.
Key words and phrases. Howland integrals, generalized Howland integrals.

integrating and then applying the Kummer transformation [1], the following relations are found for $s \geqslant 1$:

$$
(3) \qquad
\begin{aligned}
\left(\frac{2}{s+2}\right)^{k+1} - I_{k,s} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} \binom{n+k}{n} \left\{ \left(\frac{2}{s+2}\right)^{n+k+1} - I_{n+k,s-1} \right\}, \\
I^*_{k,s} - \left(\frac{2}{s+2}\right)^{k+1} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} \binom{n+k}{n} \left\{ I^*_{n+k,s-1} - \left(\frac{2}{s+2}\right)^{n+k+1} \right\}.
\end{aligned}
$$

Here the ratio of the binomial coefficient to 2^n may or may not be greater than unity. When it is, a certain amount of accuracy is lost. The loss is larger, when k is larger. For instance, the loss is 5D when $k = 20$ and 10D when $k = 36$.

It appears possible to reduce the loss of accuracy if the computation for a unit increment of s is carried out in several steps instead of a single step as in (3). Suppose that four steps are taken such that in each step the increment of s is $\frac{1}{4}$. Then, in the rth step, for $r = 1, 2, 3,$ or 4, the intermediate integrals are given by

$$
(4) \qquad
\begin{aligned}
&\left(\frac{8}{4s+r+8}\right)^{k+1} - I_{k,s+\frac{1}{4}r} \\
&\qquad = \sum_{n=0}^{\infty} \frac{(-1)^n}{8^n} \binom{n+k}{n} \left\{ \left(\frac{8}{4s+r+7}\right)^{n+k+1} - I_{n+k,s+\frac{1}{4}r-\frac{1}{4}} \right\}, \\
&I^*_{k,s+\frac{1}{4}r} - \left(\frac{8}{4s+r+8}\right)^{k+1} \\
&\qquad = \sum_{n=0}^{\infty} \frac{(-1)^n}{8^n} \binom{n+k}{n} \left\{ I^*_{n+k,s+\frac{1}{4}r-\frac{1}{4}} - \left(\frac{8}{4s+r+7}\right)^{n+k+1} \right\}.
\end{aligned}
$$

It is seen that the binomial coefficient involved is now divided by 8^n instead of 2^n. When $k = 18$, the accumulated loss of accuracy in four steps together is reduced to slightly less than 2D only. Hence, if we begin the computation with the 25D values of $I_{k,0}$ and $I^*_{k,0}$ and take four steps for each unit increment of s, we can find up to $k = 18$, 23D values of $I_{k,1}$ and $I^*_{k,1}$, 21D values of $I_{k,2}$ and $I^*_{k,2}$, and 19D values of $I_{k,3}$ and $I^*_{k,3}$, successively. However, values of the intermediate integrals at each step for $k \geqslant 19$ are also needed in the computation. These values can be found directly by developing the integrals into series [5] as follows:

$$
(5) \qquad
\begin{matrix} I_{k,s+r/4} \\ I^*_{k,s+r/4} \end{matrix} = \sum_{n=1}^{\infty} (\mp 1)^{n+1} q_n\left(k, s+\frac{r}{4}\right) \left(\frac{8}{8n+4s+r}\right)^{k+1},
$$

where, for $n \geqslant 0$,

$$
(6) \qquad
\begin{aligned}
q_{2n+1}\left(k, s+\frac{r}{4}\right) &= \sum_{t=0}^{\infty} \binom{k+2t}{k} \frac{(n+t)!}{(n-t)!} \left(\frac{16}{16n+4s+r+8}\right)^{2t}, \\
q_{2n+2}\left(k, s+\frac{r}{4}\right) &= \sum_{t=0}^{\infty} \binom{k+2t+1}{k} \frac{(n+t+1)!}{(n-t)!} \left(\frac{16}{16n+4s+r+16}\right)^{2t+1}.
\end{aligned}
$$

When $k = 19$, the series in (5) are to be carried to $n = 190$ for 23D, to $n = 100$ for 21D, and to $n = 50$ for 19D. For 10D, the corresponding value of n is 14 when $k = 15$, or 6 when $k = 19$. The convergence of the series increases with k but only slightly with s.

When the values of the integrals for $s = 0$, 1, 2, and 3 are available in high precision as described above, we can use the recurrence relations in (2) to compute $I_{k,s+2}$ and $I^*_{k,s+2}$ for $s \geqslant 2$ in terms of $I_{k,s-2}$, $I_{k+1,s}$ and $I^*_{k,s-2}$, $I^*_{k+1,s}$, respectively. Owing to the factor $2(k+1)$ associated with $I_{k+1,s}$ and $I^*_{k+1,s}$, some accuracy is always lost. The loss is 1D when $k = 4$, $1\frac{1}{2}$D when $k = 14$, or 2D when $k = 49$. Hence the integrals can be computed successively so long as the desired accuracy of 10D is still sustained. In this manner, values of a considerable number of integrals are obtained, including the leading integrals $I_{1,s}$ and $I^*_{3,s}$. There is ground to claim that the values of $I_{1,s}$ and $I^*_{3,s}$ thus obtained up to $s = 20$ and 18, respectively, are accurate to 10D.

For further evaluation of $I_{1,s}$ and $I^*_{3,s}$, consider the asymptotic expansion of these integrals. We begin by changing the variable x in the integrals with the substitution

$$e^x = 1 + y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \tag{7}$$

Reversion of the series yields

$$x = \ln(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4} + \cdots. \tag{8}$$

The powers of x can therefore be expressed as series in y. By differentiation, we also find

$$dx = dy/(1+y). \tag{9}$$

Next, find the reciprocals of the following expansions of x:

$$\begin{aligned} \frac{\sinh x + x}{x} &= 2 + \frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \cdots, \\ \frac{\sinh x - x}{x^3} &= \frac{1}{3!} + \frac{x^2}{5!} + \frac{x^4}{7!} + \frac{x^6}{9!} + \cdots. \end{aligned} \tag{10}$$

We have

$$\begin{aligned} \frac{2x}{\sinh x + x} &= 1 - \frac{x^2}{12} + \frac{x^4}{360} + \frac{x^6}{60{,}480} - \frac{11x^8}{1{,}814{,}400} + \cdots, \\ \frac{x^3}{6(\sinh x - x)} &= 1 - \frac{x^2}{20} + \frac{11x^4}{8400} - \frac{17x^6}{756{,}000} + \frac{563x^8}{2{,}328{,}480{,}000} - \cdots. \end{aligned} \tag{11}$$

Suppose that these series in x are expressed as series in y in the form:

$$\frac{2x}{\sinh x + x} = 1 + \sum_{m=2}^{\infty} p_m y^m, \qquad \frac{x^3}{6(\sinh x - x)} = 1 + \sum_{m=2}^{\infty} p^*_m y^m. \tag{12}$$

The following values are found:

(13)

m	2	3	4	5	6	7	8
p_m	$-\frac{1}{12}$	$\frac{1}{12}$	$-\frac{53}{720}$	$\frac{23}{360}$	$-\frac{3359}{60{,}480}$	$\frac{979}{20{,}160}$	$-\frac{155{,}083}{3{,}628{,}800}$
p^*_m	$-\frac{1}{20}$	$\frac{1}{20}$	$-\frac{187}{4200}$	$\frac{41}{1050}$	$-\frac{12{,}991}{378{,}000}$	$\frac{3841}{126{,}000}$	$-\frac{881{,}701}{32{,}340{,}000}$

To proceed further, let

$$y = \sin^2\theta/\cos^2\theta. \tag{14}$$

Consequently,

$$e^{-x/2} = \cos\theta, \qquad dx = 2(\sin\theta/\cos\theta)\,d\theta. \tag{15}$$

We then have

$$I_{1,s} = \frac{1}{2s} + \frac{1}{4}\sum_{m=2}^{\infty} p_m H_{s,m}, \qquad I^*_{3,s} = \frac{1}{s} + \frac{1}{2}\sum_{m=2}^{\infty} p^*_m H_{s,m}, \tag{16}$$

where

$$\begin{aligned} H_{s,m} &= 2\int_0^{\pi/2}\left(\frac{\sin\theta}{\cos\theta}\right)^{2m+1}\cos^s\theta\,d\theta \\ &= m!\,\Gamma\left(\frac{s}{2}-m\right)\Big/\Gamma\left(\frac{s}{2}+1\right) \\ &= m!\Big/\left\{\frac{s}{2}\left(\frac{s}{2}-1\right)\left(\frac{s}{2}-2\right)\cdots\left(\frac{s}{2}-m\right)\right\}. \end{aligned} \tag{17}$$

By expanding the preceding expression into inverse power series of s, the following asymptotic series are obtained:

$$\begin{aligned} I_{1,s} &\sim \frac{1}{4}\left(\frac{2}{s}\right) - \frac{1}{24}\left(\frac{2}{s}\right)^3 + \frac{1}{60}\left(\frac{2}{s}\right)^5 + \frac{1}{336}\left(\frac{2}{s}\right)^7 - \frac{11}{180}\left(\frac{2}{s}\right)^9 + \cdots, \\ I^*_{3,s} &\sim \frac{1}{2}\left(\frac{2}{s}\right) - \frac{1}{20}\left(\frac{2}{s}\right)^3 + \frac{11}{700}\left(\frac{2}{s}\right)^5 - \frac{17}{2100}\left(\frac{2}{s}\right)^7 + \frac{563}{115{,}500}\left(\frac{2}{s}\right)^9 - \cdots. \end{aligned} \tag{18}$$

The first series gives values to 10D, 11D, 12D, when $s = 16, 20, 24$, respectively, and the second when $s = 10, 13, 16$, respectively.

As described before, 10D values of $I_{1,s}$ and $I^*_{3,s}$ have been found up to $s = 20$ and 18, respectively. It is thus seen that further 10D values of these two integrals can be found from the asymptotic series in (18) alone.

Lastly, to evaluate the remaining integrals $I_{k,-1}$ and $I^*_{k,-1}$, the following series may be used:

$$\begin{aligned} \frac{1}{2^{k+1}} I_{k,-1} &= 1 - \sum_{n=0}^{\infty} \frac{1}{2^{n+k+1}}\binom{n+k}{n}(1 - I_{n+k,0}), \\ \frac{1}{2^{k+1}} I^*_{k,-1} &= 1 + \sum_{n=0}^{\infty} \frac{1}{2^{n+k+1}}\binom{n+k}{n}(I^*_{n+k,0} - 1), \end{aligned} \tag{19}$$

which are found similarly by expanding $e^{x/2}$ in (1) into a series in x. No accuracy is lost in this case since here the ratio of the binomial coefficient to 2^{n+k+1} is always less than unity.

The foregoing computation was carried out on an IBM 3032 computer with extended precision. In the course of computation, the values are generally carried with an accuracy exceeding 10D. Ample guard digits were provided whenever needed to given an extra accuracy as far as practicable. In several instances, the integrals were computed by different methods for some overlapping k or s to serve as a check. Finally, the results were rounded off to 10D and shown in Tables 1–4. For the sake of brevity, other values are not shown. Further 10D values of $I_{1,s}$ and $I^*_{3,s}$ in Table 1 are both given by the first three terms of the respective asymptotic series in (18). It may be mentioned that the first two terms of the series in (5) when $r = 0$ are

$$\begin{matrix} I_{k,s} \\ I^*_{k,s} \end{matrix} \sim \frac{2^{k+1}}{(s+2)^{k+1}} \mp \frac{(k+1)2^{k+3}}{(s+4)^{k+2}}. \tag{20}$$

They give good approximation when k is large. Further 10D values of $I_{k,1}$ and $I^*_{k,1}$ in Table 2 are given when $s = 1$ by the first term only. If both terms are used, they give 10D values from $k = 23$ onward.

TABLES 1 & 2

Values of $I_{1,s}$, $I^*_{3,s}$ and $I_{k,1}$, $I^*_{k,1}$

s or k	$I_{1,s}$	$I^*_{3,s}$	$I_{k,1}$	$I^*_{k,1}$
1	0.35726 51300	0.79021 90430	0.35726 51300	-
2	0.22011 95814	0.46071 37190	0.21089 86635	-
3	0.15623 63163	0.32021 75927	0.14605 74537	0.79021 90430
4	0.12028 34787	0.24418 64095	0.10380 05261	0.26920 28971
5	0.09749 98574	0.19694 88557	0.07348 74141	0.13201 49095
6	0.08185 80268	0.16487 60199	0.05146 00004	0.07467 03132
7	0.07048 83836	0.14171 96838	0.03562 53748	0.04529 37207
8	0.06186 52335	0.12423 36202	0.02442 08531	0.02853 83900
9	0.05510 73534	0.11057 07199	0.01660 93689	0.01837 93682
10	0.04967 20114	0.09960 49274	0.01122 95814	0.01199 26267
11	0.04520 74267	0.09061 16352	0.00755 93368	0.00788 81283
12	0.04147 59140	0.08310 38443	0.00507 28893	0.00521 42567
13	0.03831 12562	0.07674 23485	0.00339 69198	0.00345 75221
14	0.03559 38024	0.07128 37241	0.00227 12707	0.00229 71652
15	0.03323 52717	0.06654 88043	0.00151 71047	0.00152 81325
16	0.03116 91294	0.06240 28195	0.00101 26760	0.00101 73574
17	0.02934 42938	0.05874 24644	0.00067 56671	0.00067 76484
18	0.02772 09046	0.05548 72329	0.00045 06807	0.00045 15169
19	0.02626 74072	0.05257 34633	0.00030 05545	0.00030 09065
20	0.02495 85002	0.04995 01563	0.00020 04124	0.00020 05603
21	0.02377 36613	0.04757 59783	0.00013 36264	0.00013 36883
22	0.02269 60716	0.04541 70769	0.00008 90919	0.00008 91178
23	0.02171 18169	0.04344 54629	0.00005 93978	0.00005 94086
24	0.02080 92877	0.04163 77944	0.00003 95999	0.00003 96044
25	0.01997 87213	0.03997 44513	0.00002 64005	0.00002 64024
26	0.01921 18489	0.03843 88223	0.00001 76006	0.00001 76013
27	0.01850 16206	0.03701 67499	0.00001 17338	0.00001 17341
28	0.01784 19892	0.03569 60933	0.00000 78226	0.00000 78227
29	0.01722 77380	0.03446 63822	0.00000 52151	0.00000 52151
30	0.01665 43430	0.03331 85392	0.00000 34767	0.00000 34767
31	0.01611 78618	0.03224 46552	0.00000 23178	0.00000 23178
32	0.01561 48434	0.03123 78079	0.00000 15452	0.00000 15452
33	0.01514 22533	0.03029 19125	0.00000 10301	0.00000 10301
34	0.01469 74132	0.02940 15987	0.00000 06868	0.00000 06868
35	0.01427 79499	0.02856 21087	0.00000 04578	0.00000 04578
36	0.01388 17532	0.02776 92127	0.00000 03052	0.00000 03052
37	0.01350 69405	0.02701 91374	0.00000 02035	0.00000 02035
38	0.01315 18267	0.02630 85061	0.00000 01357	0.00000 01357
39	0.01281 48994	0.02563 42880	0.00000 00904	0.00000 00904
40	0.01249 47969	0.02499 37549	0.00000 00603	0.00000 00603

TABLES 3 & 4

Values of $I_{k,3}$, $I^*_{k,3}$ and $I_{k,-1}/2^{k+1}$, $I^*_{k,-1}/2^{k+1}$

k	$I_{k,3}$	$I^*_{k,3}$	$I_{k,-1}/2^{k+1}$	$I^*_{k,-1}/2^{k+1}$
1	0.15623 63163	-	0.82816 04155	-
2	0.04616 97930	-	0.89622 81338	-
3	0.01732 67749	0.32021 75927	0.94918 26604	1.15461 50481
4	0.00701 00147	0.04946 74154	0.97725 42460	1.04280 05160
5	0.00291 45579	0.01184 03416	0.99039 67899	1.01418 56891
6	0.00122 19741	0.00344 14251	0.99611 30213	1.00498 08868
7	0.00051 23332	0.00111 35709	0.99847 56980	1.00178 79993
8	0.00021 40033	0.00038 48437	0.99941 64949	1.00064 69013
9	0.00008 89236	0.00013 88317	0.99978 07591	1.00023 43666
10	0.00003 67442	0.00005 15626	0.99991 88141	1.00008 47609
11	0.00001 51021	0.00001 95430	0.99997' 02797	1.00003 05571
12	0.00000 61770	0.00000 75146	0.99998 92195	1.00001 09745
13	0.00000 25158	0.00000 29197	0.99999 61186	1.00000 39260
14	0.00000 10210	0.00000 11430	0.99999 86111	1.00000 13991
15	0.00000 04131	0.00000 04500	0.99999 95055	1.00000 04968
16	0.00000 01667	0.00000 01779	0.99999 98247	1.00000 01758
17	0.00000 00672	0.00000 00705	0.99999 99381	1.00000 00620
18	0.00000 00270	0.00000 00280	0.99999 99782	1.00000 00218
19	0.00000 00109	0.00000 00112	0.99999 99924	1.00000 00077
20	0.00000 00044	0.00000 00044	0.99999 99973	1.00000 00027
21	0.00000 00017	0.00000 00018	0.99999 99991	1.00000 00009
22	0.00000 00007	0.00000 00007	0.99999 99997	1.00000 00003
23	0.00000 00003	0.00000 00003	0.99999 99999	1.00000 00001
24	0.00000 00001	0.00000 00001	1.00000 00000	1.00000 00000

The values of the integrals for $s = 0$ and 2, or the four ordinary Howland integrals, are referred to the existing tables in the papers [3], [4], [5]. When the values of other integrals are needed, they can be found from the known values by using the recurrence relations in (2) without losing accuracy.

The values of $I_{k,-1}$ and $I^*_{k,-1}$ may be checked by the recurrence relations in (2). Those of the other integrals may be checked by the relations shown below. They are, for $s \geqslant 1$,

$$\sum_{k=0}^{\infty} I_{2k+1,s} = \frac{1}{s} - I_{1,s}, \qquad \sum_{k=1}^{\infty} kI_{2k,s} = \frac{1}{s^2} - I_{2,s}, \tag{21}$$
$$\sum_{k=1}^{\infty} I^*_{2k+1,s} = \frac{1}{s}, \qquad \sum_{k=2}^{\infty} kI^*_{2k,s} = \frac{1}{s^2},$$

and for $I_{1,s}$ and $I^*_{3,s}$,

$$\sum_{s=0}^{\infty} (-1)^s I_{1,2s+1} = \frac{\pi}{4} - V_0, \qquad \sum_{s=1}^{\infty} (-1)^{s+1} I_{1,2s} = \tfrac{1}{2}I_{1,0} - \tfrac{1}{2}III_1, \tag{22}$$
$$\sum_{s=0}^{\infty} (-1)^s I^*_{3,2s+1} = \frac{4}{3}V^*_2 - \frac{\pi^3}{24}, \qquad \sum_{s=1}^{\infty} (-1)^{s+1} I^*_{3,2s} = \tfrac{1}{2}I^*_{3,0} - \tfrac{1}{2}III^*_3,$$

where

$$
\begin{aligned}
V_0 &= \int_0^\infty \frac{\sinh x\,dx}{\sinh 2x + 2x} = 0.52685\,63984, \\
III_1 &= 2\int_0^\infty \frac{x\tanh x\,dx}{\sinh 2x + 2x} = 0.47442\,96568, \\
V_2^* &= \frac{1}{2}\int_0^\infty \frac{x^2\sinh x\,dx}{\sinh 2x - 2x} = 1.40879\,56089, \\
III_3^* &= \frac{4}{3}\int_0^\infty \frac{x^3\tanh x\,dx}{\sinh 2x - 2x} = 1.41506\,33610.
\end{aligned} \tag{23}
$$

Owing to slow convergence, the tail part of the series in (22) can be found with the aid of the Euler transformation [1] or from the values of the series of inverse powers of natural numbers. The evaluation of the four integrals in (23) was considered by the author in a previous paper [2], but the values were given to 6D only. It is straightforward to evaluate again the first three integrals. It may be more convenient to evaluate the last one from the following series:

$$III_3^* = \frac{1}{3}\sum_{n=1}^{\infty}\binom{2n+2}{2}(I_{2n+2,0}^* - U_{2n+3}), \tag{24}$$

where

$$U_n = 1 + \frac{1}{3^n} + \frac{1}{5^n} + \frac{1}{7^n} + \cdots, \quad (n \geqslant 2). \tag{25}$$

Department of Mathematics
Virginia Polytechnic Institute and State University
Blacksburg, Virginia 24061

1. T. J. I'A. Bromwich, *Theory of Infinite Series*, 2nd ed., Macmillan, New York, 1926, p. 62.
2. C. B. Ling, "Tables of values of 16 integrals of algebraic-hyperbolic type," *MTAC*, v. 11, 1957, pp. 160–166.
3. C. B. Ling & J. Lin, "A new method of evaluation of Howland integrals," *Math. Comp.*, v. 25, 1971, pp. 331–337.
4. C. B. Ling, "Further evaluation of Howland integrals," *Math. Comp.*, v. 32, 1978, pp. 900–904.
5. C. B. Ling & M. J. Wu, "Evaluation of integrals of Howland type involving a Bessel function," *Math. Comp.*, v. 38, 1982, pp. 215–222.