Density Problems Involving $p_r(n)$

By Patrick J. Costello

Abstract. Lower bounds on the density of zeros of $p_r(n)$ are provided for certain values of $r$.

If $n$ is a nonnegative integer, define $p_r(n)$ as the coefficient of $x$ in $\prod_{n=1}^{\infty} (1 - x^n)^r$; i.e.,

$$\prod_{n=1}^{\infty} (1 - x^n)^r = \sum_{n=0}^{\infty} p_r(n) x^n.$$  

Two very important number-theoretic functions occur as particular choices of $r$. $p_{-1}(n)$ is the ordinary partition function (usually written as $p(n)$) and $p_{24}(n - 1)$ is the Ramanujan $\tau$-function (i.e., $\tau(n) = p_{24}(n - 1)$). The only known explicit formulas for $p_r(n)$ are those for $p_1(n)$ and $p_3(n)$ given by the following classical results: Euler's pentagonal number theorem says

$$\prod_{n=1}^{\infty} (1 - x^n) = 1 + \sum_{n=1}^{\infty} (-1)^n x^{(3n^2 - n)/2}.$$  

An immediate consequence of Jacobi's triple product identity is

$$\sum_{n=1}^{\infty} (1 - x^n)^3 = \sum_{n=0}^{\infty} (-1)^n (2n + 1) x^{(n^2 + n)/2}.$$  

The functions $p_r(n)$ enjoy many interesting congruence properties. Ramanujan [12] was able to show the following special congruences for the partition function:

$$p(5n + 4) \equiv 0 \pmod{5},$$
$$p(7n + 5) \equiv 0 \pmod{7},$$
$$p(11n + 6) \equiv 0 \pmod{11}.$$  

Further work on the partition function has been done by Watson [13] and Atkin [2]. Bambah proved the following congruences for $\tau(n)$:

$$\tau(n) \equiv n \sigma_3(n) \pmod{5^2},$$
$$\tau(n) \equiv n \sigma_3(n) \pmod{7},$$

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where \( \sigma_k(n) \) is the sum of the \( k \)th powers of the divisors of \( n \). Newman [10] proved the following theorem that gives congruence properties for infinitely many functions \( p_r(n) \):

**Theorem.** Let \( r = 4, 6, 8, 10, 14, 26 \). Let \( p \) be a prime greater than 3 such that \( r(p + 1) \equiv 0 \) (mod 24). Set \( \Delta = r(p^2 - 1)/24 \). Then, for all \( r \equiv r \) (mod \( p \)),

\[
p_r(np + \Delta) \equiv 0 \pmod{p}.
\]

Notice that for \( R = -1 \) the choices \( r = 4, p = 5; r = 6, p = 7; r = 10, p = 11 \) give the Ramanujan congruences (3), (4) and (5).

From the known congruence properties, many people were led to investigate the asymptotic density of values \( p_r(n) \) that are divisible by some fixed modulus \( m \). If we let

\[
d_r(m) = \lim \inf_{x \to \infty} \frac{1}{x} \sum_{n \leq x, p_r(n) \equiv 0 \pmod{m}} 1,
\]

then, in particular, congruence (3) says that \( d_{-1}(5) \geq 1/5 \). For the partition function, Atkin [3] and Klove [6] have made numerous improvements on the density estimates. However, numerical evidence by MacLean [8] seems to indicate that the proven estimates might be able to be improved even further.

By reconsidering Eqs. (1) and (2), it is easy to see that \( d_1(m) = 1 \) and \( d_3(m) = 1 \) for any modulus \( m \). This is primarily because \( p_1(n) = 0 \) and \( p_3(n) = 0 \) for almost all \( n \). Hence the density of zeros of \( p_r(n) \) gives a lower bound on \( d_r(m) \) for all \( m \). The aim of this paper will be to provide some information about the density of zeros of certain \( p_r(n) \). Since \( p_{-1}(n) \) represents the number of partitions of \( n \), it will never vanish. It is still an open question (generally attributed to D. H. Lehmer) as to whether \( \tau(n) \) is ever 0. It is known that for \( n < 113, 740, 236, 287, 998 \), \( \tau(n) \neq 0 \) [7].

From a quick glance at Newman's table of values of \( p_r(n) \) [11], one might also conjecture that \( p_r(n) \neq 0 \) for \( r = 5, 7, 9, 11, 12, 13, 16 \). On the basis of unpublished numerical tabulation performed by A. O. L. Atkin and M. Newman, values of \( n \) have been found for which \( p_r(n) = 0 \), \( r = 5, 7, 9, 11 \). This implies that \( p_r(n) \) vanishes infinitely often for these values. Our work will concentrate on \( p_r(n) \) with \( r = 2, 4, 6, 8, 10, 14, 26 \). We start with the definition of the density of zeros of \( p_r(n) \).

**Definition.** \( \delta_r = \lim_{x \to \infty} \inf_{x \to \infty} \frac{1}{x} \sum_{n \leq x, p_r(n) = 0} 1 \) represents the density of zeros of \( p_r(n) \).

Our first result gives a weak statement about the density of zeros of \( p_r(n) \).

**Theorem 1.** If \( r = 2, 4, 6, 8, 10, 14, 26 \) and \( q \) is a prime greater than 3 such that \( r(q + 1) \equiv 0 \pmod{24} \), then \( \delta_r \geq 1/(q + 1) \).

**Proof.** Under the given hypotheses, Newman [9] has shown that

\[
p_r(nq + \Delta) = (-q)^{(r-2)/2} p_r(n/q)
\]

for all nonnegative \( n \) and \( \Delta = r(q^2 - 1)/24 \). Since \( p_r(a) = 0 \) when \( a \) is not integral, if we let \( n = qm + k \) with \( k = 1, 2, \ldots, q - 1 \) in Eq. (6), we get

\[
p_r(q^2m + qk + \Delta) = 0.
\]
This gives us \( q - 1 \) distinct residue classes mod \( q^2 \) which are zeros of \( p_r(n) \). Thus far we have \( \delta_r \geq (q - 1)/q^2 \). If we now let \( n = q(q^2m + qk + \Delta) \) in Eq. (6), then we get \( p_r(q^2m + q^3k + q^2\Delta + \Delta) = (-q)^{(r-2)/2}p_r(q^2m + qk + \Delta) = 0 \) by Eq. (7). Continuing to multiply the new zeros obtained by \( q \) and resubstituting into Eq. (6) leads us to the fact that for any \( t > 1 \)

\[
(8) \quad p_r(q^{2t}m + q^{2t-1}k + q^{2t-2}\Delta + q^{2t-4}\Delta + \cdots + q^2\Delta + \Delta) = 0
\]

for all \( m \) and \( k = 1, 2, \ldots, q - 1 \). Hence we have \( q - 1 \) distinct residue classes mod \( q^{2t} \) which are zeros of \( p_r(n) \). We will now show that the new zeros produced by Eq. (8) are distinct from all the zeros obtained previously.

(i) Suppose that \( q^{2t}m_1 + q^{2t-1}k_1 + q^{2t-2}\Delta + \cdots + q^2\Delta + \Delta = q^{2m_2} + q^2k_2 + \Delta \) for some \( m_1, m_2 \in \mathbb{Z} \) and \( k_1, k_2 \in \{1, 2, \ldots, q - 1\} \). Then \( qk_2 = q^{2(q^{2t-2}m_1 + q^{2t-3}k_1 + \cdots + \Delta - m_2)} \), which would imply that \( q \mid k_2 \). But this contradicts the fact that \( 1 \leq k_2 \leq q - 1 \).

(ii) Suppose that \( 1 < s < t \) and

\[
q^{2s}m_1 + q^{2s-1}k_1 + q^{2s-2}\Delta + \cdots + q^2\Delta + \Delta = q^{2s}m_2 + q^{2s-1}k_2 + q^{2s-2}\Delta + \cdots + q^2\Delta + \Delta
\]

for some \( m_1, m_2 \in \mathbb{Z} \) and \( k_1, k_2 \in \{1, 2, \ldots, q - 1\} \). Then

\[
q^{2s-1}k_2 = q^{2s}(q^{2t-2}m_1 + q^{2t-3}k_1 + \cdots + \Delta - m_2),
\]

which would again imply the impossibility that \( q \) divides \( k_2 \).

Therefore each resubstitution of zeros into Eq. (6) produces a whole new set of zeros of \( p_r(n) \). Since the \( t \)th application of this process produces \( q - 1 \) residue classes mod \( q^{2t} \) which are zeros of \( p_r(n) \) and these are different zeros from the \( q - 1 \) classes produced mod \( q^{2s} \) for all \( s < t \), we can inductively see that we have in fact accumulated \( \Sigma_{t=1}^{r}(q - 1)q^{2(r-1)} \) (where \((q - 1)q^{2(r-1)}\) comes from the \( q - 1 \) classes mod \( q^{2t} \)) distinct residue classes mod \( q^{2t} \) which are zeros of the function \( p_r(n) \). Hence

\[
\delta_r \geq \frac{q - 1}{q^2} + \frac{q - 1}{q^4} + \cdots + \frac{q - 1}{q^{2t}}.
\]

Letting \( t \to \infty \), we have \( \delta_r \geq (q - 1)/(q^2 - 1) = 1/(q + 1) \). \( \square \)

In particular, Theorem 1 says that \( \delta_2 \geq 1/12, \delta_4 \geq 1/6, \delta_6 \geq 1/8, \delta_8 \geq 1/6, \delta_{10} \geq 1/12, \delta_{14} \geq 1/12, \delta_{26} \geq 1/12 \). These bounds all come from using the smallest \( q > 3 \) which satisfies \( r(q + 1) \equiv 0 \pmod{24} \). We will now see that we can actually allow \( q \) to vary for a particular \( r \) and obtain a much better bound on the density of zeros.

**Theorem 2.** If \( r = 2, 4, 6, 8, 10, 14, 26 \) and \( q_i \) is the \( i \)th prime greater than 3 with \( r(q_i + 1) \equiv 0 \pmod{24} \), then

\[
\delta_r \geq \frac{1}{q_1 + 1} + \max_N \sum_{i=2}^{N} \left( \frac{1}{q_i + 1} - \frac{1}{q_i} \sum_{j=1}^{i-1} \frac{1}{q_j} \right).
\]
Remark. Notice that \( q_i \equiv -1 \pmod{24/r} \) for \( r = 2, 4, 6, 8 \) and \( q_i \equiv 11 \pmod{12} \) for \( r = 10, 14, 26 \). By Dirichlet's theorem there are infinitely many such \( q_i \) for each \( r \), and \( \sum_{j=1}^{i-1} \frac{1}{q_j} \) actually diverges as \( i \to \infty \) \cite{1} so eventually

\[
\frac{1}{q_i + 1} - \frac{1}{\sum_{j=1}^{i-1} \frac{1}{q_j}}
\]

is a negative number.

Proof of Theorem 2. Let \( \Delta_j = \frac{r(q_i^2 - 1)}{24}, A_j = \{ n \mid n \equiv \Delta_j \pmod{q_j} \mbox{ and } p_r(n) = 0 \}, A = \{ n \mid p_r(n) = 0 \}, \text{ and } \delta_r(S) = \lim_{x \to \infty} \inf \frac{1}{x} \sum_{n \leq x; n \in S} 1.
\]

The proof of Theorem 1 has actually shown that \( \delta_r(A_j) \geq 1/(q_i + 1) \). For all \( N, A_1 \cup \bigcup_{i=2}^{N} [A_i \setminus \bigcup_{j=1}^{i-1} (A_i \cap A_j)] \) is a disjoint union contained in \( A \), and we have

\[
\delta_r = \delta_r(A) \geq \delta_r(A_1) + \sum_{i=2}^{N} \delta_r \left( A_i \setminus \bigcup_{j=1}^{i-1} (A_i \cap A_j) \right)
\]

(9)

As we have a lower bound for \( \delta_r(A_j) \), we now attempt to find a lower bound for \( -\delta_r \left( \bigcup_{j=1}^{i-1} (A_i \cap A_j) \right) \). We have

\[
A_k \cap A_m = \{ n \mid n \equiv \Delta_k \pmod{q_k}, n \equiv \Delta_m \pmod{q_m}, p_r(n) = 0 \} \subseteq \{ n \mid n \equiv \Delta_k \pmod{q_k}, n \equiv \Delta_m \pmod{q_m} \}
\]

for some \( \Delta_{k,m} \) by the Chinese Remainder Theorem. This means \( A_k \cap A_m \) is contained inside one residue class mod \( q_kq_m \), and so

\[
\delta_r \left[ \bigcup_{j=1}^{i-1} (A_i \cap A_j) \right] \leq \sum_{j=1}^{i-1} \delta_r(A_i \cap A_j) \leq \sum_{j=1}^{i-1} \frac{1}{q_jq_i},
\]

which implies that

\[
\delta_r(A_j) - \delta_r \left[ \bigcup_{j=1}^{i-1} (A_i \cap A_j) \right] \geq \frac{1}{q_i + 1} - \frac{1}{\sum_{j=1}^{i-1} \frac{1}{q_j}}.
\]

Using this in Eq. (9), we can finally conclude that

\[
\delta_r \geq \frac{1}{q_1 + 1} + \max_{N} \sum_{i=2}^{N} \left( \frac{1}{q_i + 1} - \frac{1}{\sum_{j=1}^{i-1} \frac{1}{q_j}} \right). \quad \square
\]

The lower bounds on the density of zeros provided by Theorem 2 are quite an improvement over those of Theorem 1, as is illustrated when we compute the partial sums

\[
M_{r,N} = \frac{1}{q_1 + 1} + \sum_{i=2}^{N} \left( \frac{1}{q_i + 1} - \frac{1}{\sum_{j=1}^{i-1} \frac{1}{q_j}} \right).
\]
Table 1

Lower bounds on $\delta_r$ from Theorem 2

<table>
<thead>
<tr>
<th>$r$</th>
<th>$1/q_1 + 1$ (bound from Thm. 1)</th>
<th>$q_N$ (bound from Thm. 2)</th>
<th>$M_rN$</th>
<th>behavior of $M_rN$ at $N$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2, 10, 14, 26</td>
<td>.083</td>
<td>2560367</td>
<td>.360956</td>
<td>still increasing</td>
</tr>
<tr>
<td>4, 8</td>
<td>.16</td>
<td>85517</td>
<td>.478752</td>
<td>maximum</td>
</tr>
<tr>
<td>6</td>
<td>.125</td>
<td>473887</td>
<td>.484869</td>
<td>maximum</td>
</tr>
</tbody>
</table>

These values were computed on Ohio State's Amdahl 470 using double-precision FORTRAN.

Finally, we compare these lower bounds on $\delta_r$ with the actual densities of zeros of tabled values of $p_r(n)$ [11]. Let $\delta_{r,x} = x^{-1} \sum_{n \leq x; p_r(n) = 0} 1$.

Table 2

Densities from tabled zeros

<table>
<thead>
<tr>
<th>$r$</th>
<th>$x$</th>
<th>$\delta_{r,x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>800</td>
<td>.5037</td>
</tr>
<tr>
<td>4</td>
<td>800</td>
<td>.3325</td>
</tr>
<tr>
<td>6</td>
<td>800</td>
<td>.4412</td>
</tr>
<tr>
<td>8</td>
<td>800</td>
<td>.5162</td>
</tr>
<tr>
<td>10</td>
<td>800</td>
<td>.3200</td>
</tr>
<tr>
<td>14</td>
<td>750</td>
<td>.3613</td>
</tr>
<tr>
<td>26</td>
<td>1920</td>
<td>.1969 (•)</td>
</tr>
</tbody>
</table>

(•) this is from a table obtained from M. Newman

The bounds on $\delta_r$ given by Theorem 2 exceed these partial densities for $r = 4, 6, 10, 26$. In these cases, the zeros must occur more frequently as $x \to \infty$.

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