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# Density Problems Involving $p_r(n)$

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**Abstract.** Lower bounds on the density of zeros of $p_r(n)$ are provided for certain values of $r$.

If $n$ is a nonnegative integer, define $p_r(n)$ as the coefficient of $x$ in $\prod_{n=1}^{\infty}(1-x^n)^r$; i.e.,

$$\prod_{n=1}^{\infty}(1-x^n)^r = \sum_{n=0}^{\infty} p_r(n)x^n.$$

Two very important number-theoretic functions occur as particular choices of $r$. $p_{-1}(n)$ is the ordinary partition function (usually written as $p(n)$) and $p_{24}(n-1)$ is the Ramanujan $\tau$-function (i.e., $\tau(n) = p_{24}(n-1)$). The only known explicit formulas for $p_r(n)$ are those for $p_1(n)$ and $p_3(n)$ given by the following classical results:

Euler's pentagonal number theorem says

$$(1) \qquad \prod_{n=1}^{\infty}(1-x^n) = 1 + \sum_{n=1}^{\infty}(-1)^n x^{(3n^2 \pm n)/2}.$$

An immediate consequence of Jacobi's triple product identity is

$$(2) \qquad \sum_{n=1}^{\infty}(1-x^n)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)x^{(n^2+n)/2}.$$

The functions $p_r(n)$ enjoy many interesting congruence properties. Ramanujan [12] was able to show the following special congruences for the partition function:

$$(3) \qquad p(5n+4) \equiv 0 \pmod{5},$$

$$(4) \qquad p(7n+5) \equiv 0 \pmod{7},$$

$$(5) \qquad p(11n+6) \equiv 0 \pmod{11}.$$

Further work on the partition function has been done by Watson [13] and Atkin [2]. Bambah proved the following congruences for $\tau(n)$:

$$\tau(n) \equiv n\sigma_9(n) \pmod{5^2},$$

$$\tau(n) \equiv n\sigma_3(n) \pmod{7},$$

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where $\sigma_k(n)$ is the sum of the $k$th powers of the divisors of $n$. Newman [10] proved the following theorem that gives congruence properties for infinitely many functions $p_r(n)$:

THEOREM. *Let* $r = 4, 6, 8, 10, 14, 26$. *Let* $p$ *be a prime greater than* 3 *such that* $r(p+1) \equiv 0 \pmod{24}$. *Set* $\Delta = r(p^2-1)/24$. *Then, for all* $R \equiv r \pmod{p}$,

$$p_R(np + \Delta) \equiv 0 \pmod{p}.$$

Notice that for $R = -1$ the choices $r = 4, p = 5$; $r = 6, p = 7$; $r = 10, p = 11$ give the Ramanujan congruences (3), (4) and (5).

From the known congruence properties, many people were led to investigate the asymptotic density of values $p_r(n)$ that are divisible by some fixed modulus $m$. If we let

$$d_r(m) = \liminf_{x \to \infty} x^{-1} \sum_{\substack{n \leqslant x \\ p_r(n) \equiv 0 \pmod{m}}} 1,$$

then, in particular, congruence (3) says that $d_{-1}(5) \geqslant 1/5$. For the partition function, Atkin [3] and Klove [6] have made numerous improvements on the density estimates. However, numerical evidence by MacLean [8] seems to indicate that the proven estimates might be able to be improved even further.

By reconsidering Eqs. (1) and (2), it is easy to see that $d_1(m) = 1$ and $d_3(m) = 1$ for any modulus $m$. This is primarily because $p_1(n) = 0$ and $p_3(n) = 0$ for almost all $n$. Hence the density of zeros of $p_r(n)$ gives a lower bound on $d_r(m)$ for all $m$. The aim of this paper will be to provide some information about the density of zeros of certain $p_r(n)$. Since $p_{-1}(n)$ represents the number of partitions of $n$, it will never vanish. It is still an open question (generally attributed to D. H. Lehmer) as to whether $\tau(n)$ is ever 0. It is known that for $n < 113, 740, 236, 287, 998$ $\tau(n) \neq 0$ [7]. From a quick glance at Newman's table of values of $p_r(n)$ [11], one might also conjecture that $p_r(n) \neq 0$ for $r = 5, 7, 9, 11, 12, 13, 16$. On the basis of unpublished numerical tabulation performed by A. O. L. Atkin and M. Newman, values of $n$ have been found for which $p_r(n) = 0$, $r = 5, 7, 9, 11$. This implies that $p_r(n)$ vanishes infinitely often for these values. Our work will concentrate on $p_r(n)$ with $r = 2, 4, 6, 8, 10, 14, 26$. We start with the definition of the density of zeros of $p_r(n)$.

*Definition.* $\delta_r = \liminf_{x\to\infty} x^{-1}\sum_{n \leqslant x;\, p_r(n)=0} 1$ represents the density of zeros of $p_r(n)$.

Our first result gives a weak statement about the density of zeros of $p_r(n)$.

THEOREM 1. *If* $r = 2, 4, 6, 8, 10, 14, 26$ *and* $q$ *is a prime greater than* 3 *such that* $r(q+1) \equiv 0 \pmod{24}$, *then* $\delta_r \geqslant 1/(q+1)$.

*Proof.* Under the given hypotheses, Newman [9] has shown that

$$p_r(nq + \Delta) = (-q)^{(r-2)/2} p_r(n/q) \tag{6}$$

for all nonnegative $n$ and $\Delta = r(q^2-1)/24$. Since $p_r(a) = 0$ when $a$ is not integral, if we let $n = qm + k$ with $k = 1, 2, \ldots, q-1$ in Eq. (6), we get

$$p_r(q^2 m + qk + \Delta) = 0. \tag{7}$$

This gives us $q-1$ distinct residue classes mod $q^2$ which are zeros of $p_r(n)$. Thus far we have $\delta_r \geqslant (q-1)/q^2$. If we now let $n = q(q^2m + qk + \Delta)$ in Eq. (6), then we get $p_r(q^4m + q^3k + q^2\Delta + \Delta) = (-q)^{(r-2)/2}p_r(q^2m + qk + \Delta) = 0$ by Eq. (7). Continuing to multiply the new zeros obtained by $q$ and resubstituting into Eq. (6) leads us to the fact that for any $t > 1$

$$(8) \qquad p_r\left(q^{2t}m + q^{2t-1}k + q^{2t-2}\Delta + q^{2t-4}\Delta + \cdots + q^2\Delta + \Delta\right) = 0$$

for all $m$ and $k = 1, 2, \ldots, q-1$. Hence we have $q-1$ distinct residue classes mod $q^{2t}$ which are zeros of $p_r(n)$. We will now show that the new zeros produced by Eq. (8) are distinct from all the zeros obtained previously.

(i) Suppose that $q^{2t}m_1 + q^{2t-1}k_1 + q^{2t-2}\Delta + \cdots + q^2\Delta + \Delta = q^2m_2 + qk_2 + \Delta$ for some $m_1, m_2 \in \mathbf{Z}$ and $k_1, k_2 \in \{1, 2, \ldots, q-1\}$. Then $qk_2 = q^2(q^{2t-2}m_1 + q^{2t-3}k_1 + \cdots + \Delta - m_2)$, which would imply that $q \mid k_2$. But this contradicts the fact that $1 \leqslant k_2 \leqslant q-1$.

(ii) Suppose that $1 < s < t$ and

$$q^{2t}m_1 + q^{2t-1}k_1 + q^{2t-2}\Delta + \cdots + q^2\Delta + \Delta$$
$$= q^{2s}m_2 + q^{2s-1}k_2 + q^{2s-2}\Delta + \cdots + q^2\Delta + \Delta$$

for some $m_1, m_2 \in \mathbf{Z}$ and $k_1, k_2 \in \{1, 2, \ldots, q-1\}$. Then

$$q^{2s-1}k_2 = q^{2s}\left(q^{2t-2s}m_1 + q^{2t-2s-1}k_1 + \cdots + \Delta - m_2\right),$$

which would again imply the impossibility that $q$ divides $k_2$.

Therefore each resubstitution of zeros into Eq. (6) produces a whole new set of zeros of $p_r(n)$. Since the $t$th application of this process produces $q-1$ residue classes mod $q^{2t}$ which are zeros of $p_r(n)$ and these are different zeros from the $q-1$ classes produced mod $q^{2s}$ for all $s < t$, we can inductively see that we have in fact accumulated $\sum_{i=1}^{t}(q-1)q^{2(t-i)}$ (where $(q-1)q^{2(t-i)}$ comes from the $q-1$ classes mod $q^{2i}$) distinct residue classes mod $q^{2t}$ which are zeros of the function $p_r(n)$. Hence

$$\delta_r \geqslant \frac{q-1}{q^2} + \frac{q-1}{q^4} + \cdots + \frac{q-1}{q^{2t}}.$$

Letting $t \to \infty$, we have $\delta_r \geqslant (q-1)/(q^2-1) = 1/(q+1)$. □

In particular, Theorem 1 says that $\delta_2 \geqslant 1/12$, $\delta_4 \geqslant 1/6$, $\delta_6 \geqslant 1/8$, $\delta_8 \geqslant 1/6$, $\delta_{10} \geqslant 1/12$, $\delta_{14} \geqslant 1/12$, $\delta_{26} \geqslant 1/12$. These bounds all come from using the smallest $q > 3$ which satisfies $r(q+1) \equiv 0 \pmod{24}$. We will now see that we can actually allow $q$ to vary for a particular $r$ and obtain a much better bound on the density of zeros.

THEOREM 2. *If* $r = 2, 4, 6, 8, 10, 14, 26$ *and* $q_i$ *is the ith prime greater than* 3 *with* $r(q_i + 1) \equiv 0 \pmod{24}$, *then*

$$\delta_r \geqslant \frac{1}{q_1 + 1} + \max_N \sum_{i=2}^{N}\left(\frac{1}{q_i + 1} - \frac{1}{q_i}\sum_{j=1}^{i-1}\frac{1}{q_j}\right).$$

*Remark*. Notice that $q_i \equiv -1 \pmod{24/r}$ for $r = 2, 4, 6, 8$ and $q_i \equiv 11 \pmod{12}$ for $r = 10, 14, 26$. By Dirichlet's theorem there are infinitely many such $q_i$ for each $r$, and $\sum_{j=1}^{i-1} 1/q_j$ actually diverges as $i \to \infty$ [1] so eventually

$$\frac{1}{q_i + 1} - \frac{1}{q_i} \sum_{j=1}^{i-1} \frac{1}{q_j}$$

is a negative number.

*Proof of Theorem* 2. Let $\Delta_i = r(q_i^2 - 1)/24$, $A_i = \{n \mid n \equiv \Delta_i \pmod{q_i}$ and $p_r(n) = 0\}$, $A = \{n \mid p_r(n) = 0\}$, and $\delta_r(S) = \lim_{x\to\infty} \inf x^{-1} \sum_{n \leqslant x;\, n \in S} 1$.

The proof of Theorem 1 has actually shown that $\delta_r(A_i) \geqslant 1/(q_i + 1)$. For all $N$, $A_1 \cup \{\bigcup_{i=2}^{N} [A_i \setminus \bigcup_{j=1}^{i-1} (A_i \cap A_j)]\}$ is a disjoint union contained in $A$, and we have

$$\begin{aligned} \delta_r = \delta_r(A) &\geqslant \delta_r(A_1) + \sum_{i=2}^{N} \delta_r\left[A_i \setminus \bigcup_{j=1}^{i-1} (A_i \cap A_j)\right] \\ &= \delta_r(A_1) + \sum_{i=2}^{N} \left\{\delta_r(A_i) - \delta_r\left[\bigcup_{j=1}^{i-1} (A_i \cap A_j)\right]\right\}. \end{aligned} \tag{9}$$

As we have a lower bound for $\delta_r(A_i)$, we now attempt to find a lower bound for $-\delta_r[\bigcup_{j=1}^{i-1} (A_i \cap A_j)]$. We have

$$\begin{aligned} A_k \cap A_m &= \{n \mid n \equiv \Delta_k \pmod{q_k}, n \equiv \Delta_m \pmod{q_m}, p_r(n) = 0\} \\ &\subseteq \{n \mid n \equiv \Delta_k \pmod{q_k}, n \equiv \Delta_m \pmod{q_m}\} \\ &= \{n \mid n \equiv \Delta_{k,m} \pmod{q_k q_m}\} \end{aligned}$$

for some $\Delta_{k,m}$ by the Chinese Remainder Theorem. This means $A_k \cap A_m$ is contained inside one residue class mod $q_k q_m$, and so

$$\delta_r\left[\bigcup_{j=1}^{i-1} (A_i \cap A_j)\right] \leqslant \sum_{j=1}^{i-1} \delta_r(A_i \cap A_j) \leqslant \sum_{j=1}^{i-1} \frac{1}{q_i q_j},$$

which implies that

$$\delta_r(A_i) - \delta_r\left[\bigcup_{j=1}^{i-1} (A_i \cap A_j)\right] \geqslant \frac{1}{q_i + 1} - \frac{1}{q_i} \sum_{j=1}^{i-1} \frac{1}{q_j}.$$

Using this in Eq. (9), we can finally conclude that

$$\delta_r \geqslant \frac{1}{q_1 + 1} + \max_N \sum_{i=2}^{N} \left(\frac{1}{q_i + 1} - \frac{1}{q_i} \sum_{j=1}^{i-1} \frac{1}{q_j}\right). \quad \square$$

The lower bounds on the density of zeros provided by Theorem 2 are quite an improvement over those of Theorem 1, as is illustrated when we compute the partial sums

$$M_{r,N} = \frac{1}{q_1 + 1} + \sum_{i=2}^{N} \left(\frac{1}{q_i + 1} - \frac{1}{q_i} \sum_{j=1}^{i-1} \frac{1}{q_j}\right):$$

TABLE 1

*Lower bounds on $\delta_r$ from Theorem 2*

| $r$ | $\frac{1}{q_1+1}$ (bound from Thm. 1) | $q_N$ | $M_{r,N}$ (bound from Thm. 2) | behavior of $M_{r,N}$ at $N$ |
|---|---|---|---|---|
| 2, 10, 14, 26 | $.08\bar{3}$ | 2560367 | .360956 | still increasing |
| 4, 8 | $.1\bar{6}$ | 85517 | .478752 | maximun |
| 6 | .125 | 473887 | .484869 | maximum |

These values were computed on Ohio State's Amdahl 470 using double-precision FORTRAN.

Finally, we compare these lower bounds on $\delta_r$ with the actual densities of zeros of tabled values of $p_r(n)$ [11]. Let $\delta_{r,x} = x^{-1}\sum_{n \leq x;\, p_r(n)=0} 1$.

TABLE 2

*Densities from tabled zeros*

| $r$ | $x$ | $\delta_{r,x}$ |
|---|---|---|
| 2 | 800 | .5037 |
| 4 | 800 | .3325 |
| 6 | 800 | .4412 |
| 8 | 800 | .5162 |
| 10 | 800 | .3200 |
| 14 | 750 | .3613 |
| 26 | 1920 | .1969 (*) |

(*) this is from a table obtained from M. Newman

The bounds on $\delta_r$ given by Theorem 2 exceed these partial densities for $r = 4$, 6, 10, 26. In these cases, the zeros must occur more frequently as $x \to \infty$.

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