

On Certain Extrapolation Methods for the Numerical Solution of Integro-Differential Equations*

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Abstract. Asymptotic error expansions have been obtained for certain numerical methods for linear Volterra integro-differential equations. These results permit the application of extrapolation procedures. Computational examples are presented.

1. Introduction. Consider the linear Volterra integro-differential equation

$$(1) \quad \begin{aligned} y'(x) &= a(x) + b(x)y(x) + \int_{x_0}^x k(x, s)y(s) ds, \\ y(x_0) &= y_0, \quad x_0 \leq x \leq L, \end{aligned}$$

where $a(x)$, $b(x)$, and $k(x, s)$ are given continuous functions for $x_0 \leq x, s \leq L$, and y_0 is a given real number. Numerical solutions of more general Volterra integro-differential equations have been investigated by many authors. Methods that use finite difference and quadrature techniques have been studied by, for example, Brunner and Lambert [1], Day [2], Feldstein and Sopka [3], Goldfine [4], Linz [6], Makroglou [8], McKee [9], Mocarsky [10], Wolfe and Phillips [11]. Feldstein and Sopka [3] have also discussed asymptotic error expansion and extrapolation for their Taylor algorithms for integro-differential equations.

It is the purpose of this paper to study the asymptotic expansions for the errors associated with certain simple numerical methods. Such a study will permit the application of extrapolation procedures. As a consequence, high order of accuracy in the numerical solution of (1) can be obtained with only a modest amount of work. This will then be demonstrated by computational examples. Our work is inspired by Linz [7] in which the extrapolation, based on a simple numerical method for linear Volterra integro-differential equations of the first kind, is very effective.

In the subsequent discussion, y_n will denote an approximate value of $y(x_n)$, where $x_n = x_0 + nh$, $n = 1, 2, \dots, N$, and $h = (L - x_0)/N$. For the known functions $a(x)$, $b(x)$, and $k(x, s)$, a_i , b_i , and $k_{i,j}$ will denote $a(x_0 + ih)$, $b(x_0 + ih)$, and $k(x_0 + ih, x_0 + jh)$.

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2. The Algorithms and Asymptotic Error Expansions. Integrating (1) from x_{n-1} to x_n , we have

$$(2) \quad y(x_n) = y(x_{n-1}) + \int_{x_{n-1}}^{x_n} [a(t) + b(t)y(t)] dt + \int_{x_{n-1}}^{x_n} \int_{x_0}^t k(t, s)y(s) ds dt.$$

Replacing the integrals from x_{n-1} to x_n by the two-step Adams-Moulton rule

$$(3) \quad \int_{x_{n-1}}^{x_n} \phi(x) dx = \frac{h}{12} [5\phi(x_n) + 8\phi(x_{n-1}) - \phi(x_{n-2})] - \frac{h^4}{24} \phi'''(\xi),$$

and replacing the remaining inner integral by the Euler-Maclaurin formula (see Hildebrand [5, p. 202])

$$(4) \quad \int_{x_0}^{x_r} \phi(x) dx = h \left[\frac{1}{2} \phi(x_0) + \phi(x_1) + \dots + \phi(x_{r-1}) + \frac{1}{2} \phi(x_r) \right] - \frac{h^2}{12} [\phi'(x_r) - \phi'(x_0)] + O(h^4),$$

we obtain from (2) that

$$(5) \quad \begin{aligned} y(x_n) = & y(x_{n-1}) + \frac{h}{12} [5(a(x_n) + b(x_n)y(x_n)) \\ & + 8(a(x_{n-1}) + b(x_{n-1})y(x_{n-1})) \\ & - (a(x_{n-2}) + b(x_{n-2})y(x_{n-2}))] \\ & + \frac{h^2}{12} \left\{ 5 \left[\frac{1}{2} k(x_n, x_0)y(x_0) + \sum_{i=1}^{n-1} k(x_n, x_i)y(x_i) + \frac{1}{2} k(x_n, x_n)y(x_n) \right] \right. \\ & + 8 \left[\frac{1}{2} k(x_{n-1}, x_0)y(x_0) + \sum_{i=1}^{n-2} k(x_{n-1}, x_i)y(x_i) \right. \\ & \left. \left. + \frac{1}{2} k(x_{n-1}, x_{n-1})y(x_{n-1}) \right] \right. \\ & - \left[\frac{1}{2} k(x_{n-2}, x_0)y(x_0) + \sum_{i=1}^{n-3} k(x_{n-2}, x_i)y(x_i) \right. \\ & \left. \left. + \frac{1}{2} k(x_{n-2}, x_{n-2})y(x_{n-2}) \right] \right\} \\ & + Q_n, \end{aligned}$$

where

$$Q_n = -\frac{h^3}{144} \left\{ 5 \left[\frac{\partial}{\partial s} (k(x_n, s)y(s)) \right]_{x_0}^{x_n} + 8 \left[\frac{\partial}{\partial s} (k(x_{n-1}, s)y(s)) \right]_{x_0}^{x_{n-1}} - \left[\frac{\partial}{\partial s} (k(x_{n-2}, s)y(s)) \right]_{x_0}^{x_{n-2}} \right\} + O(h^4),$$

or, using (3) again,

$$(6) \quad Q_n = -\frac{h^2}{12} \int_{x_{n-1}}^{x_n} f(x) dx + O(h^4),$$

where

$$(7) \quad f(x) = \left[\frac{\partial}{\partial s} (k(x, s)y(s)) \right]_{x_0}^x$$

if the function $f(x)$ is C^3 in $[x_0, L]$.

From (5) we have the following algorithm.

ALGORITHM A.

Get starting values: y_0, y_1 .

Compute y_n , for $n = 2, 3, \dots, N$, according to

$$(8) \quad \begin{aligned} y_n = y_{n-1} &+ \frac{h}{12} [5(a_n + b_n y_n) + 8(a_{n-1} + b_{n-1} y_{n-1}) - (a_{n-2} + b_{n-2} y_{n-2})] \\ &+ \frac{h^2}{12} \left[5 \left(\frac{1}{2} k_{n,0} y_0 + \sum_{i=1}^{n-1} k_{n,i} y_i + \frac{1}{2} k_{n,n} y_n \right) \right. \\ &\quad \left. + 8 \left(\frac{1}{2} k_{n-1,0} y_0 + \sum_{i=1}^{n-2} k_{n-1,i} y_i + \frac{1}{2} k_{n-1,n-1} y_{n-1} \right) \right. \\ &\quad \left. - \left(\frac{1}{2} k_{n-2,0} y_0 + \sum_{i=1}^{n-3} k_{n-2,i} y_i + \frac{1}{2} k_{n-2,n-2} y_{n-2} \right) \right]. \end{aligned}$$

Now, let $e(x)$ be the solution of

$$(9) \quad e'(x) = b(x)e(x) + \int_{x_0}^x k(x, s)e(s) ds - \frac{1}{12} f(x), \quad e(x_0) = 0,$$

where $f(x)$ is given by (7). Using the same approach as before and with appropriate assumption on smoothness, we have

$$(10) \quad \begin{aligned} e(x_n) = e(x_{n-1}) &+ \frac{h}{12} [5b(x_n)e(x_n) + 8b(x_{n-1})e(x_{n-1}) - b(x_{n-2})e(x_{n-2})] \\ &+ \frac{h^2}{12} \left\{ 5 \left[\sum_{i=1}^{n-1} k(x_n, x_i)e(x_i) + \frac{1}{2} k(x_n, x_n)e(x_n) \right] \right. \\ &\quad \left. + 8 \left[\sum_{i=1}^{n-2} k(x_{n-1}, x_i)e(x_i) + \frac{1}{2} k(x_{n-1}, x_{n-1})e(x_{n-1}) \right] \right. \\ &\quad \left. - \left[\sum_{i=1}^{n-3} k(x_{n-2}, x_i)e(x_i) + \frac{1}{2} k(x_{n-2}, x_{n-2})e(x_{n-2}) \right] \right\} \\ &- \frac{1}{12} \int_{x_{n-1}}^{x_n} f(x) dx + O(h^3). \end{aligned}$$

Let $\rho_n = y(x_n) - y_n - h^2e(x_n)$, $n = 0, 1, \dots, N$. We see from (5), (8), and (10) that ρ_n satisfies the following equation.

$$(11) \quad \begin{aligned} \rho_n = & \rho_{n-1} + \frac{h}{12}(5b_n\rho_n + 8b_{n-1}\rho_{n-1} - b_{n-2}\rho_{n-2}) \\ & + \frac{h^2}{12} \left[5 \left(\sum_{i=1}^{n-1} k_{n,i}\rho_i + \frac{1}{2}k_{n,n}\rho_n \right) + 8 \left(\sum_{i=1}^{n-2} k_{n-1,i}\rho_i + \frac{1}{2}k_{n-1,n-1}\rho_{n-1} \right) \right. \\ & \left. - \left(\sum_{i=1}^{n-3} k_{n-2,i}\rho_i + \frac{1}{2}k_{n-2,n-2}\rho_{n-2} \right) \right] \\ & + O(h^4), \quad n \geq 2. \end{aligned}$$

By assumption on initial conditions, $\rho_0 = 0$. Suppose we choose the starting value y_1 so that

$$y(x_1) - y_1 = O(h^4).$$

Furthermore,

$$(12) \quad \begin{aligned} e(x_1) &= e(x_0) + e'(x_0)h + \frac{1}{2}e''(x_0)h^2 + \dots \\ &= \frac{1}{2}e''(x_0)h^2 + \dots, \end{aligned}$$

since $e(x_0) = e'(x_0) = 0$ by (9). Then we have

$$(13) \quad \rho_1 = y(x_1) - y_1 - h^2e(x_1) = O(h^4).$$

Then, it is clear that (11) implies $\rho_n = O(h^4)$, for $n \geq 2$.

We have thus proved the following theorem.

THEOREM 1. Assume that $a(x)$ and $b(x)$ are C^3 , and $k(x, s)$ is $C^{3,4}$, for $x_0 \leq x$, $s \leq L$. Then the approximations y_n , $n \geq 2$, computed from Algorithm A with $O(h^4)$ starting values, satisfy the relation

$$y(x_n) = y_n + h^2e(x_n) + O(h^4).$$

Now, the extrapolation procedure can be used. Let $Y(x, h)$ denote the approximate solution at x with step-size h . Then, by Theorem 1, we have

$$y(x) = Y(x, h) + h^2e(x) + O(h^4).$$

We then obtain immediately that

$$y(x) = \frac{1}{3} \left(4Y\left(x, \frac{h}{2}\right) - Y(x, h) \right) + O(h^4).$$

Thus a better approximate value at x is obtained with fourth order accuracy.

Now, instead of (3), let us use the three-step Adams-Moulton rule

$$(14) \quad \begin{aligned} \int_{x_{n-1}}^{x_n} \phi(x) dx &= \frac{h}{24} [9\phi(x_n) + 19\phi(x_{n-1}) - 5\phi(x_{n-2}) + \phi(x_{n-3})] \\ &\quad - \frac{19h^5}{720} \phi^{(4)}(\xi), \end{aligned}$$

together with the Euler-Maclaurin formula (4), in Eq. (2). This leads to the following algorithm.

ALGORITHM B.

Get starting values: y_0, y_1, y_2 .

Compute y_n , for $n = 3, 4, \dots, N$, according to

$$\begin{aligned}
 (15) \quad y_n = & y_{n-1} + \frac{h}{24} [9(a_n + b_n y_n) + 19(a_{n-1} + b_{n-1} y_{n-1}) \\
 & - 5(a_{n-2} + b_{n-2} y_{n-2}) + a_{n-3} + b_{n-3} y_{n-3}] \\
 & + \frac{h^2}{24} \left[9 \left(\frac{1}{2} k_{n,0} y_0 + \sum_{i=1}^{n-1} k_{n,i} y_i + \frac{1}{2} k_{n,n} y_n \right) \right. \\
 & + 19 \left(\frac{1}{2} k_{n-1,0} y_0 + \sum_{i=1}^{n-2} k_{n-1,i} y_i + \frac{1}{2} k_{n-1,n-1} y_{n-1} \right) \\
 & - 5 \left(\frac{1}{2} k_{n-2,0} y_0 + \sum_{i=1}^{n-3} k_{n-2,i} y_i + \frac{1}{2} k_{n-2,n-2} y_{n-2} \right) \\
 & \left. + \left(\frac{1}{2} k_{n-3,0} y_0 + \sum_{i=1}^{n-4} k_{n-3,i} y_i + \frac{1}{2} k_{n-3,n-3} y_{n-3} \right) \right].
 \end{aligned}$$

Again, let $e(x)$ be the solution of (9) and $\rho_n = y(x_n) - y_n - h^2 e(x_n)$, $n = 0, 1, \dots, N$. This time we find that ρ_n satisfies an equation similar to (11) but for $n \geq 3$ and with an $O(h^5)$ error term. Using an argument similar to that leading to (13), we obtain easily that $\rho_1 = \rho_2 = O(h^4)$. Then this leads again to the asymptotic error expansion

$$y(x_n) = y_n + h^2 e(x_n) + O(h^4).$$

Now, from (9) we see that

$$e''(x_0) = -\frac{1}{12} f'(x_0).$$

Suppose that

$$(16) \quad f'(x_0) = O(h).$$

Then from (12) we will have $e(x_0 + h) = O(h^3)$. This in turn will lead to $\rho_1 = \rho_2 = O(h^5)$ if we choose $O(h^5)$ starting values. Then the equation on ρ_n implies that $\rho_n = O(h^5)$, and thus the asymptotic error expansion

$$y(x_n) = y_n + h^2 e(x_n) + O(h^5).$$

By differentiating (7) we have

$$(17) \quad f'(x_0) = k_{ss}(x_0, x_0) y(x_0) + 2k_s(x_0, x_0) y'(x_0) + k(x_0, x_0) y''(x_0).$$

One sufficient condition for (16) to hold is seen to be

$$(18) \quad k(x_0, x_0) = k_s(x_0, x_0) = k_{ss}(x_0, x_0) = 0.$$

THEOREM 2. Assume that $a(x)$ and $b(x)$ are C^4 , and $k(x, s)$ is $C^{4,5}$, for $x_0 \leq x$, $s \leq L$. Then the approximations y_n , $n \geq 3$, computed from Algorithm B with $O(h^4)$ starting values, satisfy the relation

$$y(x_n) = y_n + h^2 e(x_n) + O(h^4).$$

If, furthermore, (16) is satisfied, then the values y_n , $n \geq 3$, computed from Algorithm B with $O(h^5)$ starting values, satisfy the relation

$$y(x_n) = y_n + h^2 e(x_n) + O(h^5).$$

3. Computational Examples.

Example 1.

$$y'(x) = 1 - \int_0^x y(s) ds, \quad y(0) = 0, \quad 0 \leq x \leq 1.$$

The exact solution is $y(x) = \sin x$.

Example 2.

$$y'(x) = 1 + \sin x - y(x) + \int_0^x \sin(x-s)y(s) ds,$$

$$y(0) = 0, \quad 0 \leq x \leq 1.$$

The exact solution is $y(x) = x$.

For Example 1, we see that $f'(x_0) = 0$ by (17). Both Algorithms A and B, with appropriate starting values, are used in computing the approximate solution. We list in Tables 1, 2, and 3 some of the resulting errors, before and after extrapolation. By error we mean

$$\text{error} = |\text{exact value} - \text{approximate value}|.$$

For Example 2, the approximate solution is computed using only Algorithm A. The resulting errors are listed in Tables 4 and 5. The effect of extrapolation is apparent from these tables.

The programs are written in FORTRAN in double precision for the IBM 370/158 computer at the Cleveland State University.

TABLE 1
Example 1, Algorithm A

x	$h = 0.1$	$h = 0.05$	$h = 0.025$
0.4	1.13×10^{-5}	2.55×10^{-6}	5.95×10^{-7}
0.6	3.50×10^{-5}	8.06×10^{-6}	1.92×10^{-6}
0.8	7.75×10^{-5}	1.81×10^{-5}	4.35×10^{-6}
1.0	1.42×10^{-4}	3.35×10^{-5}	8.11×10^{-6}

TABLE 2
Example 1, Algorithm B

x	$h = 0.1$	$h = 0.05$	$h = 0.025$
0.4	8.47×10^{-6}	2.21×10^{-6}	5.49×10^{-7}
0.6	2.92×10^{-5}	7.28×10^{-6}	1.81×10^{-6}
0.8	6.73×10^{-5}	1.67×10^{-5}	4.17×10^{-6}
1.0	1.26×10^{-4}	3.15×10^{-5}	7.85×10^{-6}

TABLE 3
Example 1, after extrapolation

x	Algorithm A		Algorithm B	
	h = 0.1	h = 0.05	h = 0.1	h = 0.05
0.4	3.54×10^{-7}	5.72×10^{-8}	1.25×10^{-7}	4.10×10^{-9}
0.6	9.28×10^{-7}	1.33×10^{-7}	4.46×10^{-9}	1.13×10^{-8}
0.8	1.70×10^{-6}	2.32×10^{-7}	1.18×10^{-7}	1.70×10^{-8}
1.0	2.60×10^{-6}	3.46×10^{-7}	2.06×10^{-7}	2.07×10^{-8}

TABLE 4
Example 2, Algorithm A

x	h = 0.1	h = 0.05	h = 0.025
0.4	1.14×10^{-4}	2.89×10^{-5}	7.27×10^{-6}
0.6	2.42×10^{-4}	6.11×10^{-5}	1.53×10^{-5}
0.8	4.05×10^{-4}	1.02×10^{-4}	2.54×10^{-5}
1.0	5.93×10^{-4}	1.49×10^{-4}	3.72×10^{-5}

TABLE 5
Example 2, Algorithm A, after extrapolation

x	h = 0.1	h = 0.05
0.4	7.42×10^{-7}	4.51×10^{-8}
0.6	6.16×10^{-7}	3.76×10^{-8}
0.8	5.28×10^{-7}	3.25×10^{-8}
1.0	4.78×10^{-7}	2.97×10^{-8}

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