MATHEMATICS OF COMPUTATION
VOLUME 39, NUMBER 159
JULY 1982, PAGES 173-177

On the Evaluation of Double Integrals

By Moshe Levin

Abstract. A cubature formula consisting of line integrals which is optimal on a set of functions satisfying given boundary conditions is obtained. The line integrals of this formula may be evaluated by optimal quadrature formulas. The advantage of this formula over the optimal cubature formula with a rectangular lattice of knots is shown. This approach to optimal cubatures was stimulated by the idea of blending [1], [2].

1. Notations and Definitions. Let $L_{1i}f(\cdot) = 0$, $i = 1,\dots,r_1$, and $L_{2j}f(\cdot) = 0$, $j = 1,\dots,r_2$, be linear homogeneous boundary conditions for the interval [0, 1] such that the problems

$$(1)\quad \begin{cases} y^{(r)}(x) = 0, \\ L_{1i}y(\cdot) = 0, \quad i = 1,\dots,r_1, \end{cases} \quad \textit{and} \quad \begin{cases} y^{(r)}(x) = 0, \\ L_{2j}y(\cdot) = 0, \quad j = 1,\dots,r_2, \end{cases}$$

both have the unique solution $y(x) \equiv 0$, and let $g_1(x, t)$ and $g_2(x, t)$ be the Green's functions corresponding to problems (1) [4].

Let $\mathfrak{D} = [0, 1] \times [0, 1]$, $1 < p < \infty$, $p^{-1} + q^{-1} = 1$,

$$\| f(\cdot)\|_p = \left(\int_0^1 |f(x)|^p\,dx\right)^{1/p}, \qquad \|\varphi(\cdot,\cdot)\|_p = \left(\int_0^1\int_0^1 |\varphi(x, y)|^p\,dx\,dy\right)^{1/p}.$$

We consider the following sets of functions:

$$W_{g_k}^r L_p = \Big\{ f(x)\colon f^{(r)}(x) \text{ piecewise continuous on } [0, 1],$$

$$\| f^{(r)}(\cdot)\|_p \leqslant 1,\ L_{ki}f(\cdot) = 0,\ i = 1,\dots,r_k\Big\}, \qquad k = 1, 2,$$

$$W_{g_1g_2}^{2r} L_p = \Bigg\{ f(x, y)\colon \frac{\partial^{l+s}f(x, y)}{\partial x^l \partial y^s}\ (l, s \leqslant r) \text{ piecewise continuous on } \mathfrak{D},$$

$$\left\| \frac{\partial^{2r}f(\cdot,\cdot)}{\partial x^r \partial y^r}\right\|_p \leqslant M,\ L_{1i}f(\cdot, y) \equiv 0,\ i = 1,\dots,r_1;$$

$$L_{2j}f(x, \cdot) \equiv 0, j = 1,\dots,r_2\Bigg\}.$$

The quadrature formula

$$(2)\qquad \int_0^1 f(x)\,dx = \sum_{k=1}^m A_k f(x_k) + r(f), \qquad 0 \leqslant x_1 < \cdots < x_m \leqslant 1$$

Received June 10, 1981; revised October 27, 1981.
1980 *Mathematics Subject Classification.* Primary 65D30; Secondary 65D32.

is called optimal for the set H of functions $f(x)$ if the coefficients A_k and knots x_k of the formula are chosen so that the quantity

$$\sup_{f \in H} |r(f)| \tag{3}$$

is minimal.

Designate

$$A_{p1}^{(m)}, \ldots, A_{pm}^{(m)};\ x_{p1}^{(m)}, \ldots, x_{pm}^{(m)};\ r_{p1}^{(m)}$$
$$B_{p1}^{(m)}, \ldots, B_{pm}^{(m)};\ y_{p1}^{(m)}, \ldots, y_{pm}^{(m)};\ r_{p2}^{(m)}$$

the coefficients, knots, and the value (3) of the optimal formula (2) for the sets $W_{g_1}^r L_p$ and $W_{g_2}^r L_p$, respectively. It is known [4] that $r_{p1}^{(m)} = O(m^{-r})$, $r_{p2}^{(m)} = O(m^{-r})$.

The formula

$$\begin{gathered}\int_0^1 \int_0^1 f(x, y)\, dx\, dy = \sum_{k=1}^{m} \sum_{l=1}^{m} C_{kl} f(x_k, y_l) + R(f), \\ 0 \leqslant x_1 < \cdots < x_m \leqslant 1, 0 \leqslant y_1 < \cdots < y_m \leqslant 1,\end{gathered} \tag{4}$$

is called the optimal formula for the set F of functions $f(x, y)$ if its coefficients and knots are chosen so that the quantity

$$R = \sup_{f \in F} |R(f)|$$

has the least value.

It is shown in [3], [4] that the optimal formula (4) on the set $W_{g_1 g_2}^{2r} L_2$ has the coefficients $C_{kl} = A_{2k}^{(m)} B_{2l}^{(m)}$, the knots $x_k = x_{2k}^{(m)}$, $y_l = y_{2l}^{(m)}$, and the remainder $R = O(m^{-r})$.

For the case $p \neq 2$ the optimal formula (4) on the set $W_{g_1 g_2}^{2r} L_p$ has not yet been found.

2. The Optimal Cubature Formula. We will find an optimal formula of the form

$$\begin{gathered}\int_0^1 \int_0^1 f(x, y)\, dx\, dy = \sum_{k=1}^{n} \alpha_k \int_0^1 f(x_k, y)\, dy + \sum_{j=1}^{n} \beta_j \int_0^1 f(x, y_j)\, dx \\ + \sum_{k=1}^{n} \sum_{j=1}^{n} \gamma_{kj} f(x_k, y_j) + E(f), \\ 0 \leqslant x_1 < \cdots < x_n \leqslant 1, 0 \leqslant y_1 < \cdots < y_n \leqslant 1,\end{gathered} \tag{5}$$

for the set $W_{g_1 g_2}^{2r} L_p$. In other words, we will find the formula (5) with the least value of

$$E = \sup_{f \in W_{g_1 g_2}^{2r} L_p} |E(f)|.$$

THEOREM. *The coefficients and knots*

$$\begin{gathered}\alpha_k = A_{pk}^{(n)}, \quad \beta_j = B_{pj}^{(n)}, \quad \gamma_{kj} = -A_{pk}^{(n)} B_{pj}^{(n)}, \\ x_k = x_{pk}^{(n)}, \quad y_j = y_{pj}^{(n)}, \qquad k, j = 1, \ldots, n,\end{gathered} \tag{6}$$

and the estimate

$$E = M r_{p1}^{(n)} r_{p2}^{(n)} \tag{7}$$

are the coefficients, knots, and estimate of the optimal formula (5) *on the set* $W_{g_1 g_2}^{2r} L_p$.

Proof. Let $f(x, y) \in W^{2r}_{g_1 g_2} L_p$. Then [4]

$$f(x, y) = \int_0^1 \int_0^1 \frac{\partial^{2r} f(t, u)}{\partial x^r \partial y^r} g_1(x, t) g_2(y, u)\, dt\, du.$$

Hence by (5)

$$E(f) = \int_0^1 \int_0^1 \frac{\partial^{2r} f(t, u)}{\partial x^r \partial y^r} K(t, u)\, dt\, du, \tag{8}$$

where

$$\begin{aligned} K(t, u) = \varphi_1(t)\varphi_2(u) - \varphi_2(u) \sum_{k=1}^{n} \alpha_k g_1(x_k, t) - \varphi_1(t) \sum_{j=1}^{n} \beta_j g_2(y_j, u) \\ - \sum_{k=1}^{n} \sum_{j=1}^{n} \gamma_{kj} g_1(x_k, t) g_2(y_j, u), \end{aligned}$$

$$\varphi_l(v) = \int_0^1 g_l(x, v)\, dx, \qquad l = 1, 2.$$

Using Hölder's inequality, we obtain from (8) that

$$|E(f)| \leq M \|K(\cdot, \cdot)\|_q. \tag{9}$$

Since the function

$$f_0(x, y) = \frac{M}{\|K(\cdot, \cdot)\|_q^{q/p}} \int_0^1 \int_0^1 |K(t, u)|^{q-1} \operatorname{sgn} K(t, u) g_1(x, t) g_2(y, u)\, dt\, du$$

belongs to the set $W^{2r}_{g_1 g_2} L_p$, it follows from (8) that $E(f_0) = M \|K(\cdot, \cdot)\|_q$. Therefore we have from (9) $E = M \|K(\cdot, \cdot)\|_q$.

By the result on "polynomials" of least deviation from zero [5], we have that

$$\begin{aligned} &\inf_{\{\alpha_k, \beta_j, \gamma_{kj}, x_k, y_j\}} \|K(\cdot, \cdot)\|_q \\ &= \inf_{\{\alpha_k, \beta_j, x_k, y_j\}} \Bigg\| \varphi_1(x)\varphi_2(y) - \varphi_2(y) \sum_{k=1}^{n} \alpha_k g_1(x_k, x) \\ &\qquad - \varphi_1(x) \sum_{j=1}^{n} \beta_j g_2(y_j, y) + \sum_{k=1}^{n} \sum_{j=1}^{n} \alpha_k \beta_j g_1(x_k, x) g_2(y_j, y) \Bigg\|_q \\ &= \inf_{\{\alpha_k, x_k\}} \left\| \varphi_1(\cdot) - \sum_{k=1}^{n} \alpha_k g_1(x_k, \cdot) \right\|_q \cdot \inf_{\{\beta_j, y_j\}} \left\| \varphi_2(\cdot) - \sum_{j=1}^{n} \beta_j g_2(y_j, \cdot) \right\|_q . \end{aligned} \tag{10}$$

As the equalities

$$\inf_{\{\alpha_k, x_k\}} \left\| \varphi_1(\cdot) - \sum_{k=1}^{n} \alpha_k g_1(x_k, \cdot) \right\|_q = r_{p1}^{(n)},$$

$$\inf_{\{\beta_j, y_j\}} \left\| \varphi_2(\cdot) - \sum_{j=1}^{n} \beta_j g_2(y_j, \cdot) \right\|_q = r_{p2}^{(n)}$$

hold and are achieved by the coefficients and knots (6) [4], it follows from (10) that the numbers (6) and (7) are the coefficients, knots, and estimate of the optimal formula (5) for the set $W^{2r}_{g_1 g_2} L_p$.

The theorem is proved.

Now we will obtain a cubature formula which requires less computational work than the optimal formula (4) with the same error estimate.

Let

$$\sup_{0 \leqslant y \leqslant 1} \left\| \frac{\partial^r f(\cdot, y)}{\partial x^r} \right\|_p \leqslant M_1, \qquad \sup_{0 \leqslant x \leqslant 1} \left\| \frac{\partial^r f(x, \cdot)}{\partial y^r} \right\|_p \leqslant M_2.$$

Then

$$M_1^{-1} f\left(x, y_{pj}^{(n)}\right) \in W_{g_1}^r L_p, \qquad M_2^{-1} f\left(x_{pk}^{(n)}, y\right) \in W_{g_2}^r L_p.$$

Applying optimal quadrature formulas for these sets to the integrals in (5), we obtain

$$\int_0^1 f\left(x, y_{pj}^{(n)}\right) dx = \sum_{k=1}^{n^2} A_{pk}^{(n^2)} f\left(x_{pk}^{(n^2)}, y_{pj}^{(n)}\right) + r_1(f), \tag{11}$$
$$|r_1(f)| \leqslant M_1 r_{p1}^{(n^2)},$$

$$\int_0^1 f\left(x_{pk}^{(n)}, y\right) dy = \sum_{j=1}^{n^2} B_{pj}^{(n^2)} f\left(x_{pk}^{(n)}, y_{pj}^{(n^2)}\right) + r_2(f), \tag{12}$$
$$|r_2(f)| \leqslant M_2 r_{p2}^{(n^2)}.$$

Designating

$$a_n = |A_{p1}^{(n)}| + \cdots + |A_{pn}^{(n)}|, \qquad b_n = |B_{p1}^{(n)}| + \cdots + |B_{pn}^{(n)}|$$

and substituting (11) and (12) into the optimal formula (5) for the set $W_{g_1 g_2}^{2r} L_p$, we obtain the following cubature formula

$$\begin{aligned} \int_0^1 \int_0^1 f(x, y)\, dx\, dy = {} & \sum_{k=1}^{n} \sum_{j=1}^{n^2} A_{pk}^{(n)} B_{pj}^{(n^2)} f\left(x_{pk}^{(n)}, y_{pj}^{(n^2)}\right) \\ & + \sum_{j=1}^{n} \sum_{k=1}^{n^2} B_{pj}^{(n)} A_{pk}^{(n^2)} f\left(x_{pk}^{(n^2)}, y_{pj}^{(n)}\right) \\ & - \sum_{k=1}^{n} \sum_{j=1}^{n} A_{pk}^{(n)} B_{pj}^{(n)} f\left(x_{pk}^{(n)}, y_{pj}^{(n)}\right) + E_1(f), \end{aligned} \tag{13}$$

where

$$|E_1(f)| \leqslant a_n M_2 r_{p2}^{(n^2)} + b_n M_1 r_{p1}^{(n^2)} + M r_{p1}^{(n)} r_{p2}^{(n)}. \tag{14}$$

It follows from the convergence of optimal formulas (2) that a_n and b_n are bounded as $n \to \infty$. Hence it follows from (14) that

$$|E_1(f)| = O(n^{-2r}).$$

The formula (13) has a remarkable advantage over the optimal formula (4). The optimal formula (4) with $m = n^2$ has the error estimate $O(n^{-2r})$, and it uses n^4 point values of $f(x, y)$, while the formula (13) has the same error estimate $O(n^{-2r})$ but uses only $2n^3 + n^2$ point values of the function $f(x, y)$.

3. Example. We compare the evaluation of the integral

$$I = \int_0^1 \int_0^1 \frac{(x - x^2)(y - y^2)}{0 \cdot 2 + xy} dx\, dy = 0.0701598\ldots$$

with the optimal formula (4) with $m = n^2$ and with the formula (13) obtained from the optimal formula (5). Both formulas are taken for the set $W^{2r}_{g_1 g_2} L_2$, $r = 2$.

As the integrand $f(x, y)$ satisfies the conditions

$$f(0, y) \equiv f(1, y) \equiv f(x, 0) \equiv f(x, 1) \equiv 0,$$

we can take the functions $g_1(x, t) \equiv g_2(x, t)$ as Green's functions for the problem

$$y'' = 0, \qquad y(0) = y(1) = 0.$$

Then by [4] we have

$$A^{(m)}_{2k} = B^{(m)}_{2k} = 2\varepsilon, \qquad k = 2, \ldots, m - 1,$$

$$A^{(m)}_{21} = B^{(m)}_{21} = A^{(m)}_{2m} = B^{(m)}_{2m} = \left(1 + 1.25\sqrt{2/3}\right)\varepsilon,$$

$$x^{(m)}_{2k} = y^{(m)}_{2k} = 2\varepsilon\left(\sqrt{2/3} + k - 1\right), \qquad k = 1, \ldots, m,$$

$$\varepsilon = 0.5\left(2\sqrt{2/3} + m - 1\right)^{-1}.$$

Let I_n be an approximate value of I, obtained by the optimal formula (4) with $k_n = n^4$ knots and let I'_n be the approximate value obtained by formula (13) with $q_n = 2n^3 + n^2$ knots.

We obtain for $n = 4, 7, 9$

I'_n	I_n	q_n	k_n	n
0.0701302	0.0701319	144	256	4
0.0701587	0.0701588	735	2401	7
0.0701596	0.0701596	1539	6561	9

The superiority of formula (13) over the optimal formula (4) is obvious.

Acknowledgement. I am grateful to Professor Nira Dyn for stimulating discussions on the material presented in this paper.

School of Mathematical Sciences
Tel-Aviv University
Ramat-Aviv, Tel-Aviv, Israel

1. F. J. Delvos & H. Posdorf, "N-th order blending," in *Constructive Theory of Functions of Several Variables* (W. Schempp and K. Zeller, eds.), Springer-Verlag, Berlin, Heidelberg, New York, 1977, pp. 53–64.

2. W. J. Gordon, "Distributive lattices and the approximation of multivariate functions," in *Approximations with Special Emphasis of Spline Functions* (I. J. Schoenberg, ed.), Academic Press, New York, London, 1969, pp. 223–277.

3. M. Levin, "On approximation in L_2 and optimal cubature formulae," in *Fourier Analysis and Approximation Theory*, Mathematica Societatis János Bolyai, Budapest, 1976, pp. 495–501.

4. M. Levin & J. Girshovich, *Optimal Quadrature Formulas*, Teubner-Verlag, Leipzig, 1979.

5. M. Levin & J. Girshovich, "Extremal problems for cubature formulas," *Soviet Math. Dokl.*, v. 18, 1977, pp. 1355–1358.