MATHEMATICS OF COMPUTATION
VOLUME 39, NUMBER 159
JULY 1982, PAGES 213-224

The Number of Partitions of the Integer N into M Nonzero Positive Integers

By W. J. A. Colman

Abstract. The function $p_m(n)$ is defined as the number of partitions of the integer n into exactly m nonzero positive integers where the order is irrelevant.

A series in which the leading terms alternate in sign is given for $p_m(n)$ which yields good numerical estimates.

1. Introduction. If $p_m(n)$ is the number of partitions of the integer n into exactly m positive integers and if $p_m^*(n)$ is the number of partitions into at most m parts and $p(m)$ is the usual partition function, then there are some simple known relationships between them.

$$p_m(n) - p_m(n-m) = p_{m-1}(n-1),$$
$$p_m^*(n) = p_m(n+m),$$
$$p(m) = p_m(2m).$$

Closed formulae for $p_m(n)$ are known for small m; see Gupta. In particular, we have

$$p_2(n) = \left[\frac{n}{2!\,1!}\right], \qquad p_3(n) = \left[\frac{n^2+3}{3!\,2!}\right],$$
$$p_4(n) = \left[\frac{n^3 + 3n^2 + \frac{1}{2}\{9n(-1)^n - 9n\} + 32}{4!\,3!}\right].$$

The formulae for $m = 2$ and $m = 3$ are well known and the formula for $m = 4$ is equivalent to a formula given by A. De Morgan (Dickson [2, p. 115]). Thus $p_m(n)$ is not a polynomial, but it contains a completely algebraic part which is a polynomial in n of degree $(m-1)$. If we call these polynomials $q_m(n)$ (say), then they satisfy the relationship

$$q_m(n) - q_m(n-m) = q_{m-1}(n-1).$$

2. A Series Expansion for $q_m(n)$. By writing $q_m(n) = a_{m1}n^{m-1} + a_{m2}n^{m-2} + a_{m3}n^{m-3} + \ldots$ and substituting in the recurrence relationship above, we can equate powers of n to yield the relationship

$$b_{mr} = \frac{(r-1)}{r!}\sum_{k=1}^{m} b_{k1}k^{r-1} + \frac{(r-2)}{(r-1)!}\sum_{k=1}^{m} b_{k2}b^{r-2} + \cdots + \frac{1}{2!}\sum_{k=1}^{m} b_{kr-1}k$$
$$-\left(\frac{b_{m1}m^{r-1}}{(r-1)!} + \cdots + \frac{b_{mr-1}m}{1!}\right),$$

where $a_{mr} = m!(m-r)!b_{mr}$ so that, as $a_{m1} = 1/m!(m-1)!$, we have $b_{m1} = 1$.

Received January 25, 1980; revised April 13, 1981 and November 17, 1981.
1980 *Mathematics Subject Classification.* Primary 05A17, 39A30.

This formula can be used to successively determine the a_{mr} to give the leading coefficients of $q_m(n)$. We have

$$a_{m2} = \frac{m(m-3)}{4m!\,(m-2)!}, \qquad a_{m3} = \frac{m^4 - \dfrac{58m^3}{9} + \dfrac{75m^2}{9} - \dfrac{2m}{9}}{4^2 \cdot 2!m!\,(m-3)!}.$$

This leads to the expansion

$$q_m(n) = \frac{n^{m-1}}{m!\,(m-1)!} + \frac{1}{m!\,(m-2)!}\left(\frac{m-3m}{4\cdot 1!}\right)n^{m-2}$$

$$+ \frac{1}{m!\,(m-3)!}\left(\frac{m^4 - \dfrac{58m^3}{9} + \dfrac{75m^2}{9} - \dfrac{2m}{9}}{4^2\cdot 2!}\right)n^{m-3}$$

$$+ \frac{1}{m!\,(m-4)!}\left(\frac{m^6 - \dfrac{31m^5}{3} + 29m^4 - \dfrac{65m^3}{3} + 2m^2}{4^3\cdot 3!}\right)n^{m-4} + \frac{1}{m!(m-5)!}$$

$$\cdot\left(\frac{m^8 - 14\frac{2}{3}m^7 + 66\frac{16}{27}m^6 - 107\frac{29}{225}m^5 + 55\frac{134}{135}m^4 - 10\frac{54}{135}m^3 + \frac{4}{27}m^2 - \frac{16}{225}m}{4^4\cdot 4!}\right)n^{m-5}.$$

The polynomials b_{mr} can be generated by means of a computer program where the summations are effected using the Bernoulli polynomials. This expansion although of some interest is of little use for calculating $p_m(n)$ unless n is very large compared with m.

G. J. Rieger has given the over estimate

$$p_m(n) < \frac{1}{m!\,(m-1)!}\left(n + \frac{m(m-3)}{4}\right)^{m-1} \quad \text{for } m \geq 4,$$

which exactly agrees with the first two terms of the expansion. This yields good estimates when n is large compared with $m(m-3)/4$ but gives no idea of the magnitude of the error. To determine a series expansion which will yield better numerical approximations and give some idea of the magnitude of the error, we can proceed as follows.

3. An Alternating Series Expansion. Assume there is an expansion of the form

$$q_m(n) = \sum_{r=1,3,5} \frac{F_r(m)}{m!\,(m-r)!}\left(n + \frac{m(m-3)}{4}\right)^{m-r} \quad \text{where } F_1(m) = 1,$$

we have

$$q_m(n-m) = \sum \frac{F_r(m)}{m!\,(m-r)!}\left(n + \frac{m(m-3)}{4} - m\right)^{m-r}$$

and

$$q_{m-1}(n-1) = \sum \frac{F_r(m-1)}{(m-1)!\,(m-r-1)!}\left(n + \frac{m(m-3)}{4} - \frac{m}{2}\right)^{m-r-1}.$$

Write

$$q_m(n) = \sum \frac{F_r(m)}{m!\,(m-r)!}\left(n + \frac{m(m-3)}{4} - \frac{m}{2} + \frac{m}{2}\right)^{m-r}$$

and

$$q_m(n-m) = \sum \frac{F_r(m)}{m!\,(m-r)!}\left(n + \frac{m(m-3)}{4} - \frac{m}{2} - \frac{m}{2}\right)^{m-r}.$$

Put $X = n + m(m-3)/4 - m/2$ (say) and expanding both sides in powers of X and equating, we have

$$\frac{F_1(m)m^{r-1}}{r!2^{r-1}} + \frac{F_3(m)m^{r-3}}{(r-2)!2^{r-3}} + \cdots + \frac{F_r(m)m^0}{1!2^0} = F_r(m-1),$$

where $F_1(m) = 1$ and $r = 3, 5, 7, \ldots,$

$$\therefore\ F_r(m) - F_r(m-1) = -\left(\frac{F_1(m)m^{r-1}}{r!2^{r-1}} + \frac{F_3(m)m^{r-3}}{(r-2)!2^{r-3}} + \cdots + \frac{F_{r-2}(m)m^2}{3!2^2}\right).$$

Hence the $F_r(m)$ can be determined sequentially, and the above expansion is possible. Now $F_r(m)$ is a polynomial in m of degree $(3r-3)/2$ without constant term.

Thus $F_r(0) = 0$. Putting $m = 0$, we have immediately that $F_r(-1) = 0$. Putting $m = -1$ yields $F_r(-2) = 1/r!2^{r-1}$. These values can be used as a check when explicitly calculating the $F_r(m)$. We have

$$F_r(m) - F_r(m-1) = -\left(\frac{F_1(m)m^{r-1}}{r!2^{r-1}} + \cdots + \frac{F_{r-2}(m)m^2}{3!2^2}\right),$$

$$\vdots$$

$$F_r(1) - F_r(0) = -\left(\frac{F_1(1)1^{r-1}}{r!2^{r-1}} + \cdots + \frac{F_{r-2}(1)1^2}{3!2^2}\right).$$

Adding we have as $F_r(0) = 0$ that

$$-F_r(m) = \sum_{i=1}^{m}\left(\frac{F_1(i)i^{r-1}}{r!2^{r-1}} + \cdots + \frac{F_{r-2}(i)i^2}{3!2^2}\right), \qquad r = 3, 5, 7, \ldots \text{ with } F_1(m) = 1.$$

By assuming an expansion of the form

$$F_r(m) = a_{r1}m^{(3r-3)/2} + a_{r2}m^{(3r-5)/2} + a_{r3}m^{(3r-7)/2} + \ldots,$$

it is easy but tedious to show that

$$F_r(m) = \frac{(-1)^{(r-1)/2}}{6^{r-1}2^{(r-1)/2}((r-1)/2)!} \times \left(m^{(3r-3)/2} + \frac{3(r-1)(3r+16)m^{(3r-5)/2}}{100} + \frac{(r-1)}{49\times 20{,}000}(3969r^3 + 42615r^2 - 39276r - 127870)m^{(3r-7)/2} + \ldots\right).$$

Thus the polynomials $F_r(m)$ alternate in sign. We have

$$F_1(m) = 1,$$

$$F_3(m) = -\left(\frac{m^3 + \frac{3m^2}{2} + \frac{m}{2}}{6^2\cdot 2\cdot 1!}\right),$$

$$F_5(m) = \left(\frac{m^6 + \frac{372m^5}{100} + \frac{505m^4}{100} + \frac{270m^3}{100} + \frac{25m^2}{100} - \frac{12m}{100}}{6^4 \cdot 2^2 \cdot 2!} \right).$$

These polynomials can again be determined sequentially. The computer printout is given below

$r = 5$
-0.12
0.25
2.7
5.05
3.72
1

$$\therefore F_5(m) = \frac{1}{6^4 \cdot 2^2 \cdot 2!}(m^6 + 3.72m^5 + 5.05m^4 + 2.7m^3 + 0.25m^2 - 0.12m)$$

$r = 7$
-0.293878
0.18
2.47214
-2.565
-18.3964
-26.8264
-18.6533
-6.66
-1

$$\therefore F_7(m) = \frac{-1}{6^6 \cdot 2^3 \cdot 3!}(m^9 + 6.66m^8 + 18.6533m^7 + 26.8264m^6 + 18.3964m^5 + 2.565m^4 - 2.47214m^3 - 0.18m^2 + 0.293878m)$$

$r = 9$	$r = 11$	$r = 13$
-2.22171	-36.7225	-1118
0.630955	5.90694	115.03
16.3947	258.189	7682.71
-4.82027	-36.4973	-637.779
-44.5329	-602.259	-16846
23.1309	68.4204	698.096
157.976	806.053	-19211.2
196.713	-186.351	137.842
126.587	-1638.34	-15196.3
47.8683	-1839.7	784.916
10.32	-1104.07	19823.1
1.	-416.03	20845
	-100.809	11813.2
	-14.7	4357.69
	-1	1099
		187.143
		19.8
		1

$r = 15$	$r = 17$	$r = 19$	$r = 21$
−56378.7	−0.43752E7	−0.49478E9	−0.781715E11
4028.38	230048	0.199698E8	0.25002E10
382554	0.294572E8	0.33141E10	0.521721E12
−20984.3	−0.115394E7	−0.97816E8	−0.120565E11
−812491	−0.613873E8	−0.682167E10	−0.106477E13
14732.7	548128	0.328379E8	0.296624E10
866493	0.631721E8	0.686827E10	0.105652E13
30820.2	0.235611E7	0.229987E9	0.305372E11
−583067	−0.397167E8	−0.416288E10	−0.625932E12
−55699	−0.388232E7	−0.354982E9	−0.451736E11
301160	0.174792E8	0.171547E10	0.249008E12
21659.1	0.288588E7	0.257655E9	0.317883E11
−272765	−0.626974E7	−0.525898E9	−0.718084E11
−276413	−0.10408E7	−0.118078E9	−0.141378E11
−149698	0.418598E7	0.136369E9	0.159685E11
−53845.3	0.418626E7	0.335923E8	0.447758E10
−13719.5	0.218906E7	−0.705382E8	−0.307064E10
−2504.71	768379	−0.711215E8	−0.997364E9
−318.096	195323	−0.362286E8	0.128808E10
−25.62	36850.2	−0.124395E8	0.13368E10
−1	5128.87	−0.313772E7	0.668514E9
	506.447	−600883	0.225198E9
	32.16	−87931.4	0.562438E8
	1	−9680.46	0.108307E8
		−766.529	0.163087E7
		−39.42	191397
		−1	17134.1
			1114.23
			47.4
			1

All the polynomials to $r = 21$ are of constant sign for $m \geqslant r$. This series has G. J. Rieger's estimate as its first term and being an alternating series will clearly yield successively both over and under estimates for $p_m(n)$. Some numerical examples:

	Magnitude of each term	Partial sum	
(1) $P_{20}(200) = .874388 \times 10^{11}$	$.148195 \times 10^{12}$	$.148195 \times 10^{12}$	Rieger estimate
	$.746174 \times 10^{11}$	$.735774 \times 10^{11}$	
	$.154646 \times 10^{11}$	$.890421 \times 10^{11}$	
	$.170826 \times 10^{10}$	$.873338 \times 10^{11}$	
	$.108904 \times 10^{9}$	$.874427 \times 10^{11}$	
	$.405826 \times 10^{7}$	$.874387 \times 10^{11}$	
	85496.6	$.874388 \times 10^{11}$	

The expansion in terms of the $F_r(m)$ for $p_m(n)$ when truncated at $r = 1, 3, 5$, or 7 gives upper and lower bounds for $p_m(n)$ better than those given previously for n tending to infinity.

(2) $P_{20}(500) =$	$.112794 \times 10^{18}$	
	$.127275 \times 10^{18}$	$.127275 \times 10^{18}$
	$.152099 \times 10^{17}$	$.112065 \times 10^{18}$
	$.748178 \times 10^{15}$	$.112813 \times 10^{18}$
	$.196154 \times 10^{14}$	$.112793 \times 10^{18}$
	$.296801 \times 10^{12}$	$.112794 \times 10^{18}$
	$.262506 \times 10^{10}$	$.112794 \times 10^{18}$
(3) $P_{30}(1000) =$	$.716051 \times 10^{26}$	
	$.895919 \times 10^{26}$	$.895919 \times 10^{26}$
	$.198200 \times 10^{26}$	$.697718 \times 10^{26}$
	$.194009 \times 10^{25}$	$.717119 \times 10^{26}$
	$.110784 \times 10^{24}$	$.716011 \times 10^{26}$
	$.409716 \times 10^{22}$	$.716052 \times 10^{26}$
	$.103050 \times 10^{21}$	$.716051 \times 10^{26}$
	$.180151 \times 10^{19}$	$.716051 \times 10^{26}$

4. The Relationship Between N and M for Effective Calculation of $p_m(n)$. Thus the series gives excellent results when n is suitably large compared with m. In fact this will be achieved if the terms decrease in absolute magnitude. A condition for this can be estimated approximately as follows. We have, apart from sign,

$$F_r(m) = \frac{1}{6^{r-1}2^{(r-1)/2}((r-1)/2)!} \times \left(m^{(3r-3)/2} + \frac{3(r-1)(3r+16)}{100} m^{(3r-5)/2} + \frac{(r-1)}{49 \times 20{,}000}(3969r^3 + 42615r^2 + \dots)m^{(3r-7)/2} \dots \right),$$

whereas

$$\frac{1}{6^{r-1}2^{(r-1)/2}((r-1)/2)!}\left(m + \frac{3r+16}{50}\right)^{(3r-3)/2} = \frac{1}{6^{r-1}2^{(r-1)/2}((r-1)/2)!} \times \left(m^{(3r-3)/2} + \frac{3(r-1)(3r+16)}{100} m^{(3r-5)/2} + \frac{(r-1)}{49 \times 20{,}000}(3969r^3 + 35721r^2 + \dots)m^{(3r-7)/2} \dots \right).$$

Thus we need to show that the polynomial

$$\frac{1}{6^{r-1}2^{(r-1)/2}((r-1)/2)!}\left(m + \frac{3r+16}{50}\right)^{(3r-3)/2}$$

is a good approximation for $F_r(m)$ for $m \geqslant kr$ (say), where $k \geqslant 1$. We can proceed as follows.

Consider the equation

$$F_r(m) - F_r(m-1) = -\left(\frac{F_{r-2}(m)m^2}{3!\cdot 2^2} + \frac{F_{r-4}(m)m^4}{5!\cdot 2^4} + \cdots + \frac{F_1(m)m^{r-1}}{r!\cdot 2^{r-1}}\right) \quad \text{for } m \geqslant r.$$

We will consider the function

$$G_r(m) = \frac{(-1)^{(r-1)/2}}{6^{r-1}2^{(r-1)/2}((r-1)/2)!}\left(m + \frac{3r+16}{50}\right)^{(3r-3)/2}$$

and substitute in both sides of the above equation. We have

$$\begin{aligned} G_r(m) - G_r(m-1) &= \frac{3(-1)^{(r-1)/2}}{6^{r-1}2^{(r-1)/2}((r-3)/2)!} \\ &\times\left(\left(m+\frac{3r+16}{50}\right)^{(3r-5)/2} - \frac{1}{2!}\left(\frac{3r-5}{2}\right)\left(m+\frac{3r+16}{50}\right)^{(3r-7)/2}\right. \\ &\left. + \frac{1}{3!}\left(\frac{3r-5}{2}\right)\left(\frac{3r-7}{2}\right)\left(m+\frac{3r+16}{50}-\theta\right)^{(3r-9)/2}\right), \end{aligned}$$

where $0 < \theta < 1$.

The right-hand side becomes, for $s \geqslant 2$,

$$\begin{aligned} &\frac{3(-1)^{(r-1)/2}}{6^{r-1}2^{(r-1)/2}((r-3)/2)!} \\ &\times\left(\frac{6}{3!}\left(m+\frac{3r+10}{50}\right)^{(3r-9)/2}\cdot m^2 - \frac{6^2}{5!}\left(\frac{3r-9}{2}\right)\left(m+\frac{3r+4}{50}\right)^{(3r-15)/2}\cdot m^4\right. \\ &+\cdots+\frac{(-1)^{s+1}6^s}{(2s+1)!}\left(\frac{3r-9}{2}\right)\cdots\left(\frac{3r-(6s-3)}{2}\right) \\ &\left.\times\left(m+\frac{3r-(6s-16)}{50}\right)^{(3r-(6s+3))/2}\cdot m^{2s}\cdots\right), \end{aligned}$$

where for all s $(3r-(6s+3))/2$ is positive. Thus all the factors are positive, and the series alternates in sign. We require

$$|\,s\text{th}\,| > |\,(s+1)\text{th}\,| \text{ term for } s \geqslant 2 \quad \text{(say)},$$

$$\begin{aligned} \therefore (2s+3)(2s+2)&\left(m+\frac{3r-(6s-16)}{50}\right)^{(3r-(6s+3))/2}\cdot m^{2s} \\ &> 6\left(\frac{3r-(6s+3)}{2}\right)\left(m+\frac{3r-(6s-10)}{50}\right)^{(3r-(6s+9))/2}\cdot m^{2s+2}. \end{aligned}$$

Adequate if

$$(2s+3)(2s+2)\left(m+\frac{3r-(6s-10)}{50}\right)^3 > 3(3r-(6s+3))m^2.$$

On the l.h.s. give s its smallest value when $s = 2$ if

$$42\left(m + \frac{3r - (6s - 10)}{50}\right)^3 > 3(3r - (6s + 3))m^2.$$

If $m \geqslant r$ then r.h.s $< 9m^3$ whereas l.h.s. $> 42m^3$. Clearly the first term is larger than the second for $m \geqslant r$. Thus the terms of the series alternate in sign and decrease in absolute magnitude for $m \geqslant r$. Thus the sum of the series is greater than the first two terms but less than the first three. We have that the r.h.s. series

$$> \left(m + \frac{3r + 10}{50}\right)^{(3r-9)/2} \cdot m^2 - \frac{6^2}{5!}\left(\frac{3r - 9}{2}\right)\left(m + \frac{3r + 4}{50}\right)^{(3r-15)/2} \cdot m^4.$$

We can write this as

$$\left(m + \frac{3r + 16}{50} - \frac{6}{50}\right)^{(3r-9)/2}\left(m + \frac{3r + 16}{50} - \frac{3r + 16}{50}\right)^2$$
$$- \frac{6^2}{5!}\left(\frac{3r - 9}{2}\right)\left(m + \frac{3r + 16}{50} - \frac{12}{50}\right)^{(3r-15)/2} \cdot \left(m + \frac{3r + 16}{50} - \frac{3r + 16}{50}\right)^4,$$

which is

$$> \left(m + \frac{3r + 16}{50}\right)^{(3r-5)/2} - \left(\frac{3r - 5}{4}\right)\left(m + \frac{3r + 16}{50}\right)^{(3r-7)/2} \quad \text{for } m \geqslant r.$$

We can expand the first three terms in the same way to show that the sum of the series

$$< \left(m + \frac{3r + 16}{50}\right)^{(3r-5)/2} - \left(\frac{3r - 5}{4}\right)\left(m + \frac{3r + 16}{50}\right)^{(3r-7)/2}$$
$$+ \frac{1}{24}(9.79r^2 - 44.03r + 50.00)\left(m + \frac{3r + 16}{50}\right)^{(3r-9)/2}.$$

Thus we have that the r.h.s.

$$= \frac{3(-1)^{(r-1)/2}}{6^{r-1}2^{(r-1)/2}((r-3)/2)!}$$
$$\times \left(\left(m + \frac{3r + 16}{50}\right)^{(3r-5)/2} - \left(\frac{3r - 5}{4}\right)\left(m + \frac{3r + 16}{50}\right)^{(3r-7)/2}\right.$$
$$\left. + O\left(\frac{1}{24}(9.79r^2 - 44.03r + 50.00)\left(m + \frac{3r + 16}{50}\right)^{(3r-9)/2}\right)\right),$$

whereas the l.h.s.

$$= \frac{3(-1)^{(r-1)/2}}{6^{r-1}2^{(r-1)/2}((r-3)/2)!}$$
$$\times \left(\left(m + \frac{3r + 16}{50}\right)^{(3r-5)/2} - \left(\frac{3r - 5}{4}\right)\left(m + \frac{3r + 16}{50}\right)^{(3r-7)/2}\right.$$
$$\left. + O\left(\frac{1}{24}(9r^2 - 36r + 35)\left(m + \frac{3r + 16}{50}\right)^{(3r-9)/2}\right)\right).$$

Now $F_r(m)$ is the function such that

$$\frac{\text{r.h.s}}{\text{l.h.s}} = 1.$$

We have for $G_r(m)$ that

$$\frac{\text{r.h.s}}{\text{l.h.s}} = \frac{1 - \dfrac{3r-5}{4(m+(3r+16)/50)} + O\left(\dfrac{9.79r^2 - 44.03r + 50}{24(m+(3r+16)/50)^2}\right)}{1 - \dfrac{3r-5}{4(m+(3r+16)/50)} + O\left(\dfrac{9r^2 - 36r + 35}{24(m+(3r+16)/50)^2}\right)},$$

where $0 < O(A) < A$. Let us write $m = kr$ (say), where $k \geqslant 1$ and $r \geqslant 3$. It is easy to show that for $r \geqslant 3$

$$0 < O\left(\frac{9.79r^2 - 44.03r + 50}{24(kr+(3r+16)/50)^2}\right) < \frac{9.79}{24(k+.06)^2},$$

$$0 < O\left(\frac{9r^2 - 36r + 35}{24(kr+(3r+16)/50)^2}\right) < \frac{9}{24(k+.06)^2}.$$

Hence

$$\frac{1 - \dfrac{3r-5}{4(kr+(3r+16)/50)}}{1 - \dfrac{3r-5}{4(kr+(3r+16)/50)} + \dfrac{9}{24(k+.06)^2}} < \frac{\text{r.h.s.}}{\text{l.h.s.}} < \frac{1 - \dfrac{3r-5}{4(kr+(3r+16)/50)} + \dfrac{9.79}{24(k+.06)^2}}{1 - \dfrac{3r-5}{4(kr+(3r+16)/50)}}$$

$$\therefore \frac{1}{1 + \dfrac{9}{24(k+.06)^2}\left(\dfrac{(4k+.24)r+1.28}{(4k-2.76)r+6.28}\right)} < \frac{\text{r.h.s.}}{\text{l.h.s.}} < 1 + \frac{9.79}{24(k+.06)^2}\left(\frac{(4k+.24)r+1.28}{(4k-2.76)r+6.28}\right).$$

Now

$$\max_{r\geqslant 3} \frac{(4k+.24)r+1.28}{(4k-2.76)r+6.28} = \frac{4k+.24}{4k-2.76}$$

$$\therefore \frac{1}{1 + \dfrac{9}{24(k+.06)^2}\left(\dfrac{4k+.24}{4k-2.76}\right)} < \frac{\text{r.h.s}}{\text{l.h.s}} < 1 + \frac{9.79}{24(k+.06)^2}\left(\frac{4k+.24}{4k-2.76}\right).$$

Put $k = 2$ (say),

$$\therefore \text{For } r \leqslant \frac{m}{2} \quad .8779 < \frac{\text{r.h.s}}{\text{l.h.s}} < 1.1512.$$

Put $k = 4$ (say).

$$\therefore \text{For } r \leqslant \frac{m}{4} \quad .9728 < \frac{\text{r.h.s}}{\text{l.h.s}} < 1.0304.$$

Clearly as $k \to \infty$, r.h.s/l.h.s tends to unity, which of course is apparent from the expansion for $F_r(m)$ on p. 215. It is thus clear that the function $G_r(m)$ is a good approximation for $F_r(m)$ in the sense defined above. That is, the ratio is close to unity. Thus we can say that for $r \leqslant m/2$, $G_r(m)$ is a 'good' approximation for $F_r(m)$. Thus, for $r \leqslant m/2$ at least, the functions $F_r(m)$ must be of constant sign as the functions $G_r(m)$ certainly are, and the series will be an alternating one. The leading polynomials listed to $r = 21$ are of constant sign for $r \leqslant m$ that is with $k = 1$ a property which is most probably general but is not proved here. In any event, it is only the leading terms that we require, and r of the order $m/2$ is quite sufficient. Thus for $r \leqslant m/2$ we can approximate $F_r(m)$ with $G_r(m)$ such that if a_r is the rth term of the series expansion for $q_m(r)$, where $r = 1, 2, 3, \ldots$, then

$$\frac{a_{r+1}}{a_r} = \frac{F_{2r+1}}{F_{2r-1}} \cdot \frac{\{m - (2r-1)\}(m-2r)}{(n + m(m-3)/4)^2}$$

$$\doteqdot \frac{1}{72r}\left(m + \frac{6r+19}{50}\right)^3 \frac{(m-2r)}{(n+m(m-3)/4)^2} < 1$$

$$\text{if } \left(n + \frac{m(m-3)}{4}\right) > \frac{(m-2r)}{\sqrt{72r}}\left(m + \frac{6r+19}{50}\right)^{3/2}.$$

Now the right-hand side for fixed m and positive r is monotonically decreasing. Thus the terms constantly decrease in absolute value if, putting $r = 1$, we have

$$\left(n + \frac{m(m-3)}{4}\right) > \frac{(m-2)}{\sqrt{72}}\left(m + \frac{1}{2}\right)^{3/2}.$$

Clearly $n > m^{5/2}/\sqrt{72}$ is adequate. For comparison the two estimates are

$m = 20$	$n > 112$	$n > 210$
$m = 30$	$n > 354$	$n > 581$
$m = 50$	$n > 1443$	$n > 2084$
$m = 100$	$n > 9212$	$n > 11{,}786$

The series can still be of value even when n is smaller than the above bounds as the terms must ultimately begin to decrease, as the following example shows. Consider $P_{50}(1000)$.

5. We have

$.369652 \times 10^{30}$	$.369652 \times 10^{30}$
$.617024 \times 10^{30}$	$-.247372 \times 10^{30}$
$.480117 \times 10^{30}$	$.232745 \times 10^{30}$
$.231361 \times 10^{30}$	$.138390 \times 10^{28}$
$.773674 \times 10^{29}$	$.787513 \times 10^{29}$
$.190671 \times 10^{29}$	$.596842 \times 10^{29}$
$.359008 \times 10^{28}$	$.632743 \times 10^{29}$
$.528360 \times 10^{27}$	$.627460 \times 10^{29}$
$.616781 \times 10^{26}$	$.628076 \times 10^{29}$

The underestimate will initially be negative and thus trivial, but if enough terms are retained an improved estimate is possible. We have in this case that

$$.627460 \times 10^{29} < p_{50}(1000) < .628076 \times 10^{29},$$

whereas $P_{50}(1000) = .628023 \times 10^{29}$. If the next two terms are included, we have

$$\begin{array}{ll} .576293 \times 10^{25} & .628019 \times 10^{29} \\ .433098 \times 10^{24} & .628023 \times 10^{29} \end{array}$$

On the other hand n cannot in general be too small, as the following argument will indicate. The terms of the series begin to decrease in absolute value when

$$\sqrt{72}\left(n + \frac{m(m-3)}{4}\right) > \frac{(m-2r)}{\sqrt{r}}\left(m + \frac{6r+19}{50}\right)^{3/2}.$$

The r.h.s is monotonically decreasing for $r = 1, 2, 3, \ldots$. Thus we need to determine the smallest r such that

$$\sqrt{72}\left(n + \frac{m(m-3)}{4}\right) = \frac{(m-2r)m^{3/2}}{\sqrt{r}}\left(1 + \frac{6r+19}{50m}\right)^{3/2}.$$

Now the maximum value of $1 + (6r+19)/50m = 1 + (6m+19)/50m$. Thus

$$\operatorname*{Max}_{r}\left(1 + \frac{6r+19}{50m}\right)^{3/2} < (1\cdot 5)^{3/2}.$$

Thus if we consider the simpler relationship

$$\frac{(m-2r)}{\sqrt{r}} = \frac{\sqrt{72}}{m^{3/2}}\left(n + \frac{m(m-3)}{4}\right) = X \quad \text{(say)},$$

we have that

$$r = \frac{(X^2+4m) \pm \sqrt{(X^2+4m)^2 - 16m^2}}{8},$$

$$\therefore \quad r \doteqdot \frac{m^2}{X^2+4m} \quad \text{as} \quad \frac{16m^2}{(X^2+4m)^2} < 1,$$

$$\therefore \quad r \doteqdot \frac{m^5}{72\left(n + \frac{m(m-3)}{4}\right)^2 + 4m^4}, \quad \text{where } r = 1, 2, 3, 4, \ldots.$$

Now $p(m) = p_m(2m)$, and thus we have in this case that

$$r \doteqdot \frac{m^5}{72\left(2m + \frac{m(m-3)}{4}\right)^2 + 4m^4} \doteqdot \frac{2m}{17}.$$

Thus $r \to \infty$ with m, and the series is thus of no value for calculating $p(m)$.

One final example using the above approximation for r.

6. Consider $P_{50}(500)$. We have that $r \doteqdot 2.8$. Thus the third term in the series is approximately the largest in absolute magnitude. In fact, the largest is the fourth, as the following results show.

$.329499 \times 10^{22}$	$.329499 \times 10^{22}$
$.117201 \times 10^{23}$	$-.842513 \times 10^{22}$
$.194333 \times 10^{23}$	$.110081 \times 10^{23}$
$.199553 \times 10^{23}$	$-.894714 \times 10^{22}$
$.142198 \times 10^{23}$	$.527271 \times 10^{22}$
$.746776 \times 10^{22}$	$-.219505 \times 10^{22}$
$.299625 \times 10^{22}$	$.801205 \times 10^{21}$
$.939667 \times 10^{21}$	$-.138462 \times 10^{21}$
$.233746 \times 10^{21}$	$.952837 \times 10^{20}$
$.465398 \times 10^{20}$	$.487439 \times 10^{20}$
$.745310 \times 10^{19}$	$.561970 \times 10^{20}$

Thus $.487 \times 10^{20} < p_{50}(500) < .562 \times 10^{20}$, whereas $P_{50}(500) = .553301 \times 10^{20}$.

7. Conclusion. The alternating series expansion for $q_m(n)$ provides an effective means for determining both upper and lower bounds when $n > m^{5/2}/\sqrt{72}$.

Department of Mathematics
North East London Polytechnic
Barking Precinct
Longbridge Road
Dagenham, Essex RM8 2AS, England

1. G. E. Andrews, *The Theory of Partitions*, Addison-Wesley, Reading, Mass., 1976.
2. L. E. Dickson, *History of the Theory of Numbers*, Vol. 2, Chelsea, New York, 1966.
3. H. Gupta, *Tables of Partitions*, The Royal Society Mathematical Tables, Vol. 4, Cambridge, 1958.
4. G. J. Rieger, "Über partitionen," *Math. Ann.*, v. 138, 1959, pp. 356–362.