REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

The numbers in brackets are assigned according to the revised indexing system printed in Volume 28, Number 128, October 1974, pages 1191–1194.

17[9.35].—WILLIAM B. JONES & W. J. THRON, "Continued fractions, analytic theory and applications," *Encyclopedia of Mathematics and its Applications*, Addison-Wesley, Reading, Mass., 1980, xxviii + 428 pp., 24 cm. Price $37.50.

The history of continued fractions (C.F.) is an old one since it can be viewed as beginning with Euclid's algorithm (c. 330 B.C.—c. 275 B.C.) for the g.c.d. In modern notation the algorithm can be put in the form of a three-term recurrence relation which in turn can be expressed as a C.F. True, Euclid did not present the algorithm in this fashion though he used geometrical considerations concerning the relative length of two segments. The algorithm is useful to simplify ratios as exemplified in the work of Archimedes (287 B.C.—212 B.C.), whose approximation 22/7 for π was often used before the advent of electric and hand computers. It was Theon of Alexandria (c. 365 A.D.) who used the start of the C.F.

$$(a^2+b)^{1/2} = a + \frac{b}{2a\,+}\,\frac{b}{2a\,+}\cdots$$

to find the side of a square of a given area. In this connection, in 1572, R. Bombelli in essence gave the approximate expansion

$$13^{1/2} = 3 + \frac{4}{6\,+}\,\frac{4}{6\,+}\cdots.$$

In 1613, P. Cataldi gave the analogous form for $18^{1/2}$. Indeed he gave the first 15 convergents and proved that they are alternately greater and smaller than $18^{1/2}$ and that they converge to $18^{1/2}$.

Another problem of ancient times which leads to the early development of C.F. is named after Diophantus (c. 250 A.D.) who found a rational solution of $ax + by = c$ where a, b and c are given positive integers. The problem was completely solved by the Indian mathematician Āryabhata (476–550). The solution was rediscovered by a number of workers including Lagrange (1736–1813).

In 1655, John Wallis (1616–1703) published his now famous infinite product expansion for $4/\pi$. He reports that Lord William Brouncker (1620–1684) gave the expansion which in contemporary notation reads

$$\frac{4}{\pi} = 1 + \frac{1|}{|2} + \frac{9|}{|2} + \cdots + \frac{n^2|}{|2} + \cdots.$$

Wallis uses the words (in Latin) 'continued fraction' to describe the expression and thus the genesis of the words continued fractions. From this time on, the study of C.F. moves in essentially two directions. One has to do with the theory of numbers, the other is in their use in the representation and approximation of functions.

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Contributions to number theory by use of C.F. in the period 1700–1900 were made by many well-known names in the history of mathematics. In particular, Euler (1707–1783) made many important discoveries. He proved that every rational number can be developed into a terminating C.F. and that a periodic C.F. is the root of a quadratic equation. He discussed Euclid's algorithm and the simplification of fractions. The proof that every prime of the form $4n + 1$ is the sum of two squares is due to Euler. Lagrange gave the solution of the Pell equation $u^2 - Dv^2 = 1$, where D is a positive integer, while the complete solution is due to Legendre (1752–1833). Other results were obtained by Gauss (1777–1855) and Liouville (1809–1882).

Contributions to the representation and approximation of functions by use of C.F. in the period 1700–1900 also contains a number of notable figures. Euler derived the formal divergent series

$$\sum_{k=0}^{\infty} (-1)^k k!/x^k$$

for the integral

$$x \int_0^{\infty} e^{-t}\, dt/(x+t).$$

He converted the series into a C.F. and used it to 'evaluate' the divergent series for $x = 1$. Lambert (1728–1777) expressed $\tan x$, $\arctan x$, $\ln(1 + x)$ and $(e^x - 1)/(e^x + 1)$ as C.F. In 1776, Lagrange wrote a paper which developed a technique for converting a power series into a C.F. In particular cases, these were reduced to ordinary fractions. He noticed that the latter as a power series agreed with the original power series as far as is possible. We have here the genesis of Padé approximants. Lagrange did no further work on the subject. In a letter to d'Alembert, he makes reference to the volume containing the 1776 paper and states "there is naturally something of myself in this, but nothing that merits your attention..."—a diametrically opposite view of the current interest in Padé approximations. Laplace (1749–1827) expressed the error function

$$\int_x^{\infty} e^{-t^2}\, dt$$

as a C.F. Gauss (1777–1855) derived three-term recurrence relations involving the parameters of the hypergeometric function which bears his name, and from these the corresponding C.F. for ${}_2F_1(1, a; c; -z)$ and its confluent forms are readily derived. If $P(x)$, $Q(x)$, and $R(x)$ are polynomials in x, Laguerre (1834–1886) studied the C.F. representations of some particular solutions of the differential equation

$$P(x)y' = Q(x)y + R(x).$$

Frobenius (1849–1917), Padé (1863–1953) and others studied the problem of expanding formal power series into the ratio of two polynomials. Frobenius derived certain recursion relations between the numerators and denominators of Padé fractions.

Perhaps the most important contributions to the theory of C.F. in the 19th century were made by Stieltjes (1856–1894). Indeed he is really the founder of the modern analytic theory of C.F. Stieltjes studied

$$F(x) = x\int_0^\infty \frac{d\alpha(t)}{x+t},$$

its asymptotic expansion, its C.F. representation, and the connection of the latter with the Gaussian quadrature of $F(x)$. If P_n/Q_n is the nth convergent of the C.F., he showed that the sequences $\{Q_n\}$ are orthogonal polynomials over $(0, \infty)$ with respect to $d\alpha(t)$. Stieltjes studied the expansion of arbitrary functions in series of orthogonal polynomials and the so-called moment problem. That is, determine a function $\alpha(t)$ associated with a given sequence $\{c_n\}$ defined by

$$c_n = \int_0^\infty t^n d\alpha(t)\, dt.$$

In the first half of the 20th century research in C.F. deals mostly with the analytic theory. This era is marked by some important volumes which effectively summarize research on the subject of C.F. from its beginnings. First is the 1929 volume by Perron [1] and second is the 1948 volume by Wall [2] (1902–1971). Also pertinent are two volumes (1954, 1957) by Perron [3], [4] which form a revised and improved edition of his 1929 volume. There is also the 1963 translation (by Wynn) of a 1956 work by Khovanskii [5]. Approximate cut off dates for [2]–[5] can be taken as 1950. Since that time two developments have taken place which have spawned much interest in the subject. First is the recognition that Padé approximations (though discussed by Frobenius in 1881 and Padé in 1892) play an important role in physical applications; and, second is the advent of the high speed digital computer. Clearly, an up to date exposition of the subject is long overdue. In view of their many research contributions, Jones and Thron are eminently qualified to remedy this deficiency. It is a credit to their scholarship, erudition and skill that they have produced a most worthy and useful volume—a valuable research tool. As noted from the title, emphasis is on the analytic theory.

The book is intended for all pure and applied workers in mathematics and the sciences. The only requisite knowledge is the rudiments of complex analysis.

There are twelve chapters, two appendices, author and subject indices. The bibliography contains about 370 references, with about 240 of them dating from 1950.

The book is Volume 11 of a series which is to form an encyclopedia of mathematics. The editor is Gian-Carlo Rota, who presents a description of this effort. Of more pertinence to the subject of the present volume is a Foreword by Felix E. Browder, who is the General Editor, Section on Analysis, of the projected encyclopedia and an Introduction by Peter Henrici. The Foreword and Introduction are interesting reviews of the subject of continued fractions.

Y. L. L.

1. O. Perron, *Die Lehre von den Kettenbrüchen*, Teubner, Leipzig, 1929.

2. H. S. Wall, *Analytic Theory of Continued Fractions*, Van Nostrand, New York, 1948.

3. O. Perron, *Die Lehre von den Kettenbrüchen*, Band I, Teubner, Stuttgart, 1954.

4. O. Perron, *Die Lehre von den Kettenbrüchen*, Band II, Teubner, Stuttgart, 1957.

5. A. M. Khovanskii, *The Application of Continued Fractions and Their Generalizations to Problems in Approximation Theory*, Noordhoff, Groningen, 1963.

18[5.10.3].—BRUCE IRONS & SOHRAB AHMAD, *Techniques of Finite Elements*, Ellis Harwood Ltd., Chichester & Halsted Press, a division of John Wiley & Sons, New York, 1981, 529 pp., 23cm. Price $30.95.

Let it be said at the outset that this reviewer holds Bruce Irons in great respect for his many original contributions to the understanding of finite element technology as engineers understand it. Bruce Irons stands out as the most prolific contributor to the art of finite element analysis during the 1960's and early 1970's. Several techniques Irons developed or suggested have since become standard procedures in finite element analysis or motivated significant theoretical work. Examples are: isoparametric elements, use of rational basis functions in conjunction with exactly and minimally conforming C^1 elements, hierarchic basis functions, the frontal solution technique and, of course, the patch test. These impressive accomplishments are based on Irons' almost unique ability to combine strong engineering intuition with mathematical know-how at the level of traditional engineering training.

This book was written with the approach that served Irons so well in that exciting and productive period of the 60's: intuitive reasoning, supported by sketches and analogies whenever possible, with minimal mathematical development. This approach is the source of both the strength and weakness of the book: while the intuitive treatment may provide fresh insight for the mathematically trained and untrained alike, it can lead to misinterpretations and erroneous conclusions when important conditions and qualifications are neglected. There has been very substantial progress in the theoretical understanding of the finite element method in the last ten years. By ignoring these developments the authors accepted the limitations common to all engineer-oriented books written on the subject.

Looking to the future, the authors present interesting observations concerning the users of finite element technology and their requirements: There will be "vast numbers of unsophisticated users" who will be "interested in meshes that give about $\pm 2\%$ accuracy [presumably in stresses]—not much more and certainly not much less". They will have access to "uncosted computers" in their offices and will employ finite element technology in design, submitting small ten-element jobs almost daily. Without actually saying so, the authors make a very good case for adaptive finite element technology. Clearly, unsophisticated users, who are viewing finite element technology as a modeling tool and state their requirements in terms of desired levels of precision of some computed function, will require sophisticated, foolproof software systems. On the subject of error estimation, the authors express a view which has not yet been generally accepted by engineer analysts but is likely to become conventional wisdom within a few years: "We feel that the time is past when we should confirm our answers by experiments, or compare them with other people's answers. Experiments are generally less trustworthy than calculations using an established technique. Rather, we should attempt to keep computing self-contained, to make the computer assess its own results". These requirements present an important challenge to the finite element research community.

The book deals with a wide range of topics. Its 29 chapters are divided into seven parts: 1. Introduction (basic techniques, shape functions); 2. Organization (miscellaneous topics ranging over problems of management, structural concepts, the patch test and development of elements); 3. Solution techniques; 4. Trends in element formulation (the authors predict a bright future for the hybrid and SEMILOOF type elements); 5. Trends in solution techniques; 6. Speculations (miscellaneous non-structural applications and the patch test are discussed); 7. Theoretical details (numerical integration, matrices, differential geometry).

Fracture mechanics and the computational problems associated with applications of the finite element method in fracture mechanics receive little attention. This is surprising especially because the authors make the point that they are looking to the future, which undoubtedly will bring ever increasing reliance on fracture mechanics at the expense of the traditional approach based on maximum stress design.

An interesting and unique aspect of the book is its informal style. It is replete with irrelevant and often irreverent comments, which provide welcome diversions in the otherwise demanding pursuit of the idiosyncratic processes of bright engineering minds wrestling with the finer points of finite element analysis.

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19[3.35].—ALAN GEORGE & JOSEPH W. LIU, *Computer Solution of Large Sparse Positive Definite Systems*, Prentice-Hall, Englewood Cliffs, N. J., 1981, xii + 324 pp., $23\frac{1}{2}$ cm. Price \$24.95.

This book is about efficient implementation of the Cholesky factorization method for the solution of sparse positive definite systems. The two main concerns of the book are the development of the various reordering schemes and the corresponding efficient algorithms. The basic tools for direct sparse matrix solution are described which include the fundamentals of Cholesky factorization and solution as well as graph theoretic ideas for reordering algorithms. Algorithmic efficiency is achieved by careful consideration of operation counts and storage requirements. Fortran programs implementing the algorithms are included and discussed in great detail.

The authors discuss a collection of methods which from their experience they prefer for solving sparse matrix problems. Band and envelope methods are described and the Reverse Cuthill-McKee Algorithm is proposed for the reordering problem. The minimal degree algorithm is considered for low fill reordering for general sparse matrices. Quotient tree methods for finite element and finite difference problems are also studied. The last of the methods studied are one-way and nested dissection methods for finite element problems.

This book should be of interest to numerical analysts, engineers, and anyone involved in the solution of positive definite sparse matrix systems. As a text, the book could be used along with a good text on iterative matrix techniques for a one semester course on sparse matrix solution. Exercises dealing with program modification as well as more theoretical considerations are included at the end of the chapters. The book also provides an in depth documentation of the authors sparse matrix package "SPARSPAK". The detailed study of the coding of these methods provides an example of good Fortran programming style.

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20[10.05].—V. K. B. KOTA, *Table of Reduction of U(10) Partitions into SU(3) Irreducible Components*, 112 pages of computer printout, deposited in the UMT file.

The irreducible representations (IR) of the unitary group $U(10)$ corresponding to any integer N, are denoted [1] by the Young partitions $[f]$, where

$$[f] = [f_1 f_2 \cdots f_{10}]$$

with $f_i \geqslant f_{i+1} \geqslant 0$, and $\Sigma_{i=1}^{10} f_i = N$ while the IR of $SU(3)$ (the unitary unimodular group in three dimensions) are denoted [2] by the pair of numbers $(\lambda\,\mu)$. The tabulations are for the reduction of the IR of $U(10)$ to IR of $SU(3)$ with the constraint that the partition [1] of $U(10)$ correspond to the IR $(3\,0)$ of $SU(3)$. Essentially the problem is to obtain the $SU(3)$ content of the "plethysms" $\{3\} \otimes \{f\}$. All possible $U(10)$ partitions having a maximum of four columns (i.e., all partitions of the type $[4^a 3^b 2^c 1^d]$ are considered and their reductions to $SU(3)$ contents are tabulated. A method to obtain these reductions is given in [3] and a computer code is constructed for the IBM 360/44 machine which follows this procedure step by step. The reductions for $N \leqslant 6$ are given previously by Ibrahim [4]. The tabulations give the reductions up to $N = 20$, and for the remaining partitions the reductions can be obtained using the relationship

$$[4^{10-a-b-c-d} 3^d 2^c 1^b] = \sum_\gamma A_\gamma(\mu_\gamma \lambda_\gamma),$$

if

$$[4^a 3^b 2^c 1^d] = \sum_\gamma A_\gamma(\lambda_\gamma \mu_\gamma),$$

where A_γ gives the number of occurrences of $(\lambda_\gamma \mu_\gamma)$ in the reduction of the partition $[4^a 3^b 2^c 1^d]$. The stringent dimensionality check is performed for each partition and the tabulations display all types of symmetry checks.

In the tables, the $U(10)$ partition was printed out as $f_1 f_2 f_3 \cdots f_{10}$ and below this the IR of $SU(3)$ contained in the partition are all listed as $A_1(\lambda_1 \mu_1)\, A_2(\lambda_2 \mu_2) \ldots$.

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2. J. P. ELLIOTT, "Collective motion in the nuclear shell model I and II," *Proc. Roy. Soc. London Ser. A*, v. 245, 1958, pp. 128, 562.

3. V. K. B. KOTA, "Plethysm problem of $U((n+1)(n+2)/2) \supset SU(3)$," *J. Phys. A*, v. 10, 1977, p. L39.

4. E. M. IBRAHIM, *Tables for the Plethysm of S-functions*, Roy. Soc. (London), Depository of unpublished tables, no. 1, 1950.