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Some Remarks on the Convergence of Approximate Solutions of Nonlinear Evolution Equations in Hilbert Spaces

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Abstract. Let $\partial\Phi$ be the subdifferential of some lower semicontinuous convex function Φ of a real Hilbert space H, $f \in L^2(0, T; H)$ and u_n a continouous piecewise linear approximate solution of $du/dt + \partial\Phi(u) \ni f$, obtained by an implicit scheme. If $u_0 \in \overline{\mathrm{Dom}(\Phi)}$, then du_n/dt converges to du/dt in $L^2(0, T; H)$. Moreover, if $u_0 \in \overline{\mathrm{Dom}(\partial\Phi)}$, we construct a step function $\eta_n(t)$ approximating t such that $\lim_{n\to+\infty} \int_0^T \eta_n |du_n/dt - du/dt|^2\,dt = 0$. When Φ is inf-compact and when the sequence of approximation of f is weakly convergent to f, then u_n converges to u in $C([0, T]; H)$ and $\eta_n du_n/dt$ is weakly convergent to $t\,du/dt$.

Introduction. Set H a real Hilbert space with scalar product $(\cdot, \cdot)$ and norm $|\cdot|$ and Φ a lower semicontinuous proper convex function from H into $(-\infty, +\infty]$. The subdifferential $\partial\Phi$ of Φ is the multivalued operator from H into $\mathcal{P}(H)$ defined by

$$y \in \partial\Phi(x) \text{ if and only if, } \forall u \in H, \quad \Phi(u) - \Phi(x) \geqslant (y, u - x).$$

The operator $\partial\Phi$ is a maximal monotone operator on H (cf. [4]) and the semigroup generated by $-\partial\Phi$ has strong regularizing properties which have been discovered by Brezis [6]. In particular, if $f \in L^2(0, T; H)$ and $u_0 \in \overline{\mathrm{Dom}(\partial\Phi)}$, then the weak solution of

$$(1) \qquad \begin{cases} du/dt + \partial\Phi(u) \ni f & \text{on } (0, +\infty), \\ u(0) = u_0, \end{cases}$$

is in fact a strong one. If we assume $\operatorname{Min}\Phi = 0$ (which is always possible), then the following estimate holds

$$(2) \qquad \left(\int_0^T \left|\frac{du}{dt}\right|^2 t\,dt\right)^{1/2} \leqslant \left(\int_0^T |f|^2 t\,dt\right)^{1/2} + \frac{1}{\sqrt{2}}\int_0^T |f|\,dt + \frac{1}{\sqrt{2}}\operatorname{dist}(u_0, K),$$

where $K = \{x \in H: \Phi(x) = 0\}$. Moreover if u_n is the solution of

$$(3) \qquad \begin{cases} du_n/dt + \partial\Phi(u_n) \ni f_n & \text{on } (0, +\infty), \\ u_n(0) = u_{0,n}, \end{cases}$$

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and if $\lim_{n\to+\infty} u_{0,n} = u_0$ and $\lim_{n\to+\infty}\int_0^T |f - f_n|^2\,dt = 0$, then

$$(4)\qquad \begin{cases} \text{(i)} & \lim_{n\to+\infty} \|u - u_n\|_{C([0,T];\,H)} = 0, \\ \text{(ii)} & \lim_{n\to+\infty} \int_0^T \left|\frac{du_n}{dt} - \frac{du}{dt}\right|^2 t\,dt = 0. \end{cases}$$

However, it can be noticed that if (4)(i) is an obvious consequence of the properties of weak solutions and remains valid in more general cases, (4)(ii) was only discovered in 1977 (cf. [5] and [1] for a similar result).

In this paper we introduce the approximate solutions of (1) through the implicit scheme

$$(5)\qquad \begin{cases} \dfrac{u_n^k - u_n^{k-1}}{t_n^k - t_n^{k-1}} + \partial\Phi(u_n^k) \ni f_n^k, & k = 1, 2, \ldots, N(n), \\ u_n^0 = u_0 \quad\text{and}\quad 0 = t_n^0 < t_n^1 < \cdots < t_n^{N(n)} = T. \end{cases}$$

If f_n is the step function taking the value f_n^k on (t_n^{k-1}, t_n^k), u_n the continuous piecewise linear function taking the value u_n^k at t_n^k, and η_n the step function taking the value t_n^{k-1} on (t_n^{k-1}, t_n^k), then

$$(6)\qquad \left(\int_0^T \left|\frac{du_n}{dt}\right|^2 \eta_n\,dt\right)^{1/2} \leqslant \left(\int_0^T |f_n|^2 \eta_n\,dt\right)^{1/2} + \sqrt{2}\int_0^T |f_n|\,dt + \frac{1}{\sqrt{2}}\operatorname{dist}(u_0, K).$$

Moreover if $\lim_{n\to+\infty} \operatorname{Max}_{0\leqslant k\leqslant N(n)}(t_n^k - t_n^{k-1}) = 0$ and $\lim_{n\to+\infty}\int_0^T |f_n - f|^2\,dt = 0$, then

$$(7)\qquad \begin{cases} \text{(i)} & \lim_{n\to+\infty} \|u_n - u\|_{C([0,T];\,H)} = 0, \\ \text{(ii)} & \lim_{n\to+\infty} \int_0^T \left|\frac{du_n}{dt} - \frac{du}{dt}\right|^2 \eta_n\,dt = 0. \end{cases}$$

It must be noticed that if (7)(i) is already well known in more general cases (cf. [7]) and is more or less a consequence of the theorem of Crandall and Liggett, (7)(ii) is new and could be of some use in numerical analysis. However, in the general case it appears that it is not possible to obtain error estimates for the convergence of du_n/dt.

When Φ is inf-compact and f_n weakly convergent to f in $L^2(0, T;\, H)$, then (7)(i) remains valid and $\eta_n du_n/dt$ is weakly convergent to $t\,du/dt$ as $n \to +\infty$.

We give also an extension of that type of result for a more general operator A generating a semigroup $(S(t))_{t\geqslant 0}$ which is compact for $t > 0$.

The Main Estimates. Set Φ a lower continuous proper convex function from H into $(-\infty, +\infty]$ such that $\operatorname{Min}\Phi = 0$ and $K = \{x \in H\colon \Phi(x) = 0\}$. For $T > 0$ we set $P_n = \{0 = t_n^0 < t_n^1 < \cdots < t_n^{N(n)} = T\}$ a partition of $[0, T]$, $\varepsilon_n^k = t_n^k - t_n^{k-1}$ and $\|P_n\| = \operatorname{Max}_{0\leqslant k\leqslant N(n)} \varepsilon_n^k$. We define the two step functions η_n and f_n from $[0, T]$ into $\mathbf{R}$ and H, respectively, as the functions taking the values t_n^{k-1} and f_n^k on (t_n^{k-1}, t_n^k).

Thanks to the maximal monotonicity of $\partial\Phi$, the sequence $\{u_n^k\}_{k=0,\dots,N}$ is well defined by the following relations

$$(8)\qquad \begin{cases} \dfrac{u_n^k - u_n^{k-1}}{\varepsilon_n^k} + \partial\Phi(u_n^k) \ni f_n^k, & k = 1,\dots,N, \\ u_n^0 = u_0. \end{cases}$$

If u_n is the continuous piecewise linear function taking the value u_n^k at t_n^k, then

$$u_n(t) = \frac{t_n^k - t}{\varepsilon_n^k} u_n^{k-1} + \frac{t - t_n^{k-1}}{\varepsilon_n^k} u_n^k \quad \text{and} \quad \frac{du_n}{dt} = \frac{u_n^k - u_n^{k-1}}{\varepsilon_n^k}$$

on (t_n^{k-1}, t_n^k).

THEOREM 1. *Suppose* $u_0 \in \overline{\mathrm{Dom}(\partial\Phi)}$. *Then the following estimates hold*:

$$(9)\qquad \left(\int_0^T \left|\frac{du_n}{dt}\right|^2 \eta_n\, dt\right)^{1/2} \leq \left(\int_0^T |f_n|^2 \eta_n\, dt\right)^{1/2} + \sqrt{2}\int_0^T |f_n| dt + \frac{1}{\sqrt{2}}\operatorname{dist}(u_0, K),$$

$$(10)\qquad \left(\int_\delta^T \left|\frac{du_n}{dt}\right|^2 dt\right)^{1/2} \leq \left(\int_0^T |f_n|^2 dt\right)^{1/2} + \sqrt{\frac{2}{\delta}}\int_0^\delta |f_n| dt + \frac{1}{\sqrt{2\delta}}\operatorname{dist}(u_0, K),$$

for all δ *such that* $\delta = t_n^p$, $1 \leq p \leq N(n)$. *Moreover if* $u_0 \in \mathrm{Dom}(\Phi)$, *we have*

$$(11)\qquad \left(\int_0^T \left|\frac{du_n}{dt}\right|^2 dt\right)^{1/2} \leq \left(\int_0^T |f_n|^2 dt\right)^{1/2} + \sqrt{\Phi(u_0)}.$$

Remark 1. If we no longer assume $\operatorname{Min}\Phi = 0$, we obtain inequalities similar to those of Theorem 1 in changing Φ and f_n. To get this we consider $x_0 \in \mathrm{Dom}(\partial\Phi)$ and $\xi \in \partial\Phi(x_0)$, and we set $\tilde{\Phi}(x) = \Phi(x) - \Phi(x_0) - (\xi, x - x_0)$ for $x \in \mathrm{Dom}(\Phi)$ and $\tilde{f}_n = f_n - \xi$. As $\partial\tilde{\Phi} = \partial\Phi - \xi$, $\{u_n^k\}$ satisfies (8) with Φ and f_n replaced by $\tilde{\Phi}$ and $\tilde{f}$ and $\operatorname{Min}\tilde{\Phi} = 0$.

Remark 2. If we let $f_n = 0$ in Theorem 1, we deduce from (9) that

$$(12)\qquad \left(\int_0^t \left|\frac{du_n}{dt}\right|^2 \eta_n\, dt\right)^{1/2} \leq \frac{1}{\sqrt{2}}\operatorname{dist}(u_0, K),$$

for any $0 \leq t \leq T$. If we want to have a pointwise estimate on du_n/dt, we have to suppose $\varepsilon_n^k = T/N$, for $k = 0, 1,\dots,N$. We then deduce from the monotonicity of $\partial\Phi$ that $|du_n/dt|$ is nonincreasing. Hence

$$(13)\qquad \left|\frac{du_n}{dt}(t)\right| = \frac{N}{T}\left|u_n^k - u_n^{k-1}\right| \leq \frac{1}{\sqrt{t_n^k t_n^{k-1}}}\operatorname{dist}(u_0, K),$$

for any $t_n^{k-1} < t < t_n^k$, $k > 1$.

Before proving our estimates we need the following result which is somewhat analogous to Lemma A.5 of [4].

LEMMA 1. *Suppose* $\{a_n\}$ *and* $\{b_n\}$ *are two sequences of nonnegative numbers such that*

$$\frac{1}{2}a_n^2 \leqslant \frac{1}{2}a_0^2 + \sum_{j=1}^{n} a_j b_j \quad \forall n \geqslant 0. \tag{14}$$

Then

$$a_n \leqslant a_0 + 2\sum_{j=1}^{n} b_j \quad \forall n \geqslant 0. \tag{15}$$

Proof. By induction we define the nonnegative sequence $\{\alpha_n\}_{n\geqslant 1}$ by

$$\frac{1}{2}\alpha_n^2 = \frac{1}{2}a_0^2 + \sum_{j=1}^{n} \alpha_j b_j \quad \forall n \geqslant 1. \tag{16}$$

Then $\alpha_n^2 - 2\alpha_n b_n - (a_0^2 + 2\Sigma_{j=1}^{n-1}\alpha_j b_j) = 0$. But $\alpha_{n-1}^2 = a_0^2 + 2\Sigma_{j=1}^{n-1}\alpha_j b_j$, so $\alpha_n^2 - 2\alpha_n b_n - \alpha_{n-1}^2 = 0$ and $\alpha_n = b_n + \sqrt{b_n^2 + \alpha_{n-1}^2}$. Hence $\alpha_n \leqslant 2b_n + \alpha_{n-1}$, and $\alpha_n \leqslant a_0 + 2\Sigma_{j=1}^{n} b_j$. As it is easy to see by induction that $a_n \leqslant \alpha_n$ for $n \geqslant 1$, we get (15).

Proof of Theorem 1. First we assume $u_0 \in \mathrm{Dom}(\Phi)$, and we prove (11). From the definition of $\{u_n^k\}$, we have $f_n^k - (u_n^k - u_n^{k-1})/\varepsilon_n^k \in \partial\Phi(u_n^k)$. Then

$$\left(u_n^k - u_n^{k-1}, f_n^k - \frac{u_n^k - u_n^{k-1}}{\varepsilon_n^k}\right) \geqslant \Phi(u_n^k) - \Phi(u_n^{k-1});$$

so we get

$$\frac{|u_n^k - u_n^{k-1}|^2}{\varepsilon_n^k} + \Phi(u_n^k) - \Phi(u_n^{k-1}) \leqslant (f_n^k, u_n^k - u_n^{k-1}) \quad \forall 1 \leqslant k \leqslant N. \tag{17}$$

Summing all those inequalities, we have

$$\sum_{k=1}^{N} \frac{|u_n^k - u_n^{k-1}|^2}{(\varepsilon_n^k)^2}\varepsilon_n^k + \Phi(u_n^N) - \Phi(u_0) \leqslant \sum_{k=1}^{N}\left(f_n^k, \frac{u_n^k - u_n^{k-1}}{\varepsilon_n^k}\right)\varepsilon_n^k,$$

that is,

$$\int_0^T \left|\frac{du_n}{dt}\right|^2 dt + \Phi(u_n^N) \leqslant \Phi(u_0) + \int_0^T \left(f_n, \frac{du_n}{dt}\right) dt. \tag{18}$$

So we deduce (11).

From (17) we get for $1 \leqslant k \leqslant N$,

$$\frac{|u_n^k - u_n^{k-1}|}{(\varepsilon_n^k)^2}\varepsilon_n^k t_n^{k-1} + (\Phi(u_n^k) - \Phi(u_n^{k-1}))t_n^{k-1} \leqslant \left(f_n^k, \frac{u_n^k - u_n^{k-1}}{\varepsilon_n^k}\right)\varepsilon_n^k t_n^{k-1}. \tag{19}$$

Summing those inequalities and using the fact that

$$\begin{aligned}\sum_{k=1}^{N} (\Phi(u_n^k) - \Phi(u_n^{k-1}))t_n^{k-1} &= \Phi(u_n^N)t_n^{N-1} - \sum_{k=1}^{N-1} \Phi(u_n^k)(t_n^k - t_n^{k-1}) \\ &= \Phi(u_n^N)t_n^{N-1} - \sum_{k=1}^{N-1} \Phi(u_n^k)\varepsilon_n^k,\end{aligned}$$

we deduce that

$$\sum_{k=1}^{N} \frac{\left|u_n^k - u_n^{k-1}\right|^2}{\left(\varepsilon_n^k\right)^2} \varepsilon_n^k t^{k-1} + \Phi\left(u_n^N\right) t^{N-1} \leqslant \sum_{k=1}^{N} \left(f_n^k, \frac{u_n^k - u_n^{k-1}}{\varepsilon_n^k} \right) \varepsilon_n^k t_n^{k-1} + \sum_{k=1}^{N-1} \Phi\left(u_n^k\right) \varepsilon_n^k,$$

and so, as $\Phi \geqslant 0$ in H,

$$\int_0^T \left| \frac{du_n}{dt} \right|^2 \eta_n \, dt \leqslant \int_0^T \left(f_n, \frac{du_n}{dt} \right) \eta_n \, dt + \int_0^T \Phi(\tilde{u}_n) \, dt, \tag{20}$$

where $\tilde{u}_n$ is the step function taking the value u_n^k on (t_n^{k-1}, t_n^k). By Schwarz inequality we get

$$\left(\int_0^T \left| \frac{du_n}{dt} \right|^2 \eta_n \, dt \right)^{1/2} \leqslant \left(\int_0^T \Phi(\tilde{u}_n) \, dt \right)^{1/2} + \left(\int_0^T |f_n|^2 \eta_n \, dt \right)^{1/2}. \tag{21}$$

On the other hand, we have for any $v \in \operatorname{Dom}(\Phi)$

$$\Phi(v) - \Phi\left(u_n^k\right) \geqslant \left(f_n^k - \frac{u_n^k - u_n^{k-1}}{\varepsilon_n^k}, v - u_n^k \right).$$

In particular, if $v \in K$, $\Phi(u_n^k) - \Phi(v) = \Phi(u_n^k)$ and

$$\Phi\left(u_n^k\right) + \left(\frac{u_n^k - u_n^{k-1}}{\varepsilon_n^k}, u_n^k - v \right) \leqslant \left(f_n^k, u_n^k - v \right) \quad \forall 1 \leqslant k \leqslant N. \tag{22}$$

But $(u_n^k - u_n^{k-1}, u_n^k - v) = ((u_n^k - v) - (u_n^{k-1} - v), u_n^k - v) \geqslant \frac{1}{2} \mid u_n^k - v \mid^2 - \frac{1}{2} \mid u_n^{k-1} - v \mid^2$. So we get from (22)

$$\sum_{k=1}^{N} \Phi\left(u_n^k\right) \varepsilon_n^k + \frac{1}{2} \left|u_n^N - v\right|^2 \leqslant \sum_{k=1}^{N} \left(f_n^k, u_n^k - v\right) \varepsilon_n^k + \frac{1}{2} \left|u_0 - v\right|^2,$$

that is,

$$\int_0^T \Phi(\tilde{u}_n) \, dt + \frac{1}{2} \left|u_n^N - v\right|^2 \leqslant \frac{1}{2} \left|u_0 - v\right|^2 + \int_0^T (f_n, \tilde{u}_n - v) \, dt. \tag{23}$$

Moreover, that last inequality remains true if we replace $T = t_n^N$ by t_n^k and u_n^N by u_n^k, so

$$\frac{1}{2} \left|u_n^k - v\right|^2 + \sum_{j=1}^{k} \tilde{\Phi}\left(u_n^j\right) \varepsilon_n^j \leqslant \frac{1}{2} \left|u_0 - v\right|^2 + \sum_{j=1}^{N} \varepsilon_n^j \left|f_n^j\right| \left(\left|u_n^j - v\right|^2 + 2 \sum_{i=1}^{j} \Phi\left(u_n^i\right) \varepsilon_n^i \right)^{1/2}. \tag{24}$$

By applying Lemma 1, we deduce

$$\left(\left|u_n^k - v\right|^2 + 2 \sum_{j=1}^{k} \Phi\left(u_n^j\right) \varepsilon_n^j \right)^{1/2} \leqslant \left|u_0 - v\right| + 2 \sum_{j=1}^{k} \left|f_n^j\right| \varepsilon_n^j. \tag{25}$$

In particular, for $k = N$, we have

$$\left(\int_0^T \Phi(\tilde{u}_n)\,dt\right)^{1/2} \leq \frac{1}{\sqrt{2}}|u_0 - v| + \sqrt{2}\int_0^T |f_n|\,dt. \tag{26}$$

From that inequality and (21) we get (9) when $u_0 \in \mathrm{Dom}(\Phi)$. For $u_0 \in \overline{\mathrm{Dom}(\partial\Phi)} = \overline{\mathrm{Dom}(\Phi)}$, we consider $\{u_{0,m}\} \in \mathrm{Dom}(\Phi)$ such that $u_{0,m} \to_{m\to\infty} u_0$. If $\{u_{n,m}\}$ is the sequence of continuous piecewise linear approximate solutions defined by the same implicit scheme as u_n, with initial data $u_{0,m}$ instead of u_0, then $\|u_n - u_{n,m}\|_{C([0,T];\,H)} \leq |u_0 - u_{0,m}|$. Hence we get (9) as $\lim_{m\to+\infty} du_{n,m}/dt = du_n/dt$, uniformly on $[0, T]$.

In order to get (10), set $\delta = t_n^p$ for some $p \geq 1$. We have

$$\frac{1}{\delta}\int_0^\delta \Phi(\tilde{u}_n)\,dt \in \left[\mathrm{Min}\left\{\Phi\left(u_n^i\right) \colon 1 \leq i \leq p\right\};\, \mathrm{Max}\left\{\Phi\left(u_n^i\right) \colon 1 \leq i \leq p\right\}\right].$$

If $\mathrm{Min}\{\Phi(u_n^i)\colon 1 \leq i \leq p\} = \Phi(u_n^{k_0})$, then $\Phi(u_n^{k_0}) \leq (1/\delta)\int_0^\delta \Phi(\tilde{u}_n)\,dt$. In summing the inequalities (17) for $k = k_0 + 1, \ldots, N$, we get

$$\int_{t_n^{k_0}}^T \left|\frac{du_n}{dt}\right|^2 dt + \Phi\left(u_n^N\right) - \Phi(u_0) \leq \int_{t_n^{k_0}}^T \left(f_n, \frac{du_n}{dt}\right) dt,$$

so we have

$$\left(\int_\delta^T \left|\frac{du_n}{dt}\right|^2 dt\right)^{1/2} \leq \left(\int_0^T |f_n|^2\,dt\right)^{1/2} + \left(\frac{1}{\delta}\int_0^\delta \Phi(\tilde{u}_n)\,dt\right)^{1/2}.$$

But, from (26) with T replaced by δ, we have

$$\left(\frac{1}{\delta}\int_0^\delta \Phi(\tilde{u}_n)\,dt\right)^{1/2} \leq \frac{1}{\sqrt{2\delta}}|u_0 - v| + \sqrt{\frac{2}{\delta}}\int_0^\delta |f_n|\,dt.$$

With those two last inequalities we deduce (10).

Remark 3. By changing slightly the proof of (9) we can also obtain the following inequality valid for any $v \in \mathrm{Dom}(\Phi)$:

$$\begin{aligned}\left(\int_0^T \left|\frac{du_n}{dt}\right|^2 \eta_n\,dt\right)^{1/2} &\leq \left(\int_0^T |f_n|^2 \eta_n\,dt\right)^{1/2} + \sqrt{2}\int_0^T |f_n|\,dt \\ &\quad + \frac{1}{\sqrt{2}}|u_0 - v| + \sqrt{T\Phi(v)}.\end{aligned} \tag{27}$$

To get this, we start from

$$\Phi(v) - \Phi\left(u_n^k\right) \geq \left(f_n^k - \frac{u_n^k - u_n^{k-1}}{\varepsilon_n^k}, v - u_n^k\right) \quad \forall v \in \mathrm{Dom}(\Phi),$$

and then we get

$$\int_0^T \Phi(\tilde{u}_n)\,dt + \frac{1}{2}\left|u_n^N - v\right|^2 \leq \frac{1}{2}|u_0 - v|^2 + T\Phi(v) + \int_0^T (f_n, \tilde{u}_n - v)\,dt.$$

Using (24) (with $\frac{1}{2}|u_0 - v|^2$ replaced by $\frac{1}{2}|u_0 - v|^2 + T\Phi(v)$) and Lemma 1 we deduce (27).

The Convergence Theorems. We still keep the notations of the first section and we suppose that u is the strong solution of

$$(28)\qquad \begin{cases} \dfrac{du}{dt} + \partial\Phi(u) \ni f & \text{on } (0, +\infty), \\ u(0) = u_0, \end{cases}$$

with $u_0 \in \overline{\mathrm{Dom}(\partial\Phi)}$ and $f \in L^2(0, T; H)$. Our first result is the following

THEOREM 2. *If* $u_0 \in \overline{\mathrm{Dom}(\partial\Phi)}$ *and* $\lim_{n\to\infty} \|P_n\| = \lim_{n\to+\infty} \int_0^T |f - f_n|^2\,dt = 0$, *then*

$$(29)\qquad \begin{cases} \text{(i)} & \lim\limits_{n\to+\infty} \|u_n - u\|_{C([0,T];\,H)} = 0, \\ \text{(ii)} & \lim\limits_{n\to+\infty} \displaystyle\int_0^T \left|\frac{du_n}{dt} - \frac{du}{dt}\right|^2 \eta_n\,dt = 0. \end{cases}$$

Moreover, if $u_0 \in \mathrm{Dom}(\Phi)$, *then*

$$(30)\qquad \lim_{n\to+\infty} \int_0^T \left|\frac{du_n}{dt} - \frac{du}{dt}\right|^2 dt = 0.$$

Before proving Theorem 2, first notice that (29)(i) is known in quite more general cases (cf. [7]). We first need the following result.

LEMMA 2. *Suppose* $u_0 \in \mathrm{Dom}(\partial\Phi)$ *and* $\lim_{n\to+\infty} \|P_n\| = \lim_{n\to+\infty} \int_0^T |f - f_n|^2\,dt = 0$. *Then* $\eta_n du_n/dt$ *converges weakly to* $t\,du/dt$ *in* $L^2(0, T; H)$.

Proof. As $\|P_n\| \to 0$, for every $\varepsilon > 0$, there exists $n_0 \in \mathbf{N}$ such that $\eta_n(t) > \varepsilon/2$ on $(\varepsilon, T]$ for any $n \geqslant n_0$. By Theorem 1 and classical results on maximal monotone operators (cf. [4]) du_n/dt converges weakly to du/dt in $L^2_{\mathrm{loc}}((0, T]; H)$.

Set $\psi \in L^2(0, T; H)$. For any $\alpha > 0$, we have

$$\int_0^T \left(\eta_n \frac{du_n}{dt} - t\frac{du}{dt}, \psi\right) dt = \int_0^\alpha \left(\eta_n \frac{du_n}{dt} - t\frac{du}{dt}, \psi\right) dt + \int_\alpha^T \left(\eta_n \frac{du_n}{dt} - t\frac{du}{dt}, \psi\right) dt.$$

When n tends to $+\infty$, $\eta_n(t) - t$ tends to 0, uniformly with respect to t, so

$$\lim_{n\to+\infty} \int_\alpha^T \left(\eta_n \frac{du_n}{dt} - t\frac{du}{dt}, \psi\right) dt = 0.$$

Set $\delta > 0$. There exists $v \in \mathrm{Dom}(\Phi)$ such that $|u_0 - v| < \delta$. For such a v there exists $\alpha_0 > 0$ such that for $0 < \alpha \leqslant \alpha_0$ and $n \in \mathbf{N}$ we have (see Remark 3)

$$\left(\int_0^\alpha |f_n|^2 \eta_n\,dt\right)^{1/2} + \sqrt{2}\int_0^\alpha |f_n|\,dt + \sqrt{\alpha\Phi(v)} < \delta,$$

as $|f_n|$ is uniformly equi-integrable on $(0, T)$. Hence $(\int_0^\alpha |du_n/dt|^2 \eta_n\,dt)^{1/2} < 2\delta$. If α is also fixed such that $(\int_0^\alpha |du/dt|^2 t\,dt)^{1/2} < \delta$, we deduce

$$\limsup_{n\to+\infty} \left|\int_0^T \left(\eta_n \frac{du_n}{dt} - t\frac{du}{dt}, \psi\right) dt\right| < 3\delta\|\psi\|_{L^2(0,T;\,H)}.$$

Hence $\eta_n du_n/dt - t\,du/dt$ converges weakly to 0 in $L^2(0, T; H)$ and similarly $\eta_n(du_n/dt - du/dt)$, as $\lim_{n\to+\infty} \int_0^T (t - \eta_n)\,|du/dt|^2\,dt = 0$.

LEMMA 3. *Under the hypotheses of Lemma* 2, *we have*

$$\limsup_{n\to+\infty}\int_0^T\Phi(\tilde u_n)\,dt\leqslant\int_0^T\Phi(u)\,dt, \tag{31}$$

$\tilde u_n$ *being the step function taking the value* u_n^k *on* (t_n^{k-1}, t_n^k).

Proof. We start from the following inequalities valid for $v\in \mathrm{Dom}(\Phi)$ and $1\leqslant k\leqslant N$.

$$\Phi(u(t))-\Phi(u_n^k)\geqslant\left(f_n^k-\frac{u_n^k-u_n^{k-1}}{\varepsilon_n^k}, u(t)-u_n^k\right), \tag{32}$$

$$\Phi(v)-\Phi(u_n^k)\geqslant\left(f_n^k-\frac{u_n^k-u_n^{k-1}}{\varepsilon_n^k}, v-u_n^k\right). \tag{33}$$

From (33) we deduce

$$t_n^k\Phi(v)-\int_0^{t_n^k}\Phi(\tilde u_n)\,dt\geqslant\int_0^{t_n^k}(f_n, v-\tilde u_n)\,dt-\sum_{j=1}^k\left(u_n^j-u_n^{j-1}, v-u_n^j\right).$$

But $-(u_n^j-u_n^{j-1}, v-u_n^j)\geqslant\frac12|u_n^j-v|^2-\frac12|u_n^{j-1}-v|^2$, so we get

$$\int_0^{t_n^k}\Phi(\tilde u_n)\,dt\leqslant t_n^k\Phi(v)-\int_0^{t_n^k}(f_n, v-\tilde u_n)\,dt-\frac12\left|v-u_n^k\right|^2+\frac12\left|v-u_0\right|^2. \tag{34}$$

On the other hand we have from (32) (integrating over (t_n^j, t_n^{j+1}) and summing for $j=k,\dots,N$)

$$\int_{t_n^k}^T\Phi(\tilde u_n)\,dt\leqslant\int_{t_n^k}^T\Phi(u)\,dt-\int_{t_n^k}^T(f_n, u-\tilde u_n)\,dt+\int_{t_n^k}^T\left(\frac{du_n}{dt}, u-\tilde u_n\right)dt. \tag{35}$$

From (34) and (35) we get for $v\in\mathrm{Dom}(\Phi)$

$$\begin{aligned}\int_0^T\Phi(\tilde u_n)\,dt&\leqslant t_n^k\Phi(v)-\int_0^{t_n^k}(f_n, v-\tilde u_n)\,dt-\frac12\left|v-u_n^k\right|^2+\frac12\left|v-u_0\right|^2\\&\cdots+\int_{t_n^k}^T\Phi(u)\,dt-\int_{t_n^k}^T(f_n, u-\tilde u_n)\,dt+\int_{t_n^k}^T\left(\frac{du_n}{dt}, u-\tilde u_n\right)dt.\end{aligned} \tag{36}$$

For $\varepsilon>0$, as $\|P_n\|\to_{n\to+\infty}0$, there exists a sequence $\{t_n^{k_n}\}$ such that $t_n^{k_n}\underset{n\to+\infty}{\to}\varepsilon$. As $\{f_n\}$ and $\{du_n/dt\}$ remain bounded in $L^2(t_n^{k_n}, T; H)$ independently of n and as $u-\tilde u_n$ goes to 0 in $L^\infty(0, T; H)$ when n tends to $+\infty$, we deduce from (36)

$$\begin{aligned}\limsup_{n\to+\infty}\int_0^T\Phi(\tilde u_n)\,dt&\leqslant\varepsilon\Phi(v)-\int_0^\varepsilon(f, v-u)\,dt-\frac12\left|v-u(\varepsilon)\right|^2\\&\cdots+\frac12\left|v-u_0\right|^2+\int_\varepsilon^T\Phi(u)\,dt.\end{aligned} \tag{37}$$

As u is continuous, we deduce (31) in letting $\varepsilon\to0$.

Proof of Theorem 2. First we prove (29)(ii) by supposing $u_0\in\overline{\mathrm{Dom}(\partial\Phi)}$.

From (19), as in the proof of Theorem 1, we get

$$\int_0^T\left|\frac{du_n}{dt}\right|^2\eta_n\,dt+t_n^{N-1}\Phi(u_n^N)\leqslant\int_0^T\Phi(\tilde u_n)\,dt+\int_0^T\left(f_n, \frac{du_n}{dt}\right)\eta_n\,dt. \tag{38}$$

But we also have, obviously,

$$\int_0^T \left|\frac{du}{dt}\right|^2 t\,dt + T\Phi(u(T)) = \int_0^T \Phi(u)\,dt + \int_0^T \left(f, \frac{du}{dt}\right) t\,dt. \tag{39}$$

Moreover as $\lim_{n\to+\infty} t_n^{N-1} = T$, $T\Phi(u(T)) \leq \liminf_{n\to+\infty} t_n^{N-1}\Phi(u_n^N)$. Using the fact that $\lim_{n\to+\infty}\int_0^T (f_n, du_n/dt)\eta_n\,dt = \int_0^T (f, du/dt)t\,dt$ and Lemma 2, we deduce

$$\limsup_{n\to+\infty} \int_0^T \left|\frac{du_n}{dt}\right|^2 \eta_n\,dt \leq \int_0^T \left|\frac{du}{dt}\right|^2 t\,dt. \tag{40}$$

As $\{\sqrt{\eta_n}\, du_n/dt\}$ converges weakly in $L^2(0, T; H)$ to $\sqrt{t}\, du/dt$, we conclude that

$$\lim_{n\to+\infty} \int_0^T \left|\frac{du_n}{dt}\sqrt{\eta_n} - \frac{du}{dt}\sqrt{t}\right|^2 dt = 0.$$

But

$$\left\{\int_0^T \left|\frac{du_n}{dt} - \frac{du}{dt}\right|^2 \eta_n\,dt\right\}^{1/2}$$

$$\leq \left\{\int_0^T \left|\frac{du_n}{dt}\sqrt{\eta_n} - \frac{du}{dt}\sqrt{t}\right|^2 dt\right\}^{1/2} + \left\{\int_0^T \left|\frac{du}{dt}\right|^2 \left(\sqrt{t} - \sqrt{\eta_n}\right)^2 dt\right\}^{1/2}.$$

From Lebesgue's theorem and the estimate (2),

$$\lim_{n\to+\infty} \int_0^T \left|\frac{du}{dt}\right|^2 \left(\sqrt{t} - \sqrt{\eta_n}\right)^2 dt = 0,$$

so we get (29)(ii).

To prove (30), first we notice that $\{du_n/dt\}$ remains bounded in $L^2(0, T; H)$ when $u_0 \in \mathrm{Dom}(\Phi)$. Hence, from the maximality of $\partial\Phi$, it converges weakly in $L^2(0, T; H)$ to du/dt. As we have already seen in the proof of Theorem 1, we have

$$\int_0^T \left|\frac{du_n}{dt}\right|^2 dt \leq \Phi(u_0) + \int_0^T \left(f_n, \frac{du_n}{dt}\right) dt - \Phi(u_n^N),$$

so we get

$$\limsup_{n\to+\infty} \int_0^T \left|\frac{du_n}{dt}\right|^2 dt \leq \Phi(u_0) + \int_0^T \left(f, \frac{du}{dt}\right) dt - \Phi(u(T)). \tag{41}$$

But as $|du/dt|^2 = d(\Phi(u))/dt + (f, du/dt)$ a.e. on $(0, T)$ (cf. [4]), integrating on $(0, T)$ we obtain

$$\limsup_{n\to+\infty} \int_0^T \left|\frac{du_n}{dt}\right|^2 dt \leq \int_0^T \left|\frac{du}{dt}\right|^2 dt.$$

So we have (30), which ends our proof.

Remark 4. Using similar devices, we can obtain the convergence result (29) in replacing the initial data u_0 of $\{u_n^k\}$ by a sequence $\{u_{0,n}\}$ in $\overline{\mathrm{Dom}(\partial\Phi)}$ such that $u_{0,n} \to_{n\to+\infty} u_0$. Moreover if $u_{0,n} \in \mathrm{Dom}(\Phi)$ and $\Phi(u_{0,n}) \to \Phi(u_0)$, we have (30).

Remark 5. We can also obtain some results concerning the convergence of $\{\Phi(u_n)\}$ from Theorem 2. For example, if $u_0 \in \mathrm{Dom}(\Phi)$, we have

$$\lim_{n\to+\infty} \|\Phi(u_n) - \Phi(u)\|_{C([0,T])} = 0. \tag{42}$$

In order to prove this, we start from (17) and we have for $1 \leqslant k \leqslant N$,

$$\Phi(u_n^k) \leqslant \Phi(u_0) + \int_0^{t_n^k}\left(f_n, \frac{du_n}{dt}\right) - \int_0^{t_n^k}\left|\frac{du_n}{dt}\right|^2 dt. \tag{43}$$

For $t \in [0, T]$, as $\|P_n\| \to 0$, there exists a sequence $\{t_n^{k_n}\}$ such that $\lim t_n^{k_n} = t$ and $t_n^{k_n} > t$ so $\Phi(u_n^{k_n}) = \Phi(\tilde{u}_n(t))$ and

$$\limsup_{n\to+\infty} \Phi(\tilde{u}_n(t)) \leqslant \Phi(u_0) + \int_0^t\left(f, \frac{du}{dt}\right) dt - \int_0^t\left|\frac{du}{dt}\right|^2 dt. \tag{44}$$

But from [4]

$$\Phi(u_0) + \int_0^t\left(f, \frac{du}{dt}\right) dt - \int_0^t\left|\frac{du}{dt}\right|^2 dt = \Phi(u(t)) \leqslant \liminf_{n\to+\infty} \Phi(\tilde{u}_n(t));$$

hence $\lim_{n\to+\infty}\Phi(\tilde{u}_n(t)) = \Phi(u(t))$, and it is not difficult to see that this limit is uniform on $[0, T]$ and that $\tilde{u}_n$ can be replaced by u_n.

For $k \geqslant 0$ we set $\Omega_k = \{v \in H\colon\ \Phi(v) + |v|^2 \leqslant k\}$. Our second result is the following

THEOREM 3. *Suppose Ω_k is compact for any $k \geqslant 0$ and $u_0 \in \overline{\mathrm{Dom}(\partial\Phi)}$. If $\lim_{n\to+\infty}\|P_n\| = 0$ and if f_n converges weakly to f in $L^2(0, T; H)$, then*

$$\begin{cases} \text{(i)} & \lim_{n\to+\infty} \|u_n - u\|_{C([0,T];\,H)} = 0, \\ \text{(ii)} & \eta_n \dfrac{du_n}{dt} \rightharpoonup t\dfrac{du}{dt} \text{ weakly in } L^2(0, T; H) \text{ as } n \to +\infty. \end{cases} \tag{45}$$

Moreover if $u_0 \in \mathrm{Dom}(\Phi)$, then du_n/dt converges weakly to du/dt in $L^2(0, T; H)$.

Proof. First we suppose that $u_0 \in \mathrm{Dom}(\Phi)$. From (10) $\{du_n/dt\}$ remains bounded in $L^2(0, T; H)$, so we deduce from (18), with N replaced by k, that $\{\Phi(\tilde{u}_n(t))\}$ is bounded, uniformly with respect to t and n. From the convexity of the function Φ, it is the same with $\{\Phi(u_n(t))\}$, and, as $\{u_n(t)\}$ is bounded, the set $\{u_n(t)\}$ is relatively compact in H for any $t > 0$. From (10) the set $\{u_n\}$ is equicontinuous, so by Ascoli's theorem, it is relatively compact in $C([0, T]; H)$. Hence there exist a subsequence $\{u_{n'}\}$ and $\tilde{u} \in H^1(0, T; H)$ such that $u_{n'} \to_{n'\to+\infty} \tilde{u}$ in $C([0, T]; H)$ and $du_{n'}/dt \rightharpoonup_{n'\to+\infty} du/dt$ weakly in $L^2(0, T; H)$. Hence $\tilde{u}$ satisfies (28) and is equal to u, so instead of the subsequence $\{u_{n'}\}$ we can take $\{u_n\}$ in the previous convergences.

We suppose now that $u_0 \in \overline{\mathrm{Dom}(\partial\Phi)}$. There exists a sequence $\{u_{0,m}\}$ in $\mathrm{Dom}(\Phi)$ converging to u_0 as $m \to +\infty$. If $u_{n,m}$ is the continuous piecewise linear solution of (8) with initial data $u_{0,m}$ instead of u_0, then $\|u_n - u_{n,m}\|_{C([0,T];\,H)} \leqslant |u_0 - u_{0,m}|$. Hence $\{u_n\}$ converges to u in $C([0, T]; H)$. But as Lemma 2 remains valid we have (45)(ii).

Remark 6. Using classical estimates on convex functions, it is easy to check that $\Phi((I + \lambda\partial\Phi)^{-1}x)$ $(\lambda > 0)$ remains bounded when x is bounded. If Ω_k is compact for any $k \geqslant 0$, the resolvents $(I + \lambda\partial\Phi)^{-1}$ are compact operators, and then the semigroup $(S(t))_{t\geqslant 0}$ generated by $-\partial\Phi$ is compact for $t > 0$. In the following section we give an extension of Theorem 3 to a more general situation.

Remark 7. As the convergence of $\{du_n/dt\}$ in both Theorems 2 and 3 is obtained via weak compactness and lower semicontinuity arguments, it is clear that in the general case it is not possible to obtain any error estimate of interest (see (55), Lemma 4). However, such error estimates should exist in many applications.

Extension to More General Operators. Set A a maximal monotone operator of H and $(S(t))_{t\geqslant 0}$ the semigroup of contractions of $\overline{D(A)}$ generated by $-A$. Thanks to the maximal monotonicity of A we can construct a sequence $\{u_n^k\}$ with the relations

$$\begin{cases} \dfrac{u_n^k - u_n^{k-1}}{t_n^k - t_n^{k-1}} + Au_n^k \ni f_n^k, \qquad k = 1,2,\dots,N(n), \\ u_n^0 = u_0 \in \overline{D(A)} \quad \text{and} \quad 0 = t_n^0 < t_n^1 < \cdots < t_n^{N(n)} = T. \end{cases} \tag{46}$$

We define the functions f_n and u_n as before. If f is a given function of $L^1(0, T; H)$, we set u the weak solution (cf. [4]) of

$$\begin{cases} \dfrac{du}{dt} + Au \ni f \quad \text{on } [0, T], \\ u(0) = u_0 \in \overline{D(A)}, \end{cases} \tag{47}$$

which means that $u \in C([0, T]; H)$ satisfies $u(0) = u_0$ and

$$\begin{cases} \frac{1}{2}|u(t) - x|^2 - \frac{1}{2}|u(s) - x|^2 \leqslant \int_s^t (u(\tau) - x, f(\tau) - y)\, d\tau, \\ \text{for any } x \in D(A), y \in Ax \text{ and } 0 \leqslant s \leqslant t \leqslant T. \end{cases} \tag{48}$$

If h is defined on $(0, T) \times (0, T)$, we set

$$\begin{aligned} \|h\|^* = \inf\{ &\|g\|_{L^1(0,T)} + \|f\|_{L^1(0,T)}, f \text{ and } g \in L^1(0, T), \\ &|h(\sigma, \tau)| \leqslant f(\sigma) + g(\tau) \text{ a.e. on } (0, T) \times (0, T)\}, \end{aligned} \tag{49}$$

and we call W the completion of $C([0, T] \times [0, T])$ for $\| \ \|^*$ (cf. [7]).

THEOREM 4. *Suppose the operators $S(t)$ are compact for $t > 0$, the set of real valued functions $h_n(\tau, \sigma) = |f_n(\tau) - f_n(\sigma)|$ is relatively compact in W, $\lim_{n\to+\infty} \|P_n\| = 0$ and $\{f_n\}$ converges weakly to f in $L^1(0, T; H)$. Then $\{u_n\}$ converges to the solution u of* (47) *in $C([0, T]; H)$.*

Proof. We set v_n the weak solution of

$$\begin{cases} dv_n/dt + Av_n \ni f_n \quad \text{on } (0, T), \\ v_n(0) = u_0 \in \overline{D(A)}. \end{cases} \tag{50}$$

As the set $\{|f_n|\}$ is uniformly equi-integrable on $(0, T)$, we deduce with a slight modification of the proof of Theorem 2 of [2] that $\{v_n\}$ is relatively compact in $C([0, T], H)$, so there exist $\tilde{u} \in C([0, T]; H)$ and a subsequence $\{v_{n'}\}$ of $\{v_n\}$ such that $v_{n'} \to \tilde{u}$ in $C([0, T]; H)$. For any $x \in D(A)$, $y \in Ax$, and $0 \leqslant s \leqslant t \leqslant T$, we have

$$\frac{1}{2}|v_{n'}(t) - x|^2 - \frac{1}{2}|v_{n'}(s) - x|^2 \leqslant \int_s^t (v_{n'}(\tau) - x, f_{n'}(\tau) - y)\, d\tau. \tag{51}$$

Going to the limit in (51), we see that $\tilde{u}$ is a weak solution of (47), so $\tilde{u} = u$ and $\lim_{n\to+\infty} v_n = u$ in $C([0, T]; H)$.

At this point of the proof we need the following result:

LEMMA 4. *Under the hypotheses of Theorem* 4, $u_n - v_n$ *converges to* 0 *in* $C([0,T];\, H)$ *as* $n \to +\infty$.

Proof. That result is a consequence of Theorem 2.8 of [7]. We first suppose that $u_0 \in D(A)$, and we consider a partition $\tilde{P}_m$ which is a refinement of P_n: $\tilde{P}_m = \{0 = s_m^0 < s_m^1 < \cdots < s_m^M = T\}$, and we construct the sequence $\{\tilde{v}_m^j\}$ as follows

$$(52)\qquad \begin{cases} \dfrac{\tilde{v}_m^j - \tilde{v}_m^{j-1}}{s_m^j - s_m^{j-1}} + A\tilde{v}_m^j \ni \tilde{f}_m^j, & j = 1,\ldots,M,\\ \tilde{v}_m^0 = u_0, \quad \tilde{f}_m^j = f_n^k & \text{if } (s_m^{j-1}, s_m^j) \subset (t_n^{k-1}, t_n^k). \end{cases}$$

For $y \in Au_0$ we set $\omega_n(\tau - s) = \int_0^{|\tau-s|}(|f_n(\alpha)| + |y|)\,d\alpha$ and

$$(53)\qquad G(\omega_n, h_n)(s,\tau) = \omega_n(\tau - s) + \begin{cases} \displaystyle\int_0^s h_n(\alpha, \tau - s + \alpha)\,d\alpha & \text{if } \tau \geqslant s,\\ \displaystyle\int_0^\tau h_n(s - \tau + \alpha, \alpha)\,d\alpha & \text{if } s \geqslant \tau. \end{cases}$$

Set $H^{m,n}$ the piecewise constant function on $(0,T)\times(0,T)$ taking the value $|v_m^j - u_n^k|$ on $(s_m^{j-1}, s_m^j] \times (t_n^{k-1}, t_n^k]$. For a function h defined on $(0,T)\times(0,T)$, we call h_Δ the piecewise constant function taking the value $h(s_m^j, t_n^k)$ on $(s_m^{j-1}, s_n^j] \times (t_n^{k-1}, t_n^k]$.

If $\tilde{\omega} \in C^2([-T,T])$ and $\tilde{h} \in C^2([0,T]\times[0,T])$ with $\tilde{h}(0,0) = 0$, Theorem 2.8 of [7] gives

$$(54)\qquad \begin{aligned} &\|H^{m,n} - G(\omega_n, h_n)\|_{L^\infty((0,T)\times(0,T))}\\ &\quad\leqslant 2\|\omega_n - \tilde{\omega}\|_{L^\infty(-T,T)} + 2\|h_n - \tilde{h}\|^* + \|\tilde{h}_\Delta - \tilde{h}\|^*\\ &\qquad + 2\|P_n\|\left\{T\|\tilde{\omega}''\|_{L^\infty(-T,T)} + \|\tilde{\omega}'\|_{L^\infty(-T,T)} + (1-2T)^2\|\tilde{h}\|_{C^2([0,T]\times[0,T])}\right\}. \end{aligned}$$

Moreover $\|\tilde{h}_\Delta - \tilde{h}\|^* \leqslant T\|\tilde{h}_\Delta - \tilde{h}\|_{L^\infty((0,T)\times(0,T))}$ and as $\tilde{h}$ is continuous, that last quantity goes to 0 as $n \to +\infty$. When $m \to +\infty$, the step function $v_{n,m}$ defined from the sequence $\{\tilde{v}_n^j\}$ converges to v_n uniformly on $[0,T]$. As $G(\omega_n, h_n)(s,s) = 0$, we get

$$(55)\qquad \begin{aligned} \|u_n - v_n\|_{C([0,T];\,H)} &\leqslant 2\|\omega_n - \tilde{\omega}\|_{L^\infty(-T,T)} + 2\|h_n - \tilde{h}\|^* + \limsup_{m\to+\infty}\|\tilde{h}_\Delta - \tilde{h}\|^*\\ &\quad + 2\|P_n\|\left\{T\|\omega''\|_{L^\infty(-T,T)} + \|\omega'\|_{L^\infty(-T,T)}\right.\\ &\qquad\left. + (1-2T)^2\|h\|_{C^2([0,T]\times[0,T])}\right\}. \end{aligned}$$

As $\{|f_n|\}$ is uniformly equi-integrable on $(0,T)$, the functions ω_n are equicontinuous and uniformly bounded on $[-T,T]$, hence relatively compact in $C([-T,T])$. By hypothesis the functions h_n are relatively compact in W. So there exist $\bar{\omega} \in C([-T,T])$ and $\bar{h} \in W$ such that

$$(56)\qquad \limsup_{n\to+\infty}\|u_n - v_n\|_{C([0,T];\,H)} \leqslant 2\|\bar{\omega} - \tilde{\omega}\|_{L^\infty(-T,T)} + 2\|\bar{h} - \tilde{h}\|^*.$$

That last quantity can be made as small as we want, the density of the test functions $(\tilde{\omega}, \tilde{h})$ in $C([-T,T]) \times W$ being easy to prove; so $\lim_{n\to+\infty}\|u_n - v_n\|_{C([0,T];\,H)} = 0$.

If $u_0 \in \overline{D(A)}$, we consider a sequence $u_{0,m} \in D(A)$ such that $u_{0,m} \to_{m \to +\infty} u_0$, and we get as previously

$$\limsup_{n \to +\infty} \|u_n - v_n\|_{C([0,T]; H)} \leqslant 2\|u_0 - u_{0,m}\|,$$

which ends the proof.

End of the Proof of Theorem 4. We have

$$\|u_n - u\|_{C([0,T]; H)} \leqslant \|u_n - v_n\|_{C([0,T]; H)} + \|v_n - u\|_{C([0,T]; H)}.$$

From Lemma 4, $\lim_{n \to +\infty} \|v_n - u_n\|_{C([0,T]; H)} = 0$ and, from the first part of the proof of Theorem 4, $\lim_{n \to +\infty} \|v_n - u\|_{C([0,T]; H)} = 0$, which ends the proof.

Remark 8. Our result remains true if A is an m-accretive operator of some general Banach space X when we replace a weak solution by an integral solution (cf. [3]), under the assumption that the set $\{(u, f): u \in C([0, T]; X), f \in L^1(0, T; X), u$ is an integral solution of (47)$\}$ is closed in $C([0, T]; X) \times L^1(0, T; X)$-weak, in particular if X is uniformly convex. Without that assumption we just obtain the relative compactness of the $\{u_n\}$ in $C([0, T]; X)$.

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