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Rational Chebyshev Approximations for the Bessel Functions $J_0(x)$, $J_1(x)$, $Y_0(x)$, $Y_1(x)$*

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Abstract. This report presents near-minimax rational approximations for the Bessel functions $J_0(x)$, $J_1(x)$, $Y_0(x)$, and $Y_1(x)$ for the complete range of x, with relative errors ranging down to 10^{-23}. The first thirty zeros of each function are listed to 35D. The tabulated zeros and the McMahon asymptotic formulae may be used to construct an algorithm which retains relative accuracy in the neighborhood of zeros.

1. Introduction. The Bessel functions of the first and second kinds $J_0(x)$, $J_1(x)$, $Y_0(x)$, and $Y_1(x)$ are among the most commonly used functions in scientific computations, and efficient approximations have wide applicability.

Chebyshev series expansion coefficients are given to 20D in [3] for the ranges $|x| \leqslant 8$ and $x \geqslant 8$; in [10] for the ranges $|x| \leqslant 8$ and $x \geqslant 5$; and to 15D in [14] for $|x| \leqslant 5$. [7] contains rational minimax approximations for $|x| \leqslant 8$ and $x \geqslant 8$ with absolute errors ranging down to 10^{-25}. [4] contains unpublished rational minimax approximations to $J_0(x)$ and $J_1(x)$ for $|x| \leqslant 4$ and $4 \leqslant x \leqslant 8$, and to $Y_\nu(x)$ for $0 \leqslant \nu \leqslant 1$ [5]. A number of other approximations are listed in [10].

The present report gives rational minimax approximations to $J_0(x)$, $J_1(x)$, $Y_0(x)$, and $Y_1(x)$ for small values of x, and to the modulus and phase for larger arguments. An advantage of using the modulus and phase is that fewer function evaluations are necessary. For example, the computation of $J_0(x)$ for $x \geqslant 8$ by the formulae in [7] involves the computation of $P(x)$, $Q(x)$, $\sin \chi$ and $\cos \chi$, whereas the modulus-phase approach involves the computation of $M(x)$, $\theta(x)$ and $\cos \theta$.

Following Cody [4] we have factored the zeros of the function out of the rational approximation wherever possible, thereby providing a form which retains relative accuracy in the neighborhood of zeros. For the remaining ranges, accurate zeros are either listed in the report or are obtainable by the McMahon asymptotic formulae, and a Taylor expansion about each zero provides full relative accuracy in a neighborhood containing the zero. Thus the methods described here provide $J_0(x)$, $J_1(x)$, $Y_0(x)$, and $Y_1(x)$ with full relative accuracy for the complete range of the argument.

2. Functional Properties. $J_n(x)$ and $Y_n(x)$ are linearly independent solutions of the differential equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0.$$

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Ascending series, given in [1], are

(1) $$J_n(x) = (x/2)^n \sum_{k=0}^{\infty} \frac{(-x^2/4)^k}{k!\,(n+k)!},$$

(2) $$Y_n(x) = -\frac{(x/2)^{-n}}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}(x^2/4)^k + \frac{2}{\pi}\ln(x/2)J_n(x) - \frac{(x/2)^n}{\pi} \sum_{k=0}^{\infty} [\psi(k+1) + \psi(n+k+1)] \frac{(-x^2/4)^k}{k!(n+k)!},$$

where $\psi(n) = -\gamma + \sum_{k=1}^{n-1} k^{-1}$ and γ is Euler's constant.

The modulus and phase are defined in [1] as

(3) $$M_n(x) = \left[J_n^2(x) + Y_n^2(x)\right]^{1/2},$$

(4) $$\theta_n(x) = \arctan\left[Y_n(x)/J_n(x)\right],$$

so that $J_n(x) = M_n \cos\theta_n$ and $Y_n(x) = M_n \sin\theta_n$.

An asymptotic formula for M_n for large argument is given in [1];

(5) $$M_n^2 \sim \frac{2}{\pi x}\left(1 + \sum_{k=1}^{\infty} a_k x^{-2k}\right),$$

where

$$a_k = \frac{1\cdot 3\cdot 5\cdots(2k-1)}{2\cdot 4\cdot 6\cdots 2k}\,\frac{(\mu-1^2)(\mu-3^2)\cdots\left[\mu-(2k-1)^2\right]}{2^{2k}} \quad \text{and}$$

$\mu = 4n^2$.

The corresponding asymptotic formula for θ_n is

(6) $$\theta_n \sim x - (n/2 + 1/4)\pi + \sum_{k=1}^{\infty} b_k x^{-2k+1}.$$

An explicit formula for b_k is not known, but a recursive formula can be developed by substituting (5) and (6) in the identity $M_n^2\theta_n' = 2/(\pi x)$.

The resulting formula is

(7) $$b_k = \left[a_k - \sum_{n=1}^{k-1} (2n-1)a_{k-n}b_n\right] / (2k-1).$$

If $\mathcal{C}$ denotes either J or Y, the derivatives with respect to x are given by

(8) $$\mathcal{C}_0' = -\mathcal{C}_1, \qquad x\mathcal{C}_1' = x\mathcal{C}_0 - \mathcal{C}_1.$$

Repeated differentiation gives the following formulae for the kth derivatives

(9) $$\mathcal{C}_0^{(k)} = -\mathcal{C}_1^{(k-1)}, \qquad \mathcal{C}_1^{(k)} = -\frac{k}{x}\mathcal{C}_1^{(k-1)} + \mathcal{C}_0^{(k-1)} + \frac{(k-1)}{x}\mathcal{C}_0^{(k-2)},$$

for $k = 1, 2, 3, \ldots$.

The Wronskian

(10) $$J_1(x)Y_0(x) - J_0(x)Y_1(x) = 2/(\pi x)$$

is often used for checking computations.

The McMahon asymptotic expansions for the zeros of $J_n(x)$ and $Y_n(x)$ are [1]

$$(11) \qquad j_{n,s},\, y_{n,s} \sim \beta + \sum_{k=1}^{\infty} c_k \beta^{-2k+1},$$

where $\beta = (s + n/2 - 1/4)\pi$ for $j_{n,s}$, $\beta = (s + n/2 - 3/4)\pi$ for $y_{n,s}$.

Expressions for $c_1, c_2, \ldots, c_7$ are listed in [13], but a general formula for c_k is not known. The c_k may be computed in terms of the b_k by the following algorithm

$$(12) \qquad \begin{aligned} c_k = -b_k - \sum_{l=1}^{k-1} b_{k-l} \\ \times \sum_{i=1}^{l} \frac{(-2k+2l+1)(-2k+2l)\cdots(-2k+2l+2-i)}{i!} c_{i,l}, \end{aligned}$$

for $k = 1, 2, 3, \ldots,$ where $c_{1,k} \equiv c_k$ and

$$c_{k+1,l} = \sum_{i=1}^{l-k} c_i c_{k,l-i}, \qquad l = k+1, k+2, k+3, \ldots, k = 1, 2, 3, \ldots.$$

(7) and (12) are particular cases of the algorithm in [6].

The derivatives of J_n and Y_n at the zeros may be computed from the formulae

$$J_n'(j_{n,s}) = (-1)^s 2/[\pi j_{n,s} M_n(j_{n,s})],$$
$$Y_n'(y_{n,s}) = (-1)^{s-1} 2/[\pi y_{n,s} M_n(y_{n,s})],$$

and (9).

3. Generation of Approximations. Rational minimax approximations to $J_n(x)$, $Y_n(x)$, $M_n(x)$, and $\theta_n(x)$ were computed in 29S arithmetic on a CDC 6600/CYBER 170 Model 175 computer system using a version of the second algorithm of Remes due to Ralston [9]. The error of the approximations was levelled to three digits in most cases.

The approximation forms and intervals are:

$$\begin{aligned}
J_0(x) &\simeq (x^2 - j_{0,1}^2)(x^2 - j_{0,2}^2)(x^2 - j_{0,3}^2)(x^2 - j_{0,4}^2) R_{lm}(x^2), \qquad 0 \leqslant x \leqslant 14, \\
&= M_0(x) \cos\theta_0(x), \qquad x \geqslant 14, \\
Y_0(x) &\simeq (x^2 - j_{0,1}^2) \ln\frac{x}{y_{0,1}} R_{lm}(x^2) + (x^2 - y_{0,1}^2) S_{lm}(x^2), \qquad 0 < x \leqslant 3.5, \\
&= M_0(x) \sin\theta_0(x), \qquad x \geqslant 3.5, \\
M_0(x) &\simeq x^{-1/2} R_{lm}(1/x^2), \qquad 3.5 \leqslant x \leqslant 14 \text{ and } x \geqslant 14, \\
\theta_0(x) &\simeq x - \frac{\pi}{4} + x^{-1} S_{lm}(1/x^2), \qquad 3.5 \leqslant x \leqslant 14 \text{ and } x \geqslant 14, \qquad (13) \\
J_1(x) &\simeq x(x^2 - j_{1,1}^2)(x^2 - j_{1,2}^2)(x^2 - j_{1,3}^2)(x^2 - j_{1,4}^2) R_{lm}(x^2), \qquad 0 \leqslant x \leqslant 14, \\
&= M_1(x) \cos\theta_1(x), \qquad x \geqslant 14, \\
Y_1(x) &\simeq x(x^2 - j_{1,1}^2) \ln\frac{x}{y_{1,1}} R_{lm}(x^2) + \frac{(x^2 - y_{1,1}^2)}{x y_{1,1}^2} S_{lm}(x^2), \qquad 0 < x \leqslant 5, \\
&= M_1(x) \sin\theta_1(x), \qquad x \geqslant 5, \\
M_1(x) &\simeq x^{-1/2} R_{lm}(1/x^2), \qquad 5 \leqslant x \leqslant 14 \text{ and } x \geqslant 14, \\
\theta_1(x) &\simeq x - \frac{3\pi}{4} + x^{-1} S_{lm}(1/x^2), \qquad 5 \leqslant x \leqslant 14 \text{ and } x \geqslant 14,
\end{aligned}$$

where $R_{lm}(x)$ and $S_{lm}(x)$ are rational functions of degree l in the numerator and m in the denominator. The above forms are the most efficient of a number of different alternatives that were tested.

For the low range of x an approximation to $J_n(x)$ of the form

$$x^n \prod_{i=1}^{4} (x - j_{n,i}) R_{lm}(x)$$

was tried, but was discarded because of slow convergence in the Walsh array.

In the approximation to $Y_n(x)$ in the low range, the two terms in the formula cancel more and more as $x \to y_{n,2}$, and the breakpoint between the two lower ranges had to be somewhat smaller than $y_{n,2}$. For the value chosen, the maximum cancellation is between one and two bits for $Y_0(x)$ and about one bit for $Y_1(x)$. Other forms of approximation to $Y_n(x)$ in the ranges [3.5, 14] and [5, 14], involving the factors $(x - y_{n,i})$, were tested but were discarded because of slow convergence in the Walsh array.

In addition to the approximation to $M_n(x)$ on the previous page, we tested the form $[x^{-1}R_{lm}(1/x^2)]^{1/2}$, since both forms involve the same amount of computation. The latter form proved to be less accurate, for a given degree, and was discarded.

For the rational functions $R_{lm}(x)$ and $S_{lm}(x)$ the relative error of the approximation was minimized.

The master routines are based on the ascending series (1) and (2) for $0 \leqslant x \leqslant 6$, and on the asymptotic series (5) and (6) for $x \geqslant 40$. For the intermediate range $6 \leqslant x \leqslant 40$, $J_n(x)$ and $Y_n(x)$ are computed by Taylor series expansions of the form

$$\mathcal{C}_n(x_0 + h) = \mathcal{C}_n(x_0) + \sum_{m=1}^{N} \frac{h^m}{m!} \mathcal{C}_n^{(m)}(x_0), \tag{14}$$

where $\mathcal{C}_n(x_0)$ is the closest of a set of reference values, and where the derivatives $\mathcal{C}_n^{(m)}(x_0)$ are computed by (9). The table of reference values is constructed by using the Hankel asymptotic expansion at $x = 40$, and then using (14) repeatedly with negative values of h. For $x < 40$, M_n and θ_n are computed directly from (3) and (4).

The zeros $j_{n,1}, j_{n,2}, j_{n,3}, j_{n,4}$, and $y_{n,1}$ were obtained from [11].

For the lowest range of the argument each auxiliary function is computed from the corresponding value of J_n or Y_n if the argument is not close to a zero, and, if the argument is close to a zero, from a Taylor expansion about that zero.

The accuracy of the master routines was established by comparison with values in [8] and [7], with values computed by Brent's multiple-precision package [2], with values computed at the range boundaries, and by differencing. We conclude that the master routines are accurate to at least 26D for $0 < x \leqslant 14$, and to at least 26S for $x \geqslant 14$.

As a check of the zeros $j_{0,s}$, $j_{1,s}$, $y_{0,s}$, and $y_{1,s}$ in [11], we recomputed them with Brent's MP package and found agreement to all digits quoted in [11]. The formulae

$$\sum_{s=1}^{\infty} 1/[j_{0,s} J_1(j_{0,s})] = 1/2$$

and

$$\sum_{s=1}^{\infty} 1/[j_{1,s}^2 J_0(j_{1,s})] = -1/8,$$

combined with the Euler transformation applied to the $j_{0,s}$ and $j_{1,s}$ for $s = 1,2,\ldots,50$, gave agreement to 35D.

$y_{0,s}$ and $y_{1,s}$, $s = 1,2,\ldots,50$, were substituted in the Wronskian (10), and gave agreement to at least 40D.

The tests also indicate that the McMahon expansions (11) are accurate to 29S for $j_{0,11}, y_{0,12}, j_{1,12}, y_{1,11}$ and larger zeros.

4. Results. The details of the approximations are given in Tables 1–146, in a format similar to that used in [7]. Tables 1–14 summarize the best approximations in the L_∞ Walsh arrays of the functions, and Tables 19–146 give the coefficients of selected approximations. The precision is defined as

$$-\log_{10} \max_x \left| \frac{f(x) - R_{lm}(x)}{f(x)} \right|,$$

where $f(x)$ is the function being approximated, and the maximum is taken over the appropriate interval. Tables 1–146 are in the microfiche supplement attached to the end of this issue.

For completeness we have also listed in Tables 15–18 the zeros $j_{n,s}$ and $y_{n,s}$ to 35D for $s = 1,2,\ldots,30$.

For the lowest range of each function the rational approximations are ill-conditioned, those pertaining to $J_0(x)$ and $J_1(x)$ losing up to four significant digits by cancellation, and those pertaining to $Y_0(x)$ and $Y_1(x)$ about one digit. To eliminate the cancellation each numerator was converted to minimal Newton form [12], and the resulting coefficients rounded off by an algorithm similar to that used in [7]. The cancellation also necessitated a modification to the Remes algorithm for certain cases. In particular, the error curve for the last entry in Tables 1 and 8 was levelled to only one digit.

The approximations in Tables 19–146 were verified by comparing them with the master routine for 5000 pseudo random values of the argument in each interval. The resulting precision agreed to three digits with the computed value in the Walsh array, even for the cases in which the error curve was levelled to one digit. In addition J_n and Y_n values computed by (13) were compared with values in [8] and with values computed by the MP package, and in all cases the agreement was as expected.

5. Design of a Subroutine. For the low range of x relative accuracy may be retained in the computation of J_n and Y_n if the terms $(x^2 - c^2)$ and $\ln x/c$ in (13) are evaluated carefully, where c denotes a zero.

In the formula for J_n, let c denote the zero closest to x. Then the difference $(x^2 - c^2)$ should be computed as $(x - c)(x + c)$, and the factor $(x - c)$ computed in double-precision arithmetic, with c accurate to double precision. All other operations may be performed in single precision.

In the formula for Y_n, $(x^2 - c^2)$ should be evaluated in the same manner, where c denotes the closer of $j_{n,1}$ and $y_{n,1}$. If x is close to $y_{n,1}$, $\ln x/y_{n,1}$ may be computed accurately from the series expansion of $\ln[1 + (x - y_{n,1})/y_{n,1}]$. Again, $x - y_{n,1}$ should be computed in double-precision arithmetic.

For the remaining ranges of x, the formulae (13) are not capable of retaining complete accuracy in the neighborhood of zeros, and an alternative formulation is necessary there. We can assess the accuracy attainable with (13) by the following approximate analysis, which pertains to a binary computer with t binary digits in the mantissa of floating-point numbers.

Consider the evaluation of $Y_0(x) = M_0(x)\sin\theta_0(x)$, where

$$\theta_0(x) \simeq x - \pi/4 + x^{-1}S(1/x^2).$$

If x is reduced to the range $[0, \pi)$ by a double-precision range reduction, then the error in the reduced argument is $O(2^{-2t+1}x)$. If the subtraction of $\pi/4$ is again done in double precision, the error in the resulting value is $O(2^{-2t+1}x)$. $x^{-1}S(1/x^2)$ may now be added in single precision, introducing a further error $O(2^{-t-2}/x)$, if $S(1/x^2)$ is accurate to approximately single precision. Thus the total error in the reduced argument $\theta_0(x)$ is $O(2^{-2t+1}x) + O(2^{-t-2}/x)$, and, since the reduced $\theta_0(x)$ is approximately zero, the error in $\sin\theta_0(x)$ is $O(2^{-2t+1}x) + O(2^{-t-2}/x)$.

Let $\delta x = x - c$ be the minimum difference between x and the closest zero c of $Y_0(x)$ for which $\sin\theta_0(x)$, as computed above, gives single-precision accuracy. For this argument x, $\sin\theta_0(x) \simeq \pm\sin\delta x \simeq \pm\delta x$ (assuming $|\delta x| < 1$), and so, by the definition of δx,

$$\left|\frac{2^{-t-2}}{x\,\delta x}\right| = O(2^{-t}), \qquad \therefore |\delta x| = O(1/4x).$$

Thus, if $|\delta x| > O(1/4x)$, $\sin\theta_0(x)$ is accurate to single precision. Now, for a general argument x, the minimum δx possible for a single precision argument is $O(2^{-t+1}x)$, and so it follows that, for $x > O(2^{t/2})$, the above algorithm gives single-precision accuracy at zeros.

We have shown, therefore, that if $x > O(2^{t/2})$, or if $x < O(2^{t/2})$ and $|x - c| > O(1/4x)$, then $\sin\theta_0(x)$ as evaluated above, and hence $Y_0(x)$, is accurate to single precision.

For $x < O(2^{t/2})$ and $|x - c| < O(1/4x)$, $J_n(x)$ and $Y_n(x)$ can be computed accurately by using a Taylor expansion about c. Let $s = [x/\pi - n/2 + 1/4]$ or $s = [x/\pi - n/2 + 3/4]$ in the case of $J_n(x)$ and $Y_n(x)$, respectively (where [ ] denotes "integral part"), and compute

$$\beta = (s + n/2 - 1/4)\pi \quad \text{or} \quad \beta = (s + n/2 - 3/4)\pi.$$

The expansion should be based on $j_{n,s}$ or $y_{n,s}$ if $x - (\beta \pm c_1/\beta) < O(1/4x)$, and $j_{n,s+1}$ or $y_{n,s+1}$ if $x - (\beta + c_1/\beta) > \pi - O(1/4x)$. A typical expansion is

$$J_n(x) = hJ_n'(j_{n,s}) + h^2J_n''(j_{n,s})/2! + h^3J_n'''(j_{n,s})/3! + \cdots,$$

where $h = x - j_{n,s} \cdot h$ should be computed in double-precision arithmetic, with $j_{n,s}$ accurate to double precision, but all other operations may be performed in single precision.

Double-precision values of $j_{n,s}$ and $y_{n,s}$ may be obtained from a table of values, or computed by (11).

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1. M. ABRAMOWITZ & I. A. STEGUN (Editors), *Handbook of Mathematical Functions*, Nat. Bur. Standards, Appl. Math. Series No. 55, U. S. Government Printing Office, Washington, D. C., 1965.

2. R. P. BRENT, "A FORTRAN Multiple-Precision Arithmetic Package," *ACM Trans. Math. Software*, v. 4, 1978, pp. 57–70.

3. C W. CLENSHAW & SUSAN M. PICKEN, *Chebyshev Series for Bessel Functions of Fractional Order*, Nat. Phys. Lab. Math. Tables, v. 8, Her Majesty's Stationery Office, London, 1956.

4. W. J. CODY, "The FUNPACK Package of Special Function Subroutines," *ACM Trans. Math. Software*, v. 1, 1975, pp. 13–25.

5. W. J. CODY, R. M. MOTLEY & L. W. FULLERTON, "Coefficients for the approximation of $Y_\nu(x)$," *AMD Technical Memorandum* #284, Argonne National Laboratory. (In preparation.)

6. D. DÖRING, "Über die McMahon-Entwicklungen," *Z. Angew. Math. Phys.*, v. 18, 1967, pp. 461–473.

7. J. F. HART, ET AL., *Computer Approximations*, Wiley, New York, 1968.

8. K. HAYASHI, *Tafeln der Besselschen, Theta, Kugel—und anderer Funktionen*, Springer-Verlag, Berlin, 1930.

9. J. H. JOHNSON & J. M. BLAIR, *REMES2: A FORTRAN Program to Calculate Rational Minimax Approximations to a Given Function*, Atomic Energy of Canada Limited, Report AECL-4210, Chalk River, Ontario, 1973.

10. Y. L. LUKE, *Mathematical Functions and Their Approximations*, Academic Press, New York, 1975.

11. S. MAKINOUCHI, *Zeros of Bessel Functions $J_\nu(x)$ and $Y_\nu(x)$ Accurate to Twenty-Nine Significant Digits*, Osaka University Technology Report No. 685, 1965.

12. C. MESZTENYI & C. WITZGALL, "Stable evaluation of polynomials," *J. Res. Nat. Bur. Standards Sect. B*, v. 71B, 1967, pp. 11–17.

13. F. W. J. OLVER (Ed.), *Bessel Functions, Part* III. *Zeros and Associated Values*, Royal Society Mathematical Tables, v. 7, Cambridge Univ. Press, Cambridge, 1960.

14. J. WIMP, "Polynomial expansions of Bessel functions and some associated functions," *Math. Comp.*, v. 16, 1962, pp. 446–458.