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Closed Expressions for $\int_0^1 t^{-1} \log^{n-1} t \log^p(1-t)\,dt$

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Abstract. Closed expressions for the integral $\int_0^1 t^{-1} \log^{n-1} t \log^p(1-t)\,dt$, whose general form is given elsewhere, are listed for $n = 1(1)9$, $p = 1(1)9$. A formula is derived which allows an easy evaluation of these expressions by formula manipulation on a computer.

1. Introduction. At the beginning of this century, Nielsen discussed, in a little-known monograph [9], properties of a family of functions

$$S_{n,p}(x) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 t^{-1} \log^{n-1} t \log^p(1-xt)\,dt \tag{1}$$

for positive integers n, p, and complex x. These functions include many special cases such as Euler's dilogarithm, Kummer's trilogarithm, the Spence functions and polylogarithms. As already proposed [4], it seems appropriate to call the family (1) Nielsen's generalized polylogarithms.

Although the monograph [9] contains quite a number of misprints and a few erroneous results, it does present a considerable amount of useful information, in particular transformation formulae relating $S_{n,p}(x)$ to $S_{n,p}(1/x)$ and $S_{n,p}(1-x)$. It is remarkable that these formulae, and consequently also those for $S_{n,p}(1/(1-x))$, $S_{n,p}((x-1)/x)$, and $S_{n,p}(x/(x-1))$ contain, apart from logarithms and constants, only functions $S_{\nu,\pi}(x)$. However, as far as the author knows, the important formulae of [9] have never found their way into any of the relevant handbooks.

Interest in these functions revived some time ago, at least for the case $p = 1$, in the context of multi-dimensional integration of rational functions in quantum electrodynamics (see, for example, [1], [8]). Their properties are also of interest in group theory and geometry [7]. The book of Lewin [6] gives many formulae and properties of $S_{n,1}(x)$. A general discussion of Nielsen's monograph is given in [4].

2. The Values $s_{n,p} = S_{n,p}(1)$. The purpose of this note is to give explicit expressions for the special values

$$s_{n,p} = S_{n,p}(1) = \frac{(-1)^{n+p-1}}{(n-1)!p!} \int_0^1 t^{-1} \log^{n-1} t \log^p(1-t)\,dt, \tag{2}$$

at least for some n and p. It is easy to show that $s_{n,p} = s_{p,n}$, and hence we can restrict p to $n \geqslant p$. A closed expression for $s_{n,p}$ is given in [4] (in implicit form also in

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[6]), which reads

$$s_{n,p} = \sum_{k=1}^{p} \frac{(-1)^{k+1}}{k!} \sum_{m_i} \frac{H_p(m_1,\ldots,m_k)}{m_1 \cdots m_k} \zeta(m_1) \cdots \zeta(m_k), \tag{3}$$

where

$$H_p(m_1,\ldots,m_k) = \sum_{p_i} \binom{m_1}{p_1} \cdots \binom{m_k}{p_k}. \tag{4}$$

The sum over m_i is to be taken over all sets of integers $\{m_i\}$ $(i = 1,\ldots,k)$ which satisfy

$$m_i \geqslant 2, \quad \sum_{i=1}^{k} m_i = n + p, \tag{5}$$

and the sum over p_i over all sets of integers $\{p_i\}$ $(i = 1,\ldots,k)$ which satisfy

$$1 \leqslant p_i \leqslant m_i - 1, \quad \sum_{i=1}^{k} p_i = p. \tag{6}$$

The function

$$\zeta(m) = \sum_{k=1}^{\infty} k^{-m} \tag{7}$$

is the Riemann zeta function for integer argument. Nielsen remarked that the functions $S_{n,p}(x)$ are probably the simplest analytic functions which coincide with $\zeta(m)$ for special values of its arguments. He added that he was not able to use his theory of $S_{n,p}(x)$ to find expressions for $\zeta(2\mu + 1)$ analogous to the known expressions for $\zeta(2\mu)$.

Nielsen [9] formulated a theorem about the structure of $s_{n,p}$ and gave the principle of the proof. He also calculated the cases $p \leqslant 3$. The case $p = 1$ is trivial, giving

$$\begin{aligned} &\int_0^1 t^{-1} \log^{n-1} t \log(1-t)\, dt \\ &\qquad = (-1)^n (n-1)! s_{n,1} = (-1)^n (n-1)! \zeta(n+1). \end{aligned} \tag{8}$$

The case $p = 2$ can also be handled easily, but the $k = 3$ term in (3) for $p = 3$ is somewhat more involved, and Nielsen's final expression [9, Section 18 (19), (20)] is incorrect. However, the expression for s_{73} given as an example in [9] differs from the correct expression only by a difference in the coefficient of $\zeta(2)\zeta^2(4)$ ($\frac{1}{3}$ instead of $\frac{1}{2}$), and this could be due to a misprint.

Writing (3) as

$$s_{n,p} = \sum_{k=1}^{p} \frac{(-1)^{k+1}}{k!} \alpha_k(n, p), \tag{9}$$

it is easy to find from (4) the following expressions for $\alpha_k(n, p)$ in the case of some special values of p and k:

$$\alpha_1(n, p) = \frac{(n+p-1)!}{n!p!} \zeta(n+p), \tag{10}$$

$$\alpha_2(n,2)=\sum_{\nu=2}^{n}\zeta(\nu)\zeta(n-\nu+2), \tag{11}$$

$$\alpha_n(n,n)=\zeta^n(2). \tag{12}$$

For $k=2, p=3$, we have from (4), for $\nu=2,\dots,n+1$,

$$H_3(n-\nu+3,\nu)=\varepsilon_{\nu,2}\binom{n-\nu+3}{1}\binom{\nu}{2}+\varepsilon_{\nu,n+1}\binom{n-\nu+3}{2}\binom{\nu}{1}$$

and

$$\frac{H_3(n-\nu+3,\nu)}{(n-\nu+3)\nu}=\begin{cases}\frac{1}{2}n & \text{if } \nu=2,\ \nu=n+1,\\ \frac{1}{2}(n+1) & \text{if } \nu=3,\dots,n,\end{cases} \tag{13}$$

where $\varepsilon_{\nu,\mu}=0$ for $\nu=\mu$ and $\varepsilon_{\nu,\mu}=1$ for $\nu\neq\mu$, so that

$$\begin{aligned}\alpha_2(n,3)&=n\zeta(2)\zeta(n+1)+\frac{1}{2}(n+1)\sum_{\nu=3}^{n}\zeta(\nu)\zeta(n-\nu+3)\\ &=\sum_{\nu=2}^{n}(n-\nu+2)\zeta(\nu)\zeta(n-\nu+3).\end{aligned} \tag{14}$$

In the case $k=2, p=4$, one finds for $\nu=2,\dots,n+2$,

$$\begin{aligned}H_4(n-\nu+4,\nu)=\varepsilon_{\nu,2}\varepsilon_{\nu,3}\binom{n-\nu+4}{1}\binom{\nu}{3}+\varepsilon_{\nu,2}\varepsilon_{\nu,n+2}\binom{n-\nu+4}{2}\binom{\nu}{2}\\ +\varepsilon_{\nu,n+1}\varepsilon_{\nu,n+2}\binom{n-\nu+4}{3}\binom{\nu}{1}.\end{aligned}$$

Thus

$$\begin{aligned}&\frac{H_4(n-\nu+4,\nu)}{(n-\nu+4)\nu}\\ &\quad=\begin{cases}\frac{1}{6}n(n+1) & \text{if } \nu=2,\ \nu=n+2,\\ \frac{1}{6}n(n+2) & \text{if } \nu=3,\ \nu=n+1,\\ \frac{1}{12}\left[\nu^2-(n+4)\nu+2n^2+7n+7\right] & \text{if } \nu=4,\dots,n,\end{cases}\end{aligned} \tag{15}$$

and therefore

$$\begin{aligned}\alpha_2(n,4)&=\frac{1}{3}n(n+1)\zeta(2)\zeta(n+2)+\frac{1}{3}n(n+2)\zeta(3)\zeta(n+1)\\ &\quad+\frac{1}{12}\sum_{\nu=4}^{n}\left[\nu^2-(n+4)\nu+2n^2+7n+7\right]\zeta(\nu)\zeta(n-\nu+4).\end{aligned} \tag{16}$$

For larger values of p, $\alpha_2(n,p)$ becomes more and more complicated.

For $k=p=3$, we see that $p_1=p_2=p_3=1$ and $H_3(m_1,m_2,m_3)=m_1m_2m_3$. The sum over m_i in (3) therefore equals the sum over the products $\zeta(m_1)\zeta(m_2)\zeta(m_3)$ for all partitions $\{m_1,m_2,m_3\}$ of $n+3$ satisfying $2\leqslant m_i\leqslant[(n+3)/3]$, with a weight for possible permutations, where $[\xi]$ denotes the integer part of ξ. This leads to

$$\alpha_3(n,3)=\sum_{\mu=2}^{\mu^*}\zeta(\mu)\sum_{\nu=\mu}^{\nu^*}\omega(n;\mu,\nu)\zeta(\nu)\zeta(n+3-\nu-\mu), \tag{17}$$

where $\mu^* = [(n+3)/3]$, $\nu^* = [(n-\mu+3)/2]$, and

$$\omega(n;\mu,\nu) = \begin{cases} 1 & \text{if } \mu=\nu \text{ and } 3\mu = n+3, \\ 3 & \text{if } \mu=\nu \text{ and } 3\mu \neq n+3 \text{ or} \\ & \text{if } \mu \neq \nu \text{ and } 2\mu+\nu = n+3 \text{ or} \\ & \text{if } \mu \neq \nu \text{ and } \mu + 2\nu = n+3 \\ 6 & \text{otherwise.} \end{cases} \tag{18}$$

From (1), (10), and (11) it follows that

$$\begin{aligned} \int_0^1 t^{-1}\log^{n-1} t \log^2(1-t)\,dt &= 2(-1)^{n-1}(n-1)!s_{n,2} \\ &= (-1)^{n-1}(n-1)!\left[(n+1)\zeta(n+2) - \sum_{\nu=2}^{n}\zeta(\nu)\zeta(n-\nu+2)\right], \end{aligned} \tag{19}$$

and from (10), (14), and (17),

$$\begin{aligned} \int_0^1 t^{-1}\log^{n-1} t \log^3(1-t)\,dt &= 6(-1)^{n}(n-1)!s_{n,3} \\ &= (-1)^{n}(n-1)!\Bigg[(n+1)(n+2)\zeta(n+3) \\ &\qquad -3\sum_{\nu=2}^{n}(n-\nu+2)\zeta(\nu)\zeta(n-\nu+3) \\ &\qquad +\sum_{\mu=2}^{\mu^*}\zeta(\mu)\sum_{\nu=\mu}^{\nu^*}\omega(n;\mu,\nu)\zeta(\nu)\zeta(n+3-\nu-\mu)\Bigg]. \end{aligned} \tag{20}$$

This last formula corrects formula [9, Section 18 (19)] of Nielsen.

For arbitrary n and p, it is obvious that (3) can, in practice, be evaluated only by means of a computer. Even then, the problem is complicated. The main task consists in constructing the sets $\{m_i\}$ and $\{p_i\}$. Because of the fact that all permutations have to be taken into account, the number of these sets grows rapidly with increasing values of $n+p$. We have constructed these sets up to $n=p=9$ by means of a FORTRAN program. As an example, their number is shown for $n=p=9$ in Table 1. Therefore $\sum\{m_i\}\{p_i\} = 85376$ sets would have to be analyzed in this case. Because of the condition $1 \leqslant p_i \leqslant m_i - 1$, only 12870 of these would contribute to the 88 different terms in the result (3) for $s_{9,9}$.

TABLE 1

k	1	2	3	4	5	6	7	8	9
$\{m_i\}$	1	15	91	286	495	462	210	36	1
$\{p_i\}$	1	8	28	56	70	56	28	8	1

The complicated calculations required for the evaluation of (3) may be avoided by using an alternative expression, well-adapted to evaluation by formula-manipulation systems such as REDUCE [8]. As in the derivation of (3), we start from the relation [4, 9]:

$$s_{n,p} = \frac{(-1)^{n+p-1}}{(n-1)!p!}\frac{\partial^{n+p-1}}{\partial\beta^{n-1}\partial\alpha^p}\frac{1}{\beta}\frac{\Gamma(1+\alpha)\Gamma(1+\beta)}{\Gamma(1+\alpha+\beta)}\Big|_{\alpha=\beta=0}. \tag{21}$$

We now introduce the power series [2, No. 8.321]

$$\begin{aligned}\Gamma(1+x) &= \sum_{k=0}^{\infty} b_k x^k \qquad (|x|<1),\\ 1/\Gamma(1+x) &= \sum_{k=0}^{\infty} a_k x^k,\end{aligned} \tag{22}$$

where $a_0 = b_0 = 1$, and

$$\begin{aligned}a_k &= \frac{1}{k}\sum_{m=1}^{k}(-1)^{m+1}\zeta(m)a_{k-m},\\ b_k &= -\frac{1}{k}\sum_{m=1}^{k}(-1)^{m+1}\zeta(m)b_{k-m} \qquad (k>0),\end{aligned} \tag{23}$$

with the definition $\zeta(1) = \gamma$ (Euler's constant). Then, performing the differentiations with respect to α in (21), and using the relation

$$\sum_{\rho=0}^{p} b_{p-\rho}a_\rho = 0 \qquad (p>0), \tag{24}$$

we obtain

$$\begin{aligned}&\frac{\partial^p}{\partial\alpha^p}\Gamma(1+\alpha)/\Gamma(1+\alpha+\beta)\Big|_{\alpha=0}\\ &= \sum_{\rho=0}^{p}\binom{p}{\rho}\left(\sum_{k=0}^{\infty}a_k\sum_{\kappa=0}^{k}\binom{k}{\kappa}\alpha^\kappa\beta^{k-\kappa}\right)^{(\rho)}\left(\sum_{k=0}^{\infty}b_k\alpha^k\right)^{(p-\rho)}\Big|_{\alpha=0}\\ &= p!\sum_{\rho=0}^{p}b_{p-\rho}\sum_{k=\rho+1}^{\infty}a_k\binom{k}{\rho}\beta^{k-\rho} = H(\beta).\end{aligned} \tag{25}$$

Similarly

$$\begin{aligned}&\frac{\partial^{n-1}}{\partial\beta^{n-1}}\frac{1}{\beta}H(\beta)\Gamma(1+\beta)\\ &= p!\sum_{\nu=0}^{n-1}\binom{n-1}{\nu}[H(\beta)/\beta]^{(\nu)}\left(\sum_{k=0}^{\infty}b_k\beta^k\right)^{(n-\nu-1)}\Big|_{\beta=0},\end{aligned} \tag{26}$$

and therefore, finally,

$$s_{n,p} = (-1)^{n+p-1}\sum_{\nu=0}^{n-1}b_{n-\nu-1}\sum_{\rho=0}^{p}\binom{\nu+\rho+1}{\rho}b_{p-\rho}a_{\nu+\rho+1}. \tag{27}$$

This expression, although revealing less of the structure (already inferred by Nielsen [9]) of $s_{n,p}$ than formula (3), namely that $s_{n,p}$ can be expressed as a homogeneous polynomial of "degree" $n+p$ in the terms $\zeta(m)$, $(2 \leqslant m \leqslant n+p)$, with rational coefficients, is much more suitable for actual computation. Using a formula-manipulation system, the evaluation of (27) is in fact straightforward once the expressions (23) for a_k $(0 \leqslant k \leqslant n + p)$ and b_k $[0 \leqslant k \leqslant \max(n-1, p)]$ in terms of $\zeta(m)$ have been initially established. It follows from (5) that, at least, all terms involving $\zeta(1) = \gamma$ will cancel in the final expression for (27). For example, the special cases $s_{n,1}$ and $s_{1,p}$ reduce to a single term:

$$s_{n,1} = (-1)^n \sum_{\nu=0}^{n-1} b_{n-\nu-1}\left[(\nu+2)a_{\nu+2} + b_1 a_{\nu+1}\right] = \zeta(n+1)$$

and

$$s_{1,p} = (-1)^p \sum_{\rho=0}^{p} (\rho+1) b_{p-\rho} a_{\rho+1} = \zeta(p+1). \tag{28}$$

The results obtained with REDUCE have been checked by evaluating the definition integral (2) by numerical integration, replacing the limits 0 and 1 by $\varepsilon = 10^{-8}$ and $1 - \varepsilon$, respectively, and using Stieltjes' 32 decimal table [10] of $\zeta(m)$, $m = 2(1)70$, which is reproduced in [9], for the evaluation of $s_{n,p}$.

We add here that the substitution $t = \sin^2\theta$ in (2) leads to the integral [6],

$$s_{n,p} = -\frac{(-2)^{n+p}}{(n-1)!\,p!} \int_0^{\pi/2} \cot\theta \log^n \sin\theta \log^p \cos\theta \, d\theta. \tag{29}$$

A closed expression for a similar integral,

$$R_{n,p} = \int_0^{\pi/2} \log^n \sin t \log^p \cos t \, dt \qquad (n \geqslant 0, p \geqslant 0), \tag{30}$$

has been given in [5], with examples up to $n = p = 4$.

3. A Table of the Integral. We list the expressions for $s_{n,p}$, $n = 1(1)9$, $p = 1(1)n$. The values for the integral in (2) itself,

$$r_{n,p} = \int_0^1 t^{-1} \log^{n-1} t \log^p (1-t) \, dt = (-1)^{n+p-1}(n-1)!\,p!\,s_{n,p}, \tag{31}$$

would lead for higher n or p to rather large coefficients. The reference work [2, No. 4.2912] lists only the case $n = p = 1$, whereas Lewin [6] gives (31) for $n = 2, 3, 4$, and $p = 2$.

Using the well-known relation [2, No. 9.5421],

$$\zeta(2\mu) = \frac{2^{2\mu-1}\pi^{2\mu}\,|B_{2\mu}|}{(2\mu)!}, \tag{32}$$

where $B_{2\mu}$ are the Bernoulli numbers, the expressions for $r_{n,p}$ simplify to some extent. We also give these values for $n = 1(1)7$, $p = 1(1)n$.

$$s_{11} = \zeta(2),$$
$$s_{21} = \zeta(3),$$
$$s_{22} = -\tfrac{1}{2}\zeta^2(2) + \tfrac{3}{2}\zeta(4),$$
$$s_{31} = \zeta(4),$$
$$s_{32} = -\zeta(2)\zeta(3) + 2\zeta(5),$$

$$s_{33} = \tfrac{1}{6}\zeta^3(2) - \tfrac{3}{2}\zeta(2)\zeta(4) - \zeta^2(3) + \tfrac{10}{3}\zeta(6),$$
$$s_{41} = \zeta(5),$$
$$s_{42} = -\zeta(2)\zeta(4) - \tfrac{1}{2}\zeta^2(3) + \tfrac{5}{2}\zeta(6),$$
$$s_{43} = \tfrac{1}{2}\zeta^2(2)\zeta(3) - 2\zeta(2)\zeta(5) - \tfrac{5}{2}\zeta(3)\zeta(4) + 5\zeta(7),$$
$$s_{44} = -\tfrac{1}{24}\zeta^4(2) + \tfrac{3}{4}\zeta^2(2)\zeta(4) + \zeta(2)\zeta^2(3) - \tfrac{10}{3}\zeta(2)\zeta(6) - 4\zeta(3)\zeta(5) - \tfrac{17}{8}\zeta^2(4) + \tfrac{35}{4}\zeta(8),$$
$$r_{11} = -\tfrac{1}{6}\pi^2,$$
$$r_{21} = \zeta(3),$$
$$r_{22} = -\tfrac{1}{180}\pi^4,$$
$$r_{31} = -\tfrac{1}{45}\pi^4,$$
$$r_{32} = -\tfrac{2}{3}\pi^2\zeta(3) + 8\zeta(5),$$
$$r_{33} = -\tfrac{23}{1260}\pi^6 + 12\zeta^2(3),$$
$$r_{41} = 6\zeta(5),$$
$$r_{42} = -\tfrac{1}{105}\pi^6 + 6\zeta^2(3),$$
$$r_{43} = -\tfrac{1}{2}\pi^4\zeta(3) - 12\pi^2\zeta(5) + 180\zeta(7),$$
$$r_{44} = -\tfrac{499}{12600}\pi^8 - 24\pi^2\zeta^2(3) + 576\zeta(3)\zeta(5).$$

The remaining expressions for $s_{n,p}$, $n = 5(1)9$, $p = 1(1)n$, and $r_{n,p}$, $n = 5(1)7$, $p = 1(1)n$, are given in the microfiche section at the end of this issue. Numerical values of $s_{n,p}$ with 21 digits are presented in Table 2.

TABLE 2

n	p	$s_{n,p}$
1	1	1.64493 40668 48226 43647E+00
2	1	1.20205 69031 59594 28540E+00
2	2	2.70580 80842 77845 47879E-01
3	1	1.08232 32337 11138 19152E+00
3	2	9.65511 59989 44373 44656E-02
3	3	1.74898 53169 01140 44259E-02
4	1	1.03692 77551 43369 92633E+00
4	2	4.05368 97271 51973 78290E-02
4	3	4.12316 51524 32535 53202E-03
4	4	6.02891 53283 31913 91876E-04
5	1	1.01734 30619 84449 13971E+00
5	2	1.83559 28317 49446 58780E-02
5	3	1.10762 05206 81261 04542E-03
5	4	1.06090 22891 02175 20514E-04
5	5	1.29078 86926 10006 80019E-05
6	1	1.00834 92773 81922 82684E+00
6	2	8.65052 90995 61105 50088E-03
6	3	3.20419 48118 65540 68195E-04
6	4	2.08107 99998 53278 80665E-05
6	5	1.81177 17675 49254 62907E-06
6	6	1.88257 25261 51750 84100E-07

TABLE 2 (continued)

n	p	$s_{n,p}$
7	1	1.00407 73561 97944 33938E+00
7	2	4.17024 20454 82641 20903E-03
7	3	9.70014 34407 46026 74085E-05
7	4	4.37446 80142 37467 26660E-06
7	5	2.79046 61391 18230 10386E-07
7	6	2.19761 45278 08360 14044E-08
7	7	1.99035 60428 47009 48657E-09
8	1	1.00200 83928 26082 21442E+00
8	2	2.03771 21074 18497 21127E-03
8	3	3.02392 65882 11524 77408E-05
8	4	9.63193 58629 25147 64220E-07
8	5	4.58067 37635 27237 01146E-08
8	6	2.78188 36333 42565 39146E-09
8	7	1.98864 22337 79748 87998E-10
8	8	1.59526 66865 47416 62087E-11
9	1	1.00099 45751 27818 08534E+00
9	2	1.00397 69886 51568 46827E-03
9	3	9.61339 45728 45768 87138E-06
9	4	2.19049 55625 33962 98356E-07
9	5	7.86919 99763 14613 72568E-09
9	6	3.73432 32019 05082 31190E-10
9	7	2.13492 93645 52627 26763E-11
9	8	1.39313 33415 77287 65028E-12
9	9	1.00261 68238 56214 27731E-13

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