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On the Integral $\int_0^{\pi/2} \log^n \cos x \log^p \sin x \, dx$

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Abstract. A formula is derived for the integral in the title which allows easy evaluation by formula manipulation on a computer.

1. Introduction. In a monograph on generalized polylogarithms, and in a paper on series of reciprocal powers, Nielsen [7], [8] remarked that

$$r_{np} = \int_0^{\pi/2} \log^n \cos x \log^p \sin x \, dx \qquad (n \geqslant 0,\ p \geqslant 0) \tag{1}$$

can be expressed as π times a polynomial in $\eta(q)$ $(1 \leqslant q \leqslant n+p)$, with rational coefficients, where

$$\eta(q) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^q} = \begin{cases} \log 2 & (q=1), \\ (1-2^{1-q})\zeta(q) & (q>1). \end{cases} \tag{2}$$

This polynomial is homogeneous of degree $n+p$ if one considers $\eta(q)$ to be of degree q, and if the degree of a product is the sum of the degrees of its factors. $\zeta(q)$ is the Riemann zeta function for integer arguments. Since $r_{np} = r_{pn}$, it is sufficient to consider $n \geqslant p \geqslant 0$.

In an earlier paper [4], the following complicated closed expression for r_{np} was derived:

$$r_{np} = \frac{\pi n!p!}{2^{n+p+1}} \sum_{k=1}^{n+p} \frac{1}{k!} \sum_{\{p_i\}} \sum_{\{n_i\}} f(p_1, n_1) \cdots f(p_k, n_k), \tag{3}$$

where the innermost sums run over all sets $\{p_i\}$, $\{n_i\}$, which satisfy the conditions

$$p_i \geqslant 0, \quad \sum_{i=1}^{k} p_i = p; \quad n_i \geqslant 0, \quad \sum_{i=1}^{k} n_i = n, \tag{4}$$

and where the function $f(r, s)$ is given by

$$f(r,s) = (1-\delta_{0r})(1-\delta_{0s}) \frac{(-1)^{r+s+1}}{r+s} \binom{r+s}{r} \zeta(r+s) + |\delta_{0r} - \delta_{0s}| \, \xi(r+s), \tag{5}$$

where

$$\xi(q) = \begin{cases} -2\log 2, & q=1, \\ (-1)^q \dfrac{2^q-2}{2} \zeta(q), & q>1, \end{cases} \tag{6}$$

and δ_{ij} is the Kronecker symbol.

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The actual computation of r_{np} from (3), even for small n and p, is quite complicated. Explicit expressions for r_{np} $(1 \leqslant n \leqslant 4, 0 \leqslant p \leqslant n)$ in terms of $\eta(q)$, computed from (3), have been given in [4]. It may be noted that, in the relevant handbooks, only expressions for r_{n0} or r_{0n} with $n = 1, 2$ are listed.*

2. An Expression for r_{np}. It is the purpose of this note to derive another expression for r_{np} which is well-adapted to evaluation by a formula-manipulation system such as REDUCE [2]. As in the derivation of (3), we start from the definition (1) and, after substituting $t = \cos^2 x$ in (1), express r_{np} as a derivative of Euler's beta function:

$$
\begin{aligned}
(7) \qquad r_{np} &= \frac{1}{2^{n+p+1}} \frac{\partial^{n+p}}{\partial\beta^n \partial\alpha^p} \int_0^1 t^{\beta-1/2}(1-t)^{\alpha-1/2}\,dt \,\Big|_{\alpha=\beta=0} \\
&= \frac{1}{2^{n+p+1}} \frac{\partial^{n+p}}{\partial\beta^n \partial\alpha^p} \frac{\Gamma(\frac{1}{2}+\alpha)\Gamma(\frac{1}{2}+\beta)}{\Gamma(1+\alpha+\beta)} \Bigg|_{\alpha=\beta=0}
\end{aligned}
$$

We introduce the power series [6]

$$
\begin{aligned}
(8) \qquad \Gamma(1+x) &= \sum_{k=0}^{\infty} b_k x^k \qquad (|x| < 1), \\
1/\Gamma(1+x) &= \sum_{k=0}^{\infty} a_k x^k \qquad (|x| < \infty),
\end{aligned}
$$

where $a_0 = b_0 = 1$,

$$
\begin{aligned}
(9) \qquad a_k &= -\frac{1}{k} \sum_{m=1}^{k} (-1)^m \tilde{\zeta}(m) a_{k-m}, \\
b_k &= \frac{1}{k} \sum_{m=1}^{k} (-1)^m \tilde{\zeta}(m) b_{k-m} \qquad (k > 0),
\end{aligned}
$$

and $\tilde{\zeta}(1) = \gamma$ (Euler's constant), $\tilde{\zeta}(m) = \zeta(m)$ for $m \geqslant 2$. A direct approach by setting $x = -\frac{1}{2} + \alpha$ and $x = -\frac{1}{2} + \beta$ in (8) and then carrying out the differentiations in (7) does not lead to a satisfactory result; the resulting expression would contain infinite series, and the expected quantities π and $\log 2$ would appear only implicitly. We therefore take a different route and apply the duplication formula [5]

$$
(10) \qquad \Gamma(2x) = \frac{1}{\sqrt{2\pi}} 2^{2x-1/2} \Gamma(x) \Gamma\left(\frac{1}{2} + x\right),
$$

so that from (7)

$$
(11) \qquad r_{np} = \frac{\pi}{2^{n+p+1}} \frac{\partial^{n+p}}{\partial\beta^n \partial\alpha^p} \frac{2^{-2\alpha-2\beta}}{\Gamma(1+\alpha+\beta)} \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)} \frac{\Gamma(1+2\beta)}{\Gamma(1+\beta)} \Bigg|_{\alpha=\beta=0}
$$

We have separated out the factor π, and the terms involving $\log 2$ will arise from the differentiation of $2^{-2\alpha-2\beta}$.

The differentiations with respect to α and β may now be carried out in (11). The result is a complicated six-fold sum in a form suitable for evaluation by formula

*Nielsen [8] made a similar comment.

manipulation. A much simpler expression can, however, be obtained as follows. We rewrite (11) as

(12) $$r_{np} = \frac{\pi}{2^{n+p+1}} \frac{\partial^{n+p}}{\partial\beta^n \partial\alpha^p} \frac{G(\alpha)G(\beta)}{\Gamma(1+\alpha+\beta)}\bigg|_{\alpha=\beta=0},$$

where

(13) $$G(x) = 2^{-2x} \frac{\Gamma(1+2x)}{\Gamma(1+x)},$$

and develop $G(x)$ as a power series. Using the series [6]

(14) $$\log \Gamma(1+x) = -\gamma x + \sum_{k=2}^{\infty} (-1)^k \zeta(k) x^k / k \qquad (|x| < 1),$$

we find

(15) $$\log G(x) = -(\gamma + 2\log 2)x + \sum_{k=2}^{\infty} (-1)^k \zeta(k) \frac{2^k - 1}{k} x^k \qquad \left(|x| < \frac{1}{2}\right).$$

In order to obtain a series for $G(x)$ from (15), we make use of the theorem that if

$$f(x) = \sum_{k=1}^{\infty} a_k x^k$$

is a formal power series with $a_0 = 0$, then

$$e^{f(x)} = \sum_{k=0}^{\infty} c_k x^k,$$

where the coefficients c_k are given recursively by

(16) $$c_k = \frac{1}{k} \sum_{m=1}^{k} m a_m c_{k-m} \qquad (k > 0).$$

This theorem can be proved analogously to Theorem 1.6c of Henrici [3, p. 42]. Applying this result to the series (15), we find

(17) $$G(x) = \sum_{k=0}^{\infty} b_k^* x^k \qquad \left(|x| < \frac{1}{2}\right),$$

where $b_0^* = 1$,

(18) $$b_k^* = \frac{1}{k} \sum_{m=1}^{k} (-1)^m \zeta^*(m) b_{k-m}^*$$

and

(19) $$\zeta^*(m) = \begin{cases} \gamma + 2\log 2, & m = 1, \\ (2^m - 1)\zeta(m), & m > 1. \end{cases}$$

We now differentiate (12) with respect to α and obtain

(20) $$H(\beta) = \frac{\partial^p}{\partial\alpha^p} \frac{G(\alpha)}{\Gamma(1+\alpha+\beta)}\bigg|_{\alpha=0} = \sum_{\rho=0}^{p} \binom{p}{\rho} \Gamma^{-1}(1+\alpha+\beta)^{(\rho)} G(\alpha)^{(p-\rho)}\bigg|_{\alpha=0}$$
$$= p! \sum_{\rho=0}^{p} b_{p-\rho}^* \sum_{k=\rho}^{\infty} a_k \binom{k}{\rho} \beta^{k-\rho}.$$

Similarly,

$$\frac{\partial^n}{\partial\beta^n} H(\beta)G(\beta)\bigg|_{\beta=0} = \sum_{\nu=0}^{n} \binom{n}{\nu} H(\beta)^{(\nu)} G(\beta)^{(n-\nu)}, \tag{21}$$

and therefore finally

$$r_{np} = \frac{\pi n!p!}{2^{n+p+1}} \sum_{\nu=0}^{n} b_{n-\nu}^{*} \sum_{\rho=0}^{p} \binom{\nu+\rho}{\rho} b_{p-\rho}^{*} a_{\nu+\rho}. \tag{22}$$

This expression, although still complicated and revealing less of the structure of r_{np} than formula (3), is much more suitable for actual computation. Using a formula manipulation system, the evaluation of (22) is in fact straightforward once the expressions for a_k $(0 < k < n+p)$ and b_k^* $(0 < k < \max(n, p))$ have been initially established. It follows from (5) that, at least, all terms involving $\tilde{\zeta}(1) = \gamma$ will cancel in the final expression for (22). For the case $n \geqslant 0, p = 0$, (22) reduces to

$$r_{n0} = \pi \frac{n!}{2^{n+1}} \sum_{\nu=0}^{n} b_{n-\nu}^{*} a_\nu, \tag{23}$$

which is another form of Bowman's determinant [1] for r_{n0}. The result of Bowman for r_{n0} can also be found in the book by Lewin [5].

3. A Table for r_{np}. We give here, as examples, the expressions for r_{np} for $n = 1, 2$ and $0 \leqslant p \leqslant n$. In order to complete the table of r_{np} given in [4], we also present expressions for $_{np}$ for $n = 5$ and $0 \leqslant p \leqslant 5$ in Table 1. Note that Nielsen has already given the expressions for r_{20}, r_{11}, r_{30} in [9], for r_{02}, r_{11}, r_{03} in [10], and for r_{12}, r_{22}, r_{13} in [11], derived by different methods.

$$\begin{aligned}
r_{10} &= -\frac{1}{2}\pi \log 2, \\
r_{11} &= \frac{\pi}{8}(-\zeta(2) + 4\log^2 2) = \frac{\pi}{2}\left(-\frac{1}{24}\pi^2 + \log^2 2\right), \\
r_{20} &= \frac{\pi}{4}(\zeta(2) + 2\log^2 2) = \frac{\pi}{2}\left(\frac{1}{12}\pi^2 + \log^2 2\right), \\
r_{21} &= \frac{\pi}{8}(\zeta(3) - 4\log^3 2) = \frac{\pi}{2}\left(-\log^3 2 + \frac{1}{4}\zeta(3)\right), \\
r_{22} &= \frac{\pi}{16}(-3\zeta(4) - 8\zeta(3)\log 2 + 3\zeta^2(2) + 8\log^4 2) \\
&= \frac{\pi}{2}\left(\frac{1}{160}\pi^4 + \log^4 2 - \zeta(3)\log 2\right).
\end{aligned}$$

Numerical values of r_{np} for $0 \leqslant n \leqslant 5$, $0 \leqslant n \leqslant p$, with 21 digits are given in Table 2.

TABLE 1

$$r_{50} = -\frac{\pi}{8}\big(90\zeta(5) + 105\zeta(4)\log 2 + 30\zeta(2)\zeta(3) + 60\zeta(3)\log^2 2 + 15\zeta^2(2)\log 2 + 20\zeta(2)\log^3 2 + 4\log^5 2\big)$$

$$r_{51} = \frac{\pi}{32}\big(-30\zeta(6) + 300\zeta(5)\log 2 - 135\zeta(2)\zeta(4) - 60\zeta^2(3) + 360\zeta(4)\log^2 2 - 60\zeta(2)\zeta(3)\log 2 - 15\zeta^3(2) + 200\zeta(3)\log^3 2 + 60\zeta(2)\log^4 2 + 16\log^6 2\big)$$

$$\begin{aligned} r_{52} = \frac{\pi}{32}\big(&90\zeta(7) + 210\zeta(6)\log 2 - 150\zeta(2)\zeta(5) + 165\zeta(3)\zeta(4) \\ &- 120\zeta(5)\log^2 2 + 90\zeta(2)\zeta(4)\log 2 + 210\zeta^2(3)\log 2 - 105\zeta^2(2)\zeta(3) \\ &- 240\zeta(4)\log^3 2 + 120\zeta(2)\zeta(3)\log^2 2 - 30\zeta^3(2)\log 2 \\ &- 140\zeta(3)\log^4 2 - 48\zeta(2)\log^5 2 - 16\log^7 2\big) \end{aligned}$$

$$\begin{aligned} r_{53} = \frac{\pi}{64}\big(&-630\zeta(8) - 1440\zeta(7)\log 2 - 165\zeta(2)\zeta(6) + 990\zeta(3)\zeta(5) - 180\zeta^2(4) \\ &- 1680\zeta(6)\log^2 2 + 720\zeta(2)\zeta(5)\log 2 + 360\zeta(3)\zeta(4)\log 2 \\ &- 360\zeta^2(2)\zeta(4) + 405\zeta(2)\zeta^2(3) - 600\zeta(5)\log^3 2 + 180\zeta(2)\zeta(4)\log^2 2 \\ &+ 360\zeta^2(2)\zeta(3)\log 2 - 60\zeta^4(2) + 60\zeta(4)\log^4 2 + 120\zeta(2)\zeta(3)\log^3 2 \\ &+ 60\zeta^3(2)\log^2 2 + 168\zeta(3)\log^5 2 + 60\zeta^2(2)\log^4 2 + 88\zeta(2)\log^6 2 + 32\log^8 2\big) \end{aligned}$$

$$\begin{aligned} r_{54} = \frac{\pi}{64}\big(&2520\zeta(9) + 5670\zeta(8)\log 2 + 540\zeta(2)\zeta(7) + 210\zeta(3)\zeta(6) \\ &- 4230\zeta(4)\zeta(5) + 6480\zeta(7)\log^2 2 + 1260\zeta(2)\zeta(6)\log 2 \\ &- 3960\zeta(3)\zeta(5)\log 2 - 4455\zeta^2(4)\log 2 - 450\zeta^2(2)\zeta(5) + 990\zeta(2)\zeta(3)\zeta(4) \\ &+ 630\zeta^3(3) + 5040\zeta(6)\log^3 2 - 720\zeta(2)\zeta(5)\log^2 2 \\ &- 6120\zeta(3)\zeta(4)\log^2 2 + 270\zeta^2(2)\zeta(4)\log 2 + 1260\zeta(2)\zeta^2(3)\log 2 \\ &- 210\zeta^3(2)\zeta(3) + 2280\zeta(5)\log^4 2 - 1440\zeta(2)\zeta(4)\log^3 2 \\ &- 1680\zeta^2(3)\log^3 2 + 360\zeta^2(2)\zeta(3)\log^2 2 - 45\zeta^4(2)\log 2 \\ &+ 432\zeta(4)\log^5 2 - 840\zeta(2)\zeta(3)\log^4 2 - 112\zeta(3)\log^6 2 \\ &- 144\zeta^2(2)\log^5 2 - 96\zeta(2)\log^7 2 - 32\log^9 2\big) \end{aligned}$$

$$\begin{aligned} r_{55} = \frac{\pi}{256}\big(&-45360\zeta(10) - 100800\zeta(9)\log 2 - 9450\zeta(2)\zeta(8) - 3600\zeta(3)\zeta(7) \\ &+ 5400\zeta(4)\zeta(6) + 73800\zeta^2(5) - 113400\zeta(8)\log^2 2 - 21600\zeta(2)\zeta(7)\log 2 \\ &- 8400\zeta(3)\zeta(6)\log 2 + 169200\zeta(4)\zeta(5)\log 2 - 1800\zeta^2(2)\zeta(6) \\ &+ 39600\zeta(2)\zeta(3)\zeta(5) - 33075\zeta(2)\zeta^2(4) - 33300\zeta^2(3)\zeta(4) \\ &- 86400\zeta(7)\log^3 2 - 25200\zeta(2)\zeta(6)\log^2 2 + 79200\zeta(3)\zeta(5)\log^2 2 \\ &+ 89100\zeta^2(4)\log^2 2 + 18000\zeta^2(2)\zeta(5)\log 2 - 39600\zeta(2)\zeta(3)\zeta(4)\log 2 \\ &- 25200\zeta^3(3)\log 2 - 6750\zeta^3(2)\zeta(4) + 15300\zeta^2(2)\zeta^2(3) \\ &- 50400\zeta(6)\log^4 2 + 9600\zeta(2)\zeta(5)\log^3 2 + 81600\zeta(3)\zeta(4)\log^3 2 \\ &- 5400\zeta^2(2)\zeta(4)\log^2 2 - 25200\zeta(2)\zeta^2(3)\log^2 2 + 8400\zeta^3(2)\zeta(3)\log 2 \\ &- 765\zeta^5(2) - 18240\zeta(5)\log^5 2 + 14400\zeta(2)\zeta(4)\log^4 2 \\ &+ 16800\zeta^2(3)\log^4 2 - 4800\zeta^2(2)\zeta(3)\log^3 2 + 900\zeta^4(2)\log^2 2 \\ &- 2880\zeta(4)\log^6 2 + 6720\zeta(2)\zeta(3)\log^5 2 + 640\zeta(3)\log^7 2 \\ &+ 960\zeta^2(2)\log^6 2 + 480\zeta(2)\log^8 2 + 128\log^{10} 2\big) \end{aligned}$$

TABLE 2

n	p	r_{np}
0	0	1.57079 63267 94896 61923 E + 00
1	0	-1.08879 30451 51801 06525 E + 00
1	1	1.08729 73195 40018 85202 E − 01
2	0	2.04662 20244 72740 64617 E + 00
2	1	-5.10672 58055 97432 35017 E − 02
2	2	1.01152 43866 77157 15298 E − 02
3	0	-6.04188 29097 75093 52215 E + 00
3	1	4.38609 79674 10765 45916 E − 02
3	2	-4.26758 40717 27544 48003 E − 03
3	3	1.02686 60292 42556 73947 E − 03
4	0	2.40528 86060 94622 94399 E + 01
4	1	-5.42161 86358 95971 92894 E − 02
4	2	2.78905 89366 54715 61317 E − 03
4	3	-4.11669 99312 01587 89077 E − 04
4	4	1.08779 55160 65204 48571 E − 04
5	0	-1.20085 59155 37857 31652 E + 02
5	1	8.67856 96358 57061 30428 E − 02
5	2	-2.46254 98260 35781 37595 E − 03
5	3	2.32819 02210 25192 62212 E − 04
5	4	-4.23231 23761 73142 19272 E − 05
5	5	1.18188 78861 42353 10303 E − 05

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