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Applications of a Computer Implementation of Poincaré's Theorem on Fundamental Polyhedra*

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Abstract. Poincaré's Theorem asserts that a group Γ of isometries of hyperbolic space $\mathbb{H}$ is discrete if its generators act suitably on the boundary of some polyhedron in $\mathbb{H}$, and when this happens a presentation of Γ can be derived from this action. We explain methods for deducing the precise hypotheses of the theorem from calculation in Γ when Γ is "algorithmically defined", and we describe a file of Fortran programs that use these methods for groups Γ acting on the upper half space model of hyperbolic 3-space $\mathbb{H}^3$. We exhibit one modest example of the application of these programs, and we summarize computations of repesentations of groups PSL(2, $\mathcal{O}$) where $\mathcal{O}$ is an order in a complex quadratic number field.

In the early 1880's H. Poincaré discovered a general theorem allowing one to deduce the discreteness of, and a presentation for, a group G of isometries of hyperbolic space from its action on a hyperbolic polyhedron under certain conditions. Theorems of this sort are part of the foundations of his theories of Fuchsian and Kleinian groups that have become very popular again, and H. Seifert has recently given us a modern proof of a fairly general version of Poincaré's Theorem in [12]; see also [7]. This theorem has been little used over the past century, perhaps partly because its hypotheses have seemed very difficult to verify for a given group G except in very special circumstances. One reason for doubting that Poincaré's Theorem is unreasonably difficult to apply to fairly general discrete groups is that no general alternative method for accomplishing its tasks has been proposed. The present paper is devoted to demonstrating that Poincaré's Theorem can indeed be applied to given groups in apparently difficult cases and that much of the work can be done by a computer. Our experience suggests that the theorem is really very helpful in guiding the user to an understanding of the details of the action of G starting from a state of near ignorance.

An outline of this paper is as follows. In Section 1 we begin by stating Seifert's version of Poincaré's Theorem and explaining how we would apply it to an "algorithmically defined" group G of isometries of hyperbolic space $\mathbb{H}^n$. In Section 2 we specialize to the situation, $\mathbb{H}^3$, for which we wrote our file, *Poincaré*, of Fortran

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programs. Poincaré uses an identification of the sense preserving hyperbolic isometries of the unit ball or half space models of $\mathbb{H}^3$ with $\mathrm{PSL}(\mathbb{C}) := \mathrm{PSL}(2,\mathbb{C})$, and it embodies the methods of Section 1 as far as was practicable. In Section 3 we present one modest example of the application of this file to a group of the kind it was developed for. This group is generated by parabolics, is not free, and is discrete but not obviously so. We conclude in Section 4 with a summary of an entirely different application of the Poincaré file. By a *Bianchi group* we mean $\mathcal{G}_d = \mathrm{PSL}(\mathbb{Z}[\omega_d])$ where d is a negative integer and

$$\omega_d = \frac{1+\sqrt{d}}{2} \quad \text{if } d \equiv 1 \pmod 4, \qquad \omega_d = \sqrt{d} \quad \text{otherwise.}$$

The integral domain $\mathbb{Z}[\omega_d]$ is an order in the complex quadratic number field $\mathbb{Q}(\sqrt{d})$ and is a discrete subset of $\mathbb{C}$. Hence the Bianchi groups are automatically discrete and they are considered to be of interest for arithmetical reasons, cf. Swan [14]. We shall describe how we found the presentations of 30 Bianchi groups, give the complete presentation for $\mathcal{G}_{-43}$, and summarize the corresponding results for the other groups in Table 1. We also state a few small conjectures inferred from our examination of the computer printout. Incidently, the Bianchi project is only intended as an advertisement for the Poincaré file which was developed for entirely different applications. This author thinks that many of the torsion-free subgroups of Bianchi groups are likely to be much more exciting than the full groups.

The present paper is one part of a long term project studying the projective representations of knot groups. We have tried to make it independent of the previous papers of the project because we feel that things like the Poincaré file will be of more widespread interest. Our main application is establishing the existence of the excellent hyperbolic structure on knot complements by direct calculation, cf. [8], [10], [11]. Another related application is to Kleinian groups G which either are homomorphic images of knot groups πK, or the other way around, and this also will be reported on elsewhere.

Our methods for applying Poincaré's Theorem to specific groups were devised in 1974 while the author was an official visitor to Southampton University and had been granted the use of an office and the standard facilities, except the computer. This was during a $2\frac{1}{2}$ year period of unemployment when our main financial support was the savings from a 6 month visit in 1973 to the Université de Strasbourg that was very generously funded by the C.N.R.S. The author was a Research Fellow on a project supervised by Dr. David Singerman at Southampton University during the 4 years 1976–1979. This project was also very generously funded by the Science Research Council, and we are most grateful to Dr. Singerman for his invaluable assistance during the grim years, and to Professors H. B. Griffiths and S. A. Robertson of Southampton University for their continued support since 1968. During the first half of 1980 we wrote up the first version of this paper while enjoying the warm hospitality of the Institute for Advanced Study. From September 1980 our project was funded first by Professor W. P. Thurston from his Waterman Fellowship and then by the NSF grant supporting the Sullivan-Thurston project of 1980–81 at the University of Colorado at Boulder.

1. How We Applied the Fundamental Theorem.

1.1. H. Seifert's account [12] of this theorem is in the context of isometries of a complete simply connected Riemannian manifold G^n of constant sectional curvature and dimension n. We begin by stating his version of this theorem in approximately his notation, and we shall change over to our notation as we get deeper into our special case. A *k-blob* (k-dimensionalen Raumstück) of G^n is the closure of an open connected subset of a k-plane of G^n. Let K be a finite nonempty collection of blobs with union $|K|$ in G^n. Seifert calls K a *complex* when the following assertions hold for K.

K1: Each $p \in |K|$ is an interior point of precisely one blob of K, denoted $Z(p) = Z(p, K)$.

K2: If p belongs to the blob A of K, then $Z(p) \subset A$.

He soon restricts attention to the complexes satisfying two further conditions.

K3: Each blob of K belongs to at least one n-blob of K.

K4: Each $(n-1)$-blob of K belongs to precisely one n-blob of K.

The blobs of dimension less than n are called *sides* of K.

Let $S_1, \ldots, S_r$ be the $(n-1)$-sides of such a complex K. We suppose that for each side S_j there is an isometry τ_j of G^n that maps S_j on some other side, say on S_l. (My τ_j is Seifert's τ_j^{-1}.) We suppose also that $\tau_l = \tau_j^{-1}$, and that for each interior point $p \in S_j$, τ_j maps the inward normal to S_j (pointing into $|K|$) to the outward normal at $\tau_j(p) \in S_l$. Call $p \in S_j$ *directly equivalent to* $\tau_j(p)$, and say that points p, q in $\operatorname{bdry}|K|$ are *equivalent* when p, q are the ends of a finite sequence of points of $\operatorname{bdry} K$ each directly equivalent to its immediate neighbors. Finally, suppose that no side of K contains equivalent interior points. Then Seifert calls K a *complex with side pairing* and writes $(K; \tau_1, \ldots, \tau_r)$ or just $(K; \tau)$ for it. A *cross line* of K is a curve v: $[0, \lambda] \to |K|$ for some $\lambda > 0$ whose image lies entirely in interior $|K|$, except that $v(0)$ and $v(\lambda)$ are interior points of $(n-1)$-sides. A *cross line chain* is a sequence $\{v_\rho\}$ of cross lines such that the endpoint of v_ρ is equivalent to the beginning of $v_{\rho+1}$. It has *finite length* when each v_ρ has finite length, say l_ρ, and $\Sigma l_\rho < \infty$. A complex with side pairing $(K; \tau)$ satisfies the *cross line condition* when each cross line chain of finite length lies in a compact subset of $|K|$.

Consider an interior point p_1 of the $(n-2)$-side $Z(p_1)$ of the complex with side pairing $(K; \tau)$, and let κ be a circle of radius ε and center p_1 lying in a 2-plane of G^n perpendicular to $Z(p_1)$ at p_1. We suppose the radius ε is so small that κ is near only $Z(p_1)$ in the $(n-2)$-skeleton of K. Orient κ arbitrarily, and let b_1 be an arc of κ whose interior lies in an n-blob of K so that its beginning and endpoints lie in $(n-1)$-sides. Let τ_1 be the pairing transformation for the side containing the endpoint of b_1, and let p_2 be $\tau_1(p_1)$. Then τ_1 maps $\kappa = \kappa_1$ to an oriented circle κ_2 about $Z(p_2)$, and the endpoint of $\tau_1(b_1)$ is the beginning of an arc b_2 of κ_2 whose interior lies in an n-blob of K. Then the endpoint of b_2 belongs to an $(n-1)$-side of K with transformation τ_2 that maps p_2 on p_3 and κ_2 on κ_3, a circle about $Z(p_3)$. We continue in this manner and generate sequences $p_1, p_2, \ldots$ and $b_1, b_2, \ldots$. When K satisfies the cross line condition, p_1 is equivalent to only finitely many points of $|K|$, so these sequences are periodic. Let λ be the least positive integer such that

$b_1 = b_{\lambda+1}$. Then $Z(p_1),\dots,Z(p_{\lambda+1})$ form a cycle of $(n-2)$-sides. The two $(n-1)$-sides of K meeting along $Z(p_j)$ that contain the boundary of b_j make an angle of, say, α_j in $|K|$. The *angle sum* for this cycle is $\alpha = \alpha_1 + \cdots + \alpha_\lambda$. The *angle sum condition* on K requires that for every such cycle the angle sum α is $2\pi/l$ where l is a positive integer depending on the cycle. If $\tau_1,\dots,\tau_\lambda$ are the transformations used in the cycle in their proper order, then $(\tau_\lambda \cdots \tau_1)^l$ is the identity isometry of G^n. We call the equation $(\tau_\lambda \cdots \tau_1)^l = E$ the *cycle relation* and the product $\tau_\lambda \cdots \tau_1$ the *cycle transformation* of the cycle.

It might happen that a side pairing transformation τ_j for $(K;\tau)$ is the reflection of G^n in the $(n-1)$-plane carrying the corresponding side S_j. Then $\tau_j^2 = E$, and this relation is called a *reflection relation*. Such a relation does not naturally correspond to an $(n-2)$-cycle of $(K;\tau)$, although it is certainly possible to subdivide K and get a new complex with side pairing in which the old S_j has been split in two by a new $(n-2)$-side.

Let $(K;\tau)$ be a complex with side pairing, and let Γ be the group generated by the τ_j. Seifert calls $(K;\tau)$ a *gapless cover* of G^n when

$$G^n = \bigcup_{\gamma\in\Gamma} \gamma(|K|).$$

He also calls $(K;\tau)$ *simple* when $\gamma_1(|K|)$ meets $\gamma_2(|K|)$ at most in boundary points, for distinct γ_1, γ_2 in Γ. Then Seifert's version of the fundamental theorem can be stated as follows.

THEOREM. *For some $n \geqslant 2$ let $(K;\tau)$ be a complex with side pairing in G^n whose side pairing transformations generate the group Γ. When $(K;\tau)$ satisfies the cross line condition, then $(K;\tau)$ is a gapless cover of G^n. When the interior of $|K|$ is connected and $(K;\tau)$ satisfies both the cross line condition and the angle sum condition, the cover is both gapless and simple. In this case Γ is a discrete group of isometries acting properly discontinuously on G^n, and the cycle relations and reflection relations of $(K;\tau)$ present Γ on the generators τ_j.*

1.2. When G^n is hyperbolic space $\mathbb{H}^n$, it is convenient to replace the cross line condition by the more manageable cusp condition. This is best stated in the context of the two standard conformal models of $\mathbb{H}^n$ that we will use later, viz. the unit ball model $\mathfrak{B}^n$ and the upper half space model $\mathfrak{U}^n$. Start with Euclidean space $\mathbb{R}^n$ with its standard metric, the *E-metric*, which we use both as a distance $(d(x,y) = |x-y|)$ and as a Riemannian metric $(ds_E = |dx|)$. Let $\overline{\mathbb{R}}^n$ be the one point compactification of $\mathbb{R}^n$ with compactifying point ∞. Write

$$\mathfrak{B}^n = \{(x_1,\dots,x_n) \in \mathbb{R}^n : |x| < 1\},$$
$$\mathfrak{U}^n = \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_n > 0\},$$
$$\Pi_h = \{(x_1,\dots,x_n) \in \mathbb{R}^n : x_n = h\}, \qquad \Pi_h^* = \Pi_h \cup \{\infty\}.$$

The sphere at infinity, SP^∞, is the boundary of $\mathfrak{B}^n$ and Π_0^* in the two cases, and its points are called *points at infinity*. $\mathbb{H}^n$ carries a hyperbolic metric, the H-*metric*, which for our two models depends only on the E-metric and the distance to SP^∞, viz.

$$ds_H = \frac{2ds_E}{1-|x|^2}, \qquad ds_H = \frac{ds_E}{x_n}, \tag{1.1}$$

respectively, cf. [1], [2] for more detail. Most objects or relations in either model are described with respect to one of these two metrics, and we use the appropriate prefix, E- or H-, to denote which. In cases where the metrics always give the same result, notably angles, we do not use a prefix, and we shall often omit the E-prefix. (Therefore the H-prefix can only be omitted when the meaning is very clear.) For example, the H-*topology* on $\mathbb{H}^n$ is the restriction of the E-topology, and $\mathbb{H}^n$ is H-complete but not E-complete. Let $\overline{\mathcal{B}}^n$, $\overline{\mathcal{U}}^n$ be the E-closures of our two models.

A *horosphere* for either model is an E-sphere S of $\overline{\mathcal{B}}^n$ or $\overline{\mathcal{U}}^n$ which is tangent to SP^∞ at some point p, with the convention that if $p = \infty$, then $S = \Pi_h^*$ for some h. One component of $\mathbb{H}^n - S$ meets SP^∞ only at p and is called a *horoball*. A horoball may be thought of as a deleted H-neighborhood of the point p at infinity. Let $(K; \tau)$ be a complex with side pairing in our model, and suppose the E-closure of $|K|$ meets SP^∞ in a nonempty set L. An isolated point of L is called an *ideal vertex* of K. At each ideal vertex z_α we choose a horoball V_α which is so E-small that distinct horoballs do not meet and so that V_α meets only those sides of K that have z_α on their E-boundaries. We call the portion of $|K|$ inside V_α a *cusp* of $|K|$. The *cusp condition* on $(K; \tau)$ requires that these horoballs V_α for the cusps of $|K|$ can be chosen so that, if $p \in |K|$ belongs to some V_α, then all points of $|K|$ equivalent to p also belong to the horoballs. Seifert calls the H-distance of $P \in V_\alpha$ to the horosphere bounding V the *niveau* of p and shows that when the cusp condition holds the horoballs can be chosen so that corresponding points of $|K|$ in the horoballs have the same niveau.

ADDENDUM TO THE THEOREM. *The cusp condition for* $(K; \tau)$ *implies the cross line condition. If the horoballs for the cusps of* $|K|$ *can be chosen so that each* $V_\alpha \cap |K|$ *is connected, then the two conditions are equivalent.*

1.3. We are now ready to discuss the mechanics of actually using Poincaré's Theorem to demonstrate the discreteness of an explicitly defined isometry group Γ and to determine a presentation of Γ. One possible meaning of "explicitly defined" is the following. An *algorithmically defined group* is a triple $(\{S_j\}, \mathrm{AD}(1), \mathrm{AD}(2))$, where $\{S_j\}$ is a (finite in this paper) set of isometries of G^n generating a group Γ, AD(1) is an algorithm which solves the word problem for Γ on $\{S_j\}$, and AD(2) is an algorithm that computes the action of each $S_j \in \{S_j\}$ on G^n to any requested accuracy, with respect to some fixed atlas of charts of G^n. For brevity we shall call the set $\{S_j\}$ *algorithmic* when these algorithms are provided, and we shall regard them as provided when the definition of $\{S_j\}$ suggests ways of providing them. The standard case for us is where we are using some model of G^n that gives a convenient faithful representation of the isometry group of G^n by a matrix group $M \subset \mathrm{GL}(m, \mathbb{C})$ for some m. Suppose that each S_j is represented by a matrix whose entries belong to an algebraic number field $F = \mathbb{Q}(\omega)$, where ω is a root of a known integral polynomial $f(x)$. Then AD(1) is reduced to arithmetic in F, and this boils down to calculations with polynomials in $\mathbb{Q}[x]$ modulo $f(x)$ which can be done algorithmically. Similarly, AD(2) boils down to Newton's iteration to the specific root ω starting from a sufficiently good first approximation. Not all finite sets $\{S_j\}$ of isometries actually permit AD(1) or AD(2), for instance, because it is easy to give cute examples of cyclic subgroups of the additive group $\mathbb{Q}$ which make either algorithm depend on the truth of Fermat's Last Theorem.

Many algorithmic sets $\{S_j\}$ generate indiscrete groups Γ, and Poincaré's Theorem does not apply to them. We are told that for every G^n there is in principle an effective criterion for indiscreteness of Γ in terms of inequalities associated with sets of elements of Γ. The main difficulty in applying such a criterion is finding the right subset of Γ for it. Our basic plan of attack on $\{S_j\}$ is to try to construct a complex with side pairing $(K; \tau)$ where τ generates Γ by some kind of a reasonably efficient search for suitable transformations τ_j. Whenever a new transformation is proposed it is fed to the indiscreteness criterion so that the search can be stopped immediately when Γ is proved indiscrete. The idea is that the early guesses at a fundamental domain for Γ are too big, the search is looking for ways to reduce them, and Γ is indiscrete when the guesses get too small.

Our methods for dealing with isometry groups of $\mathbb{H}^n$ apply to the models $\mathcal{B}^n$ and $\mathcal{U}^n$ for all $n \geqslant 2$. The advantages of these models are the two metrics, the conformality, and the convenient matrix representation of the groups. A good reference for the background is Chapter II of Ahlfors [1] or Thurston [15]. We shall henceforth make the tacit assumption that all mentioned H-isometries of our model preserve orientation. Therefore if our original set $\{S_j\}$ contains orientation reversing elements we first produce an appropriate set of products of the S_j generating the orientation-preserving subgroup Γ^+ of Γ, and then study Γ^+ by the methods below. It is easy to derive the desired results for Γ from those for Γ^+.

We shall use a geometric description of the $(n-1)$- and $(n-2)$-planes and the action of H-isometries for our model, $\mathcal{B}^n$ or $\mathcal{U}^n$, of $\mathbb{H}^n$. We wish to use our system of carefully distinguishing the two metrics, but because this gets clumsy for general n, we shall describe everything primarily for $n = 3$ and indicate the notational changes for other n. It is well known that a 2-plane Π of (our model of) $\mathbb{H}^3$ has the form $\mathbb{H}^3 \cap S$, where S is an E-sphere or E-plane perpendicular to SP^∞. We shall call Π an H-*plane*, or an EH-*plane* when S is an E-plane. A 1-plane is the intersection of two intersecting H-planes. We shall call it an H-*line*, or an EH-*line* when the two planes are EH. In dimension n the corresponding objects have the same codimension, so "H-$(n-1)$-plane" and "H-$(n-2)$-plane" replaces "H-plane" and "H-line", respectively, in the following discussion.

An EH-*transformation* or EH-*isometry* is an H-isometry which is also an E-isometry. Let T be an H-isometry of $\mathcal{U}^n$. Then the nth root of the Jacobian $\mathrm{Jcb}(T)$ evaluated at a point P of $\mathcal{U}^n$ gives the expansion of the E-metric at P by T, so T is an E-isometry of $\mathcal{U}^n$ exactly on the locus $\mathrm{Jcb}(T)(P) = 1$. When $\mathrm{Jcb}(T)$ is nonconstant, then this locus is an H-hyperplane which we call the *isometric sphere* of T and denote by $I(T)$. If we denote the H-hyperplane that perpendicularly H-bisects a segment (a, b) of $\mathbb{H}^n$ by $Eq(a, b)$, then $I(T)$ (when it exists) is

$$\lim_{P \to \infty} Eq\big(T^{-1}(P), P\big),$$

and the E-centre of $I(T)$, denoted $cn(T)$, is $T^{-1}(\infty) \in \mathrm{SP}^\infty$. The cases where $I(T)$ is not defined for $\mathcal{U}^n$ are when $\mathrm{Jcb}(T)$ is a constant function. When $\mathrm{Jcb}(T) \equiv 1$, then T is EH, and when $\mathrm{Jcb}(T) \equiv c \neq 1$, then T is an E-similarity of a type that we will have to exclude from the discussion below. So if we encounter a group Γ acting on $\mathcal{U}^n$ containing such an element, we shall have to switch to the ball model. In the next section we shall give the explicit formulae for $I(T)$, $cn(T)$, and the E-radius, $rd(T)$, of $I(T)$ for $\mathcal{U}^3$ using the standard complex notation.

An EH-isometry T of $\mathcal{B}^n$ is an H-isometry which fixes the Euclidean center 0 of $\mathcal{B}^n$, cf. (1.1). When T is not an EH-isometry, we define the *isometric sphere* of T, again denoted $I(T)$, by $I(T) = Eq(T^{-1}(0), 0)$. Then, both for $\mathcal{U}^n$ and $\mathcal{B}^n$, if T admits an isometric sphere, there is an EH-hyperplane $Ref(T)$ that E-bisects $I(T)$ such that the action of T is the product of an E-inversion of $\mathbb{H}^3$ in $I(T)$, an E-reflection (inversion) in $Ref(T)$, and an EH-transformation that carries $I(T)$ on $I(T^{-1})$. It follows from this description that T is an E-magnification E-inside $I(T)$, an E-isometry exactly on $I(T)$, and an E-contraction E-outside $I(T)$. Therefore, if U is an EH-isometry we have

$$I(UT) = I(T), \qquad I(TU) = U^{-1}(I(T)). \tag{1.2}$$

We shall not need to give a rule to determine the *reflecting plane* $Ref(T)$ except when T is an involution. If T has fixed points in $\mathbb{H}^n$, then T is an H-rotation about an axis $ax(T)$ which is an H-$(n-2)$-plane. If $T^2 = E$ and $I(T)$ exists, we choose $Ref(T)$ to be the EH-plane containing $ax(T)$. If T is both EH and an H-rotation, we call T an EH-*rotation*. For $\mathcal{B}^n$ the E-centre, still $cn(T)$, of $I(T)$ for a non-EH-isometry lies outside $\mathcal{B}^n$. If R is an EH-rotation of $\mathbb{H}^n$ (either model) such that the E-$(n-2)$-plane containing $ax(R)$ contains $cn(T)$, we shall say merely that $ax(R)$ contains $cn(T)$.

1.5. Suppose we are given an algorithmic set $\{S_j\}$ generating a discrete group Γ of (orientation-preserving) H-isometries. We wish to produce a complex with side pairing $(K; \tau)$ such that τ also generates Γ, and we begin by selecting a model, $\mathcal{B}^n$ or $\mathcal{U}^n$, of $\mathbb{H}^n$. Let Γ_{EH} denote the subgroup of all EH-transformations of Γ. Then Γ_{EH} is a Euclidean group of a simple type that we may suppose has been completely classified. Hence, if the study of Γ is to present a challenge, we suppose that $\Gamma \neq \Gamma_{EH}$. Let $\mathcal{O}$ be the open region of $\mathbb{H}^n$ which is E-outside all isometric spheres of $\Gamma - \Gamma_{EH}$. Then $\mathcal{O}$ is nonempty if we are in $\mathcal{B}^n$, but $\mathcal{O}$ could be empty if we are in $\mathcal{U}^n$. If this happens, we must switch to $\mathcal{B}^n$. So suppose $\mathcal{O}$ is nonempty. Each $(n-1)$-side (face) of $\mathcal{O}$ lies on an isometric sphere, say $I(T)$, and T maps the face of $\mathcal{O}$ on $I(T)$ onto that on $I(T^{-1})$. When Γ_{EH} is trivial, then $\mathcal{O}$ is a fundamental domain for Γ. We then set $\mathcal{D} = \mathcal{O}$ and call $\mathcal{D}$ a *Ford domain* for Γ. It is associated with the set $\{T_j\}$ of elements of Γ whose isometric spheres carry faces of $\mathcal{D}$.

When Γ_{EH} is nontrivial, the region $\mathcal{O}$ is bigger than a fundamental domain, and we have to describe a rule for getting a good domain in some detail. Let pr: $\mathcal{U}^n \to \Pi_0 \subset \mathrm{SP}^\infty$ be the orthogonal projection $(x_1, \ldots, x_n) \to (x_1, \ldots, x_{n-1})$, and let pr: $\mathcal{B}^n - \{0\} \to \mathrm{SP}^\infty$ be the radial projection from 0. Our method for getting a Ford domain $\mathcal{D}$ for Γ is to select a fundamental domain $\mathcal{D}_\infty \subset \mathrm{SP}^\infty$ for Γ_{EH}, and then set $\mathcal{D} = \mathcal{O} \cap \mathrm{pr}^{-1}(\mathcal{D}_\infty)$. It will simplify the later discussion if we always choose $\mathcal{D}_\infty$ so that the number of spherical faces of $\mathcal{D}$ is minimal, and we do this as follows. Each face of $\mathcal{O}$ is an H-polygon on some isometric sphere. Suppose first that Γ does not contain an EH-rotation which rotates a face of $\mathcal{O}$ on itself. Then there is a collection $\{T_k\}$ of elements of $\Gamma - \Gamma_{EH}$ such that each $I(T_k)$ and $I(T_k^{-1})$ carries a face of $\mathcal{O}$, these faces are all distinct except for the possibility that $T_k = T_k^{-1}$, and the interior of the closure of the union of the projections on SP^∞ of these faces is a fundamental domain for Γ_{EH}. The modification needed when the EH-rotation R rotates a face F of $\mathcal{O}$ on itself is to replace F by a suitable wedge W on F. If R is a rotation of exact order $r \geq 2$, the vertex angle of W will be $2\pi/r$, and the "vertex" of W will be $ax(R) \cap I(T)$ where $F \subset I(T)$. One of the sides of W to this "vertex" will be an

H-$(n-2)$-line to an $(n-3)$-side of bdry F. Also TRT^{-1} is an EH-rotation whose axis contains $cn(T^{-1})$, and we select a wedge W' on $I(T^{-1})$ for it by $W' = T(W)$. Then $\mathcal{D}_\infty$ will be the interior of the union of the projections of the complete faces and wedges on the $I(T_k^{\pm 1})$, and is a fundamental domain for Γ_{EH}. The resulting domain $\mathcal{D}$ is called a *Ford domain* for Γ, and $\mathcal{D}$ is determined by Γ_{EH}, the collection $\{T_k\}$, and, perhaps, the choice of certain wedges. These wedges will be tacitly taken for granted and will be mentioned only when needed. So, in all cases, our Ford domain $\mathcal{D}$ is associated with a collection $\{T_k\}$ such that each T_k is some word on the original generators $\{S_j\}$, and T_k maps the face of $\mathcal{D}$ on $I(T_k)$ onto that on $I(T_k^{-1})$.

When Γ_{EH} is nontrivial, the above rule for producing $\mathcal{D}_\infty$ can be refined to ensure that $\mathcal{D}$ be connected and even that the number of EH-edges be least possible. While these features may seem desirable, they do not enter our analysis at all, and it turns out to be rather difficult to achieve them in practice. It is very unusual for one of our Ford domains to be H-convex, and typically an H-convex fundamental domain (when Γ_{EH} is nontrivial) is more complicated than a Ford domain in ways that really matter. Incidentally, the Beardon-Maskit theory for H-convex domains in dimension 3 applies directly to our Ford domains in spite of the nonconvexity.

1.6. To get a complex with side pairing $(K; \tau)$ from a Ford domain associated with $\{T_k\}$ take $|K|$ to be the H-closure of $\mathcal{D}$ and the sides of K to be the minimal collection of blobs on bdry $|K|$ that is consistent with the H-polyhedral structure and satisfies Seifert's restrictions K1,...,K4. (When Γ contains involutions we may have to bisect some faces of K so that no side of K contains equivalent interior points.) The side pairing transformations τ are $\{T_k\}$ and any necessary EH-transformations for EH-sides. It is clear that the E-distances to SP^∞ of equivalent points of $|K|$ are the same and that this implies that the cusp condition holds for $(K; \tau)$. Hence Seifert's Addendum shows that the cross line condition holds. Because $\mathcal{D}$ is a fundamental domain for Γ we know that $(K; \tau)$ is a gapless simple cover of $\mathbb{H}^n$, and then item 9.3 of [12] tells us that the angle sum condition also holds. The only new information that Poincaré's Theorem might give us now is that the cycle relations of $(K; \tau)$ present Γ on the generators τ_j.

However, our usual starting point is the algorithmic set $\{S_j\}$ generating Γ, and intiially we may not know whether Γ is discrete, or know the subgroup Γ_{EH}, or the set $\{T_k\}$ for a suitable Ford domain. These three items are to be part of the conclusion. Suppose that we have found a candidate corresponding to $\{T_k\}$ for being a Ford domain for Γ by some kind of search. The T_k and the proposed EH-pairing transformations are supposed to be explicit words on $\{S_j\}$. Our problem is then reduced to proving that $\mathcal{D}$ actually is a Ford domain for Γ. The hypothesis that $\mathcal{D}$, $\{T_k\}$ is a candidate Ford domain means first of all that we have used algorithm AD(2) to calculate roughly the intersections of the $I(T_k^{\pm 1})$ and the effect of the EH-transformations, and that we have lists of the sides of the candidate $(K; \tau)$ (defined as above for $\mathcal{D}$, $\{T_k\}$). These lists give us the proposed incidence relations of K and the approximate E-boundary of each k-plane carrying a side of K. The edges (codimension 2 sides) of K have been sorted into tentative edge cycles, and the cycle transformation for each cycle has been reduced to a word on $\{S_j\}$. The angle sum for each cycle has been calculated very accurately and found to be very nearly $2\pi/a$ for some definite integer a. Then the cycle relation $X^a = E$ for the cycle has been

verified by algorithm AD(1). It is assumed that no inconsistency in $(K; \tau)$ was found during these checks, so that $\mathcal{D}$ is pretty certain to be correct.

The easiest condition to settle is the angle sum condition for each proposed edge cycle. Say the cycle has cycle transformation X, that $X^a = E$, and that the angle sum estimate really proves that the angle sum is less than $4\pi/a$. Now $X^a = E$ implies that the angle sum is exactly $2\pi m/a$ for some integer $m \geqslant 1$ which is relatively prime to a. Our upper bound implies that $m = 1$, so the angle sum condition holds for this cycle. We call this argument the *angle sum trick*.

The really hard part of the verification is proving that the side pairing transformations really do pair the faces of K in the manner indicated by the approximate calculation. We do not have a general method for doing this, so we have to take advantage of the special circumstances of the definition $\langle\{S_j\}\rangle$ of Γ. There is a direct method for solving the pairing problem when the S_j are defined by a faithful matrix representation of the group of H-isometries and the entries of the matrices generate a known algebraic number field F. It is then possible to compute the coordinates of the vertices of the E-closure of $|K|$ as explicit algebraic numbers. Then if τ_k seems to send a side A on a side A', one can compute the vertices of $\tau_k(A)$ and A' and compare them as algebraic numbers. If A' and $\tau_k(A)$ have the same vertices, they are equal because they are both H-convex. This method is so grim that we call it the *method of last resort*.

There are two important simplifications of this pairing problem that help even the method of last resort. The first is that when a tentative Ford domain $\mathcal{D}$ is constructed by the above rules, its EH-faces are automatically paired correctly by EH-transformations and hence they can be omitted from the pairing verifications. First of all, the two EH-sides that meet along the axis of an EH-rotation through the apex of a wedge on a face of $\mathcal{O}$ are paired by R because we explicitly arranged it. For the other EH-faces recall from (1.2) that the EH-transformations permute the isometric spheres of $\Gamma - \Gamma_{EH}$, and hence permute the edges of our candidate for $\mathcal{O}$. An edge e of $\mathcal{O}$ gives rise to an EH-strip $\mathrm{pr}^{-1}(\mathrm{pr}(e))$, so Γ_{EH} permutes these strips. All the EH-faces of $\mathcal{D}$ except those from wedges are such strips, so the EH-elements of τ must pair them correctly. Note that this argument works even when $\mathcal{D}$ is wrong provided that we keep to the rules for defining $\mathcal{D}$ given a candidate $\mathcal{O}$.

The second simplification of the pairing problem is the recognition that Poincaré's Theorem really does not require hypotheses on the sides of K of codimension $\geqslant 3$. Let $T \in \tau$ and let F, F' be faces of $\mathcal{D}$ such that T approximately maps F on F'. Let $l_1, \ldots, l_m$ and $l'_1, \ldots, l'_m$ be the H-lines (or H-$(n-2)$-planes) containing the edges of F and F'. Suppose we can prove that T maps the H-lines l_j on the H-lines l'_j. Then $T(F)$ and F' are nearly coincident H-polygons on the same H-plane which are bounded by segments of the same set of H-lines. Therefore $T(F) = F'$, and we have solved the pairing problem for F, F'. The H-line $T(l_j)$ is determined by the effect of T on the E-endpoints (or E-boundary in higher dimensions), so in the method of last resort the pairing problem is reduced to calculations of the effect of the T_k on SP^∞.

We are now ready to explain our preferred methods for solving the pairing problem efficiently. We shall consider the effect of the T_k on the H-lines carrying non-EH-edges of K on a cycle by cycle basis rather than on a face by face basis. Consider a nontorsion cycle of length 6 whose cycle transformation X is the product

$U_3V_3U_2V_2U_1V_1$, where the U_j are EH-transformations and the V_j belong to $\{T_k^{\pm 1}\}$. Of course, the cycle relation $X = E$ was verified by AD(1). The H-lines carrying the edges of this cycle are

$$\pi_1 = U_3\big(I(V_3^{-1})\big) \cap I(V_1), \quad \pi_2 = I(V_1^{-1}) \cap U_1^{-1}\big(I(V_2)\big),$$
$$\pi_3 = U_1\big(I(V_1^{-1})\big) \cap I(V_2), \quad \pi_4 = I(V_2^{-1}) \cap U_2^{-1}\big(I(V_3)\big),$$
$$\pi_5 = U_2\big(I(V_2^{-1})\big) \cap I(V_3), \quad \pi_6 = I(V_3^{-1}) \cap U_3^{-1}\big(I(V_1)\big).$$

Clearly $U_1(\pi_2) = \pi_3$, $U_2(\pi_4) = \pi_5$, and $U_3(\pi_6) = \pi_1$, and we need to prove that

$$\pi_2 = V_1(\pi_1), \quad \pi_4 = V_2(\pi_3), \quad \pi_6 = V_3(\pi_5). \tag{1.3}$$

Let $\pi_1' := U_3V_3(\pi_5)$, $\pi_2' := U_1^{-1}V_2^{-1}U_2^{-1}(\pi_5)$. Because $\pi_5 \subset I(V_3)$ and $U_2^{-1}(\pi_5) \subset I(V_2^{-1})$, the maps U_3V_3 and $U_1^{-1}V_2^{-1}U_2^{-1}$ are E-isometries of π_5 on π_1' and π_2'. Hence the composite $U_1^{-1}V_2^{-1}U_2^{-1}\cdot(U_3V_2)^{-1} = Y$ is an E-isometry of π_1' on π_2', whence $\pi_1' \subset I(Y)$ and $\pi_2' \subset I(Y^{-1})$. But the cycle relation implies that $Y = V_1$, so

$$\pi_1' = I(V_1) \cap U_3\big(I(V_3^{-1})\big) = \pi_1, \quad \text{and} \quad \pi_2' = I(V_1^{-1}) \cap U_1^{-1}\big(I(V_2)\big) = \pi_2.$$

Hence $\pi_2 = V_1(\pi_1)$. Next

$$\pi_4 = U_2^{-1}(\pi_5) = V_2U_1(\pi_2') = V_2U_1(\pi_2) = V_2(\pi_3) = \pi_4,$$

and the proof of the last assertion of (1.3) is similar. The conclusion is that the H-lines carrying the edges of a torsion free cycle whose cycle transformation has exactly three non-EH-factors are automatically mapped on their successors correctly. We call this little swindle the *closing trick*. The closing trick even helps the method of last resort, because if we have a torsion free cycle with more than three non-EH-factors in its cycle transformation, the trick permits the omission of the calculation of the effect of one non-EH-factor on an H-line.

We have not found a similar good trick for most torsion cycles, and when we encounter one we rely on ad hoc arguments, cf. the example in Section 3. However, if the cycle relation is a direct power $\tau_1^r = E$, then the cycle can contain only one edge and this edge is a segment of $ax(\tau_1)$. So there is nothing to do here, but for the sake of reference we call this observation the *torsion trick*.

1.7. Experience suggests that these tricks are completely reliable except in cases where some alternative method is not especially unpleasant. One common situation where the closing trick is prone to fail is where we have two isometry groups $\Gamma \subset \Delta$, so that information about one group can be used for the other. For example, when the bigger group is proved discrete the smaller group is a fortiori discrete, and a Ford domain for Γ is the interior of the H-closure of the union of Δ-translates of one for Δ. If Γ has finite index in Δ, then Δ is discrete when Γ is, and this time we subdivide a Ford domain for Γ to get one for Δ. A third important case is where Δ is generated by Γ and EH-isometries and Γ is normal in Δ. Then Δ is discrete if Γ is, because both groups have the same lattice $\mathcal{O}$ of isometric spheres. In any of these cases, once we have produced a complex with side pairing $(K; \tau)$, where the interior of $|K|$ is a Ford domain for one group, the approximate calculations of AD(2) suffice to determine all we need to know about the other group.

One further advantage of Ford domains is that when one of the torsion free cycles admits the closing trick the angle sum α for the cycle is bounded above by 3π,

whence $\alpha = 2\pi$. More generally, suppose the cycle relation is $\tau_m \cdots \tau_1 = E$. If an edge of the cycle is a segment of the intersection of two non-EH-faces, then the angle here is $< \pi$. When one face at an edge is an EH-face paired by τ_c, we use

$$\text{angle in } |K| \text{ of } I(\tau_a) \cap \tau_c^{-1}(I(\tau_b)) + \text{angle in } |K| \text{ of } \tau_c(I(\tau_a)) \cap I(\tau_b) = \text{angle of } I(\tau_a) \cap I(\tau_b\tau_c) < \pi.$$

Hence if exactly r of the factors τ_j are not EH, we get the bound $\alpha < r\pi$. Therefore we do not have to consider α when $r = 3$ or 4 because then $\alpha = 2\pi$ automatically.

A final verification that may sometimes be necessary after $(K; \tau)$ has been found is that τ generates Γ itself, and not some proper subgroup Γ_1. For each generator $S \in \{S_j\}$ of Γ that is not known to be in Γ_1 we construct an approximation to $S(|K|)$ and connect an interior point of this to an interior point of $|K|$ by an H-polygonal path that avoids the Γ_1-images of the edges of K. This path determines a word T on τ such that $T(|K|)$ overlaps $S(|K|)$. Then we use AD(1) to check that $T = S$.

2. The Poincaré Library File. The Poincaré file is a collection of Fortran subroutines that is to be combined with a main Fortran program to make a system implementing something like the methods of Section 1 for dimension 3. The file has grown over the years and is now getting near its natural limits. There are still certain restrictions on the groups it can handle, and there will always be machine dependent practical limitations, but the system can now do most of what one could expect and want. The file is on magnetic tape and will be available for copying by other matematicians desiring to use it. We shall also prepare a manual describing Poincaré in more detail and explaining how to use it. Here we will just outline what it does and give a few hints about its methods, which will support the discussion of its applications in the next two sections.

First of all, Poincaré uses the standard complex notation for $\mathfrak{U}^3$, so now we will write

$$\mathfrak{U}^3 = \{(z, h) \in \mathbb{C} \times \mathbb{R} : h > 0\}, \qquad \mathbb{P}^1(\mathbb{C}) = \Pi_0^* = \mathrm{SP}^\infty.$$

This facilitates an identification of the group of orientation-preserving H-isometries of $\mathfrak{U}^3$ with $\mathrm{PSL}(\mathbb{C}) = \mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(\mathbb{C})/\langle -E \rangle$, where $\mathrm{SL}(\mathbb{C}) = \mathrm{SL}(2, \mathbb{C})$. Let

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad ad - bc = 1. \tag{2.1}$$

We shall consistently minimize the notational distinction between SL and PSL and express an H-isometry by one of the matrices representing it. It is also convenient to pick out the entries of the matrix (2.1) by the functions

$$a_{11}(T) = a, \quad b_{12}(T) = b, \quad c_{21}(T) = c, \quad d_{22}(T) = d.$$

Our T acts on $\mathbb{P}^1(\mathbb{C})$ by

$$T(z) = \frac{az + b}{cz + d}, \qquad T(\infty) = \frac{a}{c}, \tag{2.2}$$

where $a/c = \infty$ when $c = 0$. This action extends to $\mathfrak{U}^3$ by a quarternion formula which is equivalent to Poincaré's original formulae, cf. [1], [2]. We can identify

$\mathbb{C} \times \mathbb{R}$ with the space of quarternions $p = x_1 + x_2 i + x_3 j + 0 \cdot k$ by $(x + iy, h) \leftrightarrow p = x + yi + hj$. Then T acts on $\mathcal{U}^3$ by

$$T(p) = (ap + b) \cdot (cp + d)^{-1}. \tag{2.3}$$

It follows from (1.1) and (2.3) that T is an H-isometry of $\mathcal{U}^3$, and a standard argument shows that every orientation-preserving H-isometry T can be represented in this way. It also follows that T is EH on $\mathcal{U}^3$ exactly when $c_{21}(T) = 0$ and $|a_{11}(T)| = 1$. All EH-transformations are either EH-*translations*

$$A_\eta = A\{\eta\} = \begin{bmatrix} 1 & \eta \\ 0 & 1 \end{bmatrix} : (z, h) \mapsto (z + \eta, h) \qquad (\eta \in \mathbb{C}),$$

or EH-rotations

$$R = \begin{bmatrix} \omega & b \\ 0 & \omega^{-1} \end{bmatrix} : (z, h) \mapsto (\omega^2 z + \omega b, h) \qquad (|\omega| = 1, \omega \neq \pm 1),$$

which E-rotate $\mathcal{U}^3$ about the EH-line $ax(R)$ whose finite E-endpoint is

$$ax_0(R) = \frac{b}{\omega^{-1} - \omega}.$$

If $c_{21}(T) \neq 0$, then T has an isometric circle $I_0(T) \subset \mathbb{C}$ where $I_0(T)$ is the locus $|cz + d| = 1$. The isometric sphere $I(T)$ is the H-plane whose E-boundary is $I_0(T)$. The specific formulae for radius and center are

$$rd(T) = rd(T^{-1}) = \frac{1}{|c|}, \quad cn(T) = -\frac{d}{c}, \quad cn(T^{-1}) = \frac{a}{c}.$$

We fix a definite choice for the reflecting plane $Ref(T)$ (with E-boundary the *reflecting line* $Ref_0(T)$) by taking the action of T to be the product of inversion in $I(T)$, reflection in $Ref(T)$, and EH-translation (not rotation) of $I(T)$ on $I(T^{-1})$.

To identity the group of orientation-preserving H-isometries of $\mathcal{B}^3$ with PSL($\mathbb{C}$) we define a conformal map F: $\mathcal{U}^3 \to \mathcal{B}^3$ using the above quarternionic representation of $\mathcal{U}^3$ and an identification of $\mathbb{R}^3$ with another space of quaternions:

$$(x_1, x_2, x_3) \leftrightarrow q = x_1 + x_2 j + x_3 k.$$

Then

$$F: p \mapsto (p - j)(p + j)^{-1} = q \qquad (p \in \mathcal{U}^3)$$

has inverse

$$F^{-1}: q \mapsto (1 - q)^{-1}(1 + q)j = p \qquad (q \in \mathcal{B}^3).$$

Note that F sends $(0, 1) = j$ to $0 \in \mathcal{B}^3$ and F extends to the spheres at infinity so that $F(\infty) = (1, 0, 0) = 1$. We transfer the action of T on $\overline{\mathcal{U}}^3$ to an action on $\overline{\mathcal{B}}^3$ by

$$T(q) := F \circ T \circ F^{-1}(q).$$

The Poincaré file computes $rd(T)$ and $cn(T^{\pm 1})$ for $\mathcal{B}^3$ in a straightforward manner, but leaves $Ref(T)$ undefined because $Ref(T)$ is only used for Calcomp plots of Ford domains in $\mathcal{U}^3$.

Poincaré uses two criteria for indiscreteness. The first is Shimizu's Lemma, a special case of Jørgensen's Inequality, which asserts that

$$\left\langle \begin{bmatrix} 1 & \eta \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\rangle \qquad (ad - bc = 1)$$

is indiscrete when $0 < |\eta c| < 1$, cf. [2]. (We have been told that Shimizu's Lemma was known earlier, notably by H. Petersson, but on looking into this we did not find an account giving an explicit criterion for indiscreteness.) The other criterion is Jørgensen's Inequality, which asserts that $\langle X, Y \rangle \subset \mathrm{SL}(\mathbb{C})$ is indiscrete when

$$|tr(X)^2 - 4| + |tr(XYX^{-1}Y^{-1}) - 2| < 1,$$

except when $\langle X, Y \rangle$ is an elementary group of three explicitly noted types; cf. [6]. Whenever there is a choice, Poincaré relies solely on Shimizu's Lemma because it is easier to manage.

In using Poincaré one first gives it values for several parameters that determine the type of groups to be considered. The input data for each group G is a collection $\{S_j\}$ of unimodular matrices that generates G. The program then attempts to set up a Ford domain $\mathfrak{D}$ in $\mathfrak{U}^3$ or $\mathfrak{B}^3$ for G, and when it thinks it has succeeded it sorts the edges of $\mathfrak{D}$ into cycles and works out the non-EH-cycle relations. These, together with data for the non-EH-side pairing transformations $\{T_k\}$ of $\mathfrak{D}$ are then sent to the output routines. The official record of the calculation is the printout. This includes a list of the $\{T_k\}$ expressed as words on the $\{S_j\}$, $rd(T)$ and $cn(T^{\pm 1})$, and the matrix entries of T, for each $T \in \{T_k\}$. Then comes a count of the number of non-EH-edges, and finally the non-EH-cycle relations from $\mathfrak{D}$. Each torsion-free cycle which does not admit the closing trick is noted. If G turned out to be indiscrete, some information about the details may be given. Poincaré also allows the option of producing a Calcomp plot of the orthogonal projection of $\mathfrak{D}$ on $\mathbb{C}$ for $\mathfrak{U}^3$ or an orthogonal projection of $\mathfrak{D}$ on some plane for $\mathfrak{B}^3$. The plots for $\mathfrak{U}^3$ are usable as working diagrams that give insight into the action of G, they use three colors to make them easy to interpret, and many are rather pleasant works of art. The plots for $\mathfrak{B}^3$ are a recent development that is not yet complete, but it seems that they will have to be regarded solely as artwork because they are too difficult to use as working diagrams.

The large size of the output per group means that Parkinson's Laws usually will apply before the practical and theoretical limitations do. The only current theoretical restriction on the gorup G is that it must admit a Ford domain $\mathfrak{D}$ with only a finite number of sides, i.e. be *geometrically finite*. Two simple examples of *geometrically infinite* groups are

$$\left\langle \begin{bmatrix} 1+\omega & 1 \\ \omega & 1 \end{bmatrix}, \begin{bmatrix} 1+\omega & -1 \\ -\omega & 1 \end{bmatrix} \right\rangle, \quad \text{where } 1 + \omega + \omega^2 = 0,$$

$$\left\langle A\{2+2i\}, \begin{bmatrix} i & -1 \\ 1-i & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1-i & i \end{bmatrix} \right\rangle, \quad \text{where } 1 + i^2 = 0.$$

These are the J-groups of [10], after T. Jørgensen, who found the first examples of this type. Feeding J-groups to Poincaré would lead to disaster, and yet a casual inspection of these input matrices would not arouse suspicion. There are also certain avoidable restrictions on the subgroup G_{EH} of a group G that Poincaré accepts. The system was originally intended for parabolic representations of knot groups, cf. [8], [9], [10], [11], for which one wants G_{EH} to be allowed to contain two independent EH-translations and an EH-involution, using the model $\mathfrak{U}^3$. The very considerable programming effort needed to provide for this involution discouraged us from providing for EH-rotations of higher order. For the ball model, G_{EH} must be trivial.

To avoid these restrictions, replace G by a suitable conjugate XGX^{-1} for which G_{EH} is trivial, and use the ball model.

The practical limitations of running time and array storage space are not especially serious, because Poincaré is rather efficient, and the arrays are so large already that most groups that exceed them are too complicated to be interesting. Of greater moment is the fact that algorithms AD(1) and AD(2) are not part of the system for obvious reasons. The lack of AD(1) means that the user will have to verify the cycle relations himself, and this is where Parkinson's Laws usually come in. For each sufficiently restricted class of input groups for Poincaré one might try to write a batch of subroutines that provide AD(1). We once did this for parabolic representations of 2-bridge knot groups, and the programming effort required suggested that one will need to be very strongly motivated indeed to do this for other such classes. Probably free groups are the most that one should ask for. Instead of AD(2) we used the ordinary floating-point arithmetic for the computer, which had the equivalent of 11, 14, or 16 decimal figures for the machines on which we implemented Poincaré. Such accuracy is not overly generous because the arithmetic calculations of Poincaré lose accuracy at a shocking rate. The system uses the input quantity ε to test for presumed equality of two calculated floating point numbers, and the run will fail when ε is simultaneously too large and too small. In the first important class of groups that we studied via Poincaré we found a simple minded geometrically convergent (cf. [15]) sequence $G^{(n)}$ of discrete groups with a very simple geometric limit $G^{(\infty)}$ that is maximally unfavorable for the Poincaré system if it is based on floating-point arithmetic of any fixed accuracy, cf. [11]. Another good way of getting into accuracy trouble is to try and get near a general point on the boundary of a space of nonrigid Kleinian groups. What will happen is that the Ford domains will get progressively more complicated and the expression of the $\{T_k\}$ as words on the $\{S_j\}$ will get longer. The accuracy is lost in the resulting long chains of matrix multiplications.

Poincaré has three quirks that should be noted. The first is that if G is a proper Kleinian group such that G_{EH} for $\mathfrak{U}^3$ is nontrivial, then the EH-sides of the resulting Ford domain $\mathfrak{D}$ over the regular set of G in $\mathbb{P}^1(\mathbb{C})$ are left undefined because they do not come into consideration. The second is that when G_{EH} is nontrivial, the domain $\mathfrak{D}$ we get may be disconnected. This is difficult to recognize from the printout, and only the Calcomp plots brought this quirk to light. The only way to prevent this would be to add subroutines which check for connectedness after a good $\mathfrak{D}$ has been found, and redefine the set $\{T_k\}$ when it is not. This is contrary to the general flow of activity of the system, so it should only be done for a compelling reason. The third quirk is that when G_{EH} contains an EH-rotation which rotates the face F of $\mathfrak{O}$ on $I(T)$ on itself, Poincaré does not actually choose a wedge on F in defining the domain $\mathfrak{D}$ as described in Section 1. Recall that TRT^{-1} is an EH-rotation rotating the face F' of $\mathfrak{O}$ on $I(T^{-1})$ on itself. Poincaré marks the edges of F, F' as "live" or "ghost" according to a complicated rule, and when it sets up the cycle relations it arranges that only the live edges are used. We chose the marking rule to ensure that the resulting presentation is correct, and we did not try to ensure that the live edges actually are edges of some Ford domain. Perhaps they always are, but we do not know. One cannot easily decide this from a Calcomp plot, because the projections of F, F' appear without any indication of which edges are live.

We conclude this section with an outline of the search procedures to find $\{T_k\}$ and $\mathcal{D}$ given $\{S_j\}$. One first approximation to $\mathcal{D}$ could be to use just the isometric spheres and EH-transformations from $\{S_j\}$, but we often can do better. Suppose that we have a set $\{W_l\}$ of words on $\{S_j\}$ which are likely to be related to $\mathcal{D}$, e.g. relations of G that are known in advance or words found to be of interest from previous computer runs for G. Subroutine BWORDS cuts each word W of $\{W_l\}$ into all possible segments which neither begin nor end with an EH-transformation and sends the matrices for the segments to subroutine TEST which decides whether a matrix might actually contribute something to $\mathcal{D}$. If one knows an element of $P\Gamma L(\mathbb{C})$ (the group of all H-isometries) which normalizes G, but does not belong to G, it can be taken into account in building approximations to $\mathcal{D}$.

The heart of Poincaré is subroutine LELIM, which takes a collection $\{T_k\}$ of matrices and the known generators of G_{EH} and sets up a trial Ford domain $\mathcal{D}$ from them by the rules of Section 1. LELIM was the hardest subroutine to write, and for about 2 years the first runs for each new class of groups fed to Poincaré promptly detected complex errors in LELIM that took a fair bit of trouble to fix. LELIM sets up the lists of edges of $\mathcal{D}$ and associates with the edge e the isometric spheres which meet in the H-line l containing e, the vertices of $\mathcal{D}$ on e, the E-endpoints of l, and the dihedral angle of $\mathcal{D}$ along e. In the process the set $\{T_k\}$ is refined to remove unneeded elements.

The output of LELIM might be sent to subroutine VXCLN which checks for any obvious defects in $\mathcal{D}$ and attempts to set them right. For each vertex v of $\mathcal{D}$ and each $T \in \{T_k^{\pm 1}\}$ such that v lies on $I(T)$, VXCLN computes $T(v)$ and sees whether it is E-inside some other $I(V)$ for V in $\{T_k^{\pm}\}$. If so, v is inside $I(VT)$, and VT can be sent to TEST in building up the next approximation to $\mathcal{D}$. VXCLN actually uses this idea more aggressively, and when it is finished the calculation is returned to LELIM if the list $\{T_k\}$ has been changed. VXCLN may be called several times if our early approximations to $\mathcal{D}$ are bad. Alternatively, the output of LELIM might go to EDGCYC, which tries to set up the non-EH-edge cycles and their relations. If a cycle is found to be incomplete in a usable way, it is stored and later fed to SLUIT which deals with it in analogy with BWORDS. The calculation would then proceed to LELIM and EDGCYC, and perhaps return to SLUIT. When this is finished either the non-EH-cycle relations all seem to be correct, or some limit on recycling has been reached, or the process did not seem to be getting anywhere. The results are then sent to the output routines. To make all this practical and convenient, numerous bookkeeping subroutines are included, and some of these are long and complex. The total length of Poincaré as of July 1981 is 3936 Fortran card images, but this length will definitely change with time.

3. A Modest Example. We illustrate the application of the Poincaré file by one example, a certain nonfree discrete group G generated by three parabolics. To define G let

$$h(y) = y - 3y^2 + 2y^3 - 4y^4 + y^5 - y^6, \tag{3.1}$$

and let ω be the root of $3 + h(y)^2 = 0$, which is the limit of Newton's iteration starting from

$$0.3726019174 + 1.6959107370i. \tag{3.2}$$

Then $G := \langle A, B, C \rangle$ where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ -\omega^2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1-\omega & 1 \\ -\omega^2 & 1+\omega \end{bmatrix}. \tag{3.3}$$

PROPOSITION. *The group G just defined is a geometrically finite Kleinian group which is presented by the following relations*:

$$(AB^{-1}AB^{-1}A^{-1}BA^{-1}CA^{-1}CAC^{-1})^3 = E, \tag{3.4}$$

$$(AB^{-1}A^{-1}BA^{-1}BC^{-1}BCB^{-1}CB^{-1})^3 = E, \tag{3.5}$$

$$(A^{-1}CAC^{-1}AC^{-1}BC^{-1}B^{-1}CB^{-1}C)^3 = E. \tag{3.6}$$

Proof. We begin by verifying that these relations do hold in G. A straightforward calculation in $\mathrm{PSL}(\mathbb{Z}[y])$ modulo $3 + h(y)^2$ shows that any one of these relations holds, say (3.4). Next, notice that if

$$R = \begin{bmatrix} i & -i\omega^{-1} \\ 0 & -i \end{bmatrix}, \qquad S = \begin{bmatrix} 0 & \omega^{-1} \\ -\omega & 1 \end{bmatrix}, \tag{3.7}$$

then $SAS^{-1} = B$, $S^{-1}AS = C$, $SBS^{-1} = C$, and

$$R^2 = E, \quad RAR = A^{-1}, \quad RBR = C^{-1}. \tag{3.8}$$

(These relations depend only on the normal form (3.3) and not on the particular value of ω.) Write the relations (3.4), (3.5), (3.6) as $V_1^3 = V_2^3 = V_3^3 = E$, where V_j is the displayed word, then

$$V_2^{-1} = SV_1S^{-1}, \qquad V_3 = RV_2R.$$

Hence the relations all hold, as asserted.

When we first encountered G all we knew was what we have just proved, and this information was not especially helpful to the search for a Ford domain for G when we were relying on BWORDS and SLUIT without VXCLN. It took several attempts and the analysis of a diagram for an incomplete domain to find the collection $\{W_j\}$ of words listed below which BWORDS could use to generate the list of all non-EH-side pairing transformations for $\mathfrak{D}$. (A recent attempt using only VXCLN and SLUIT without any guidance from the relations of G or the rotation R was immediately successful. The account below is based on an earlier run.)

$$W_1 = BA^{-1}CA^{-1}C, \quad W_2 = CB^{-1}CA^{-1}C, \quad W_3 = BA^{-1}BA^{-1}C,$$
$$W_4 = C^{-1}AC^{-1}BC^{-1}B, \quad W_5 = BA^{-1}BC^{-1}BCB^{-1}CA^{-1}C,$$
$$W_6 = BA^{-1}BC^{-1}BCB^{-1}CB^{-1}AB^{-1}.$$

The EH-rotation R of (3.7) which induces an automorphism of G by conjugation was taken into account by the computer on this run.

We list below the non-EH side-pairing transformations T_j that the computer found for its chosen $\mathfrak{D}$. For each T_j we give, perhaps in abbreviated form, the expression of T_j as a word on A, B, C, and also $cn(T_j)$, $cn(T_j^{-1})$, $rd(T_j)$. These three numbers will be essential for interpreting Figure 1, a diagram of the E-projection on $\mathbb{C}$ of $\mathfrak{D}$. We have not labelled the projected sides of $\mathfrak{D}$ in the diagram because that would clutter it too much. Incidentally, the computer uses a different subscript for T_j^{-1} (when $T_j^2 \neq E$), but we only list the pair T_j, T_j^{-1} once. Hence the missing subscripts, here

the even subscripts, are used for the inverses of the listed transformations. When it got to the cycle transformations it changed notation by using $T_{(-j)}$ for the inverse of a listed transformation. We might have reindexed to make the Poincaré presentation look a little simpler, but we refrained for fear of introducing clerical errors.

n	T_n	$cn(T_n)$	$cn(T_n^{-1})$	$rd(T_n)$
1	B	$-0.301 - 0.139i$	$0.301 + 0.139i$	0.332
3	C	$-0.178 - 0.702i$	$0.425 - 0.423i$	0.332
5	$A^{-1}BA^{-1}C$	$-0.456 - 0.719i$	$-0.421 + 0.156i$	0.278
7	$A^{-1}W_1$	$-0.018 - 1.008i$	$-0.358 + 0.391i$	0.290
9	$CA^{-1}C$	$-0.199 - 0.927i$	$0.446 - 0.198i$	0.227
11	W_2	$-0.294 - 1.185i$	$0.132 - 0.189i$	0.232
13	$B^{-1}CA^{-1}C$	$-0.278 - 1.043i$	$-0.169 - 0.408i$	0.205
15	$BA^{-1}B$	$-0.323 - 0.365i$	$0.323 + 0.365i$	0.227
17	W_3A^{-1}	$0.481 - 0.954i$	$0.142 + 0.445i$	0.290
19	W_4	$-0.011 - 0.207i$	$-0.155 - 1.229i$	0.208
21	$BA^{-1}BC^{-1}$	$0.292 - 0.154i$	$0.402 + 0.480i$	0.205
23	$BA^{-1}BC^{-1}B$	$-0.009 - 0.373i$	$0.417 + 0.623i$	0.232
25	$A^{-1}W_5A^{-1}$	$0.477 - 1.155i$	$-0.353 + 0.593i$	0.232
27	$A^{-1}W_6$	$0.043 + 0.851i$	$-0.290 + 0.851i$	0.333
29	$CB^{-1}CB^{-1}AB^{-1}$	$0.279 + 0.667i$	$0.135 - 0.355i$	0.208
31	$RT_{27}R$	$0.081 - 1.4141$	$0.414 - 1.414i$	0.333

The computer found that the Ford domain generated by the rules of Section 1 from these transformations has 96 non-EH-edges which it arranged into 27 edge cycles. The cycle transformations for these cycles are as follows.

cycle		cycle	
1	$A^{-1}T_1T_{15}^{-1}T_1$	2	$A^{-1}T_3T_5^{-1}A^{-1}T_1$
3	$T_5A^{-1}T_{17}^{-1}T_1$	4	$A^{-1}T_9T_7^{-1}A^{-1}T_1$
5	$T_{13}T_9^{-1}T_1$	6	$T_{19}^{-1}T_{11}^{-1}T_1$
7	$T_{23}^{-1}T_{21}T_1$	8	$A^{-1}T_3T_9^{-1}T_3$
9	$T_7^{-1}T_5A^{-1}T_3$	10	$T_{13}T_{11}^{-1}T_3$
11	$T_{15}^{-1}T_{21}T_3$	12	$A^{-1}T_{17}^{-1}T_{15}A^{-1}T_3$
13	$T_{23}^{-1}T_{29}^{-1}T_3$	14	$T_{11}^{-1}T_{21}^{-1}AT_7$
15	$T_{13}^{-1}T_{15}^{-1}AT_7$	16	$T_{17}^{-1}T_7$
17	$T_{19}T_{23}^{-1}AT_7$	18	$T_{31}^{-1}T_{25}^{-1}T_7$
19	$A^{-1}T_{17}^{-1}T_{21}T_9$	20	$A^{-1}T_{17}^{-1}T_{29}^{-1}T_{11}$
21	$A^{-1}T_{25}^{-1}A^{-1}T_{23}T_{11}$	22	$A^{-1}T_{31}T_{19}T_{11}$
23	$A^{-1}T_{17}^{-1}T_{23}T_{13}$	24	$T_{25}^{-1}T_{27}T_{17}$
25	$T_{29}T_{27}^{-1}A^{-1}T_{23}$	26	T_{27}
27	T_{31}		

The computer also found that the angle sum for each cycle is approximately 2π, except for cycles 16, 26, 27 where the sum is near $2\pi/3$. Hence the presumed Poincaré presentation for G on A, $T_1,\ldots,T_{31}$ sets all cycle transformations to E except for cycles 16, 26, 27 where the relations are

$$(T_{17}^{-1}T_7)^3 = T_{27}^3 = T_{31}^3 = E.$$

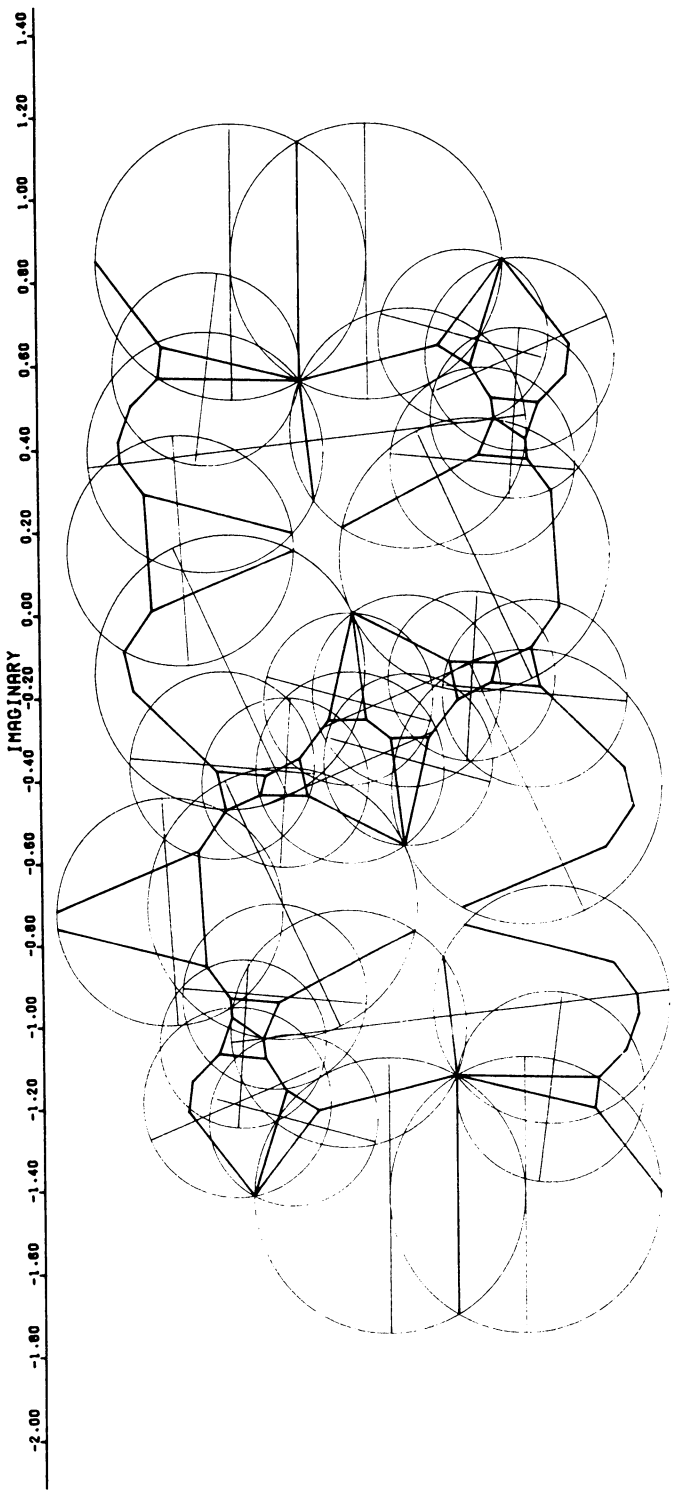

FIGURE 1

The projection of $\mathfrak{D}$ on $\mathbb{C}$. The circles are isometric circles, and the lines bisecting them suggest the reflecting planes. The bold lines are the projections of edges of $\mathfrak{D}$ where two isometric spheres meet on a segment of an H-line outside all other isometric spheres for G. The portion of $\mathfrak{D}_\infty$ in the regular set of G is left undefined, and the axes of the EH-rotations normalizing G should be clear on inspection.

We have reached a situation of the type discussed in Section 1. We have a candidate $\mathcal{D}$ for being a complex with side-pairing, and we must now apply our special arguments to complete the proof that G really is discrete and that $\mathcal{D}$ really is a fundamental domain for it. The first part is to verify that all the relations of the above Poincaré presentation actually hold in G. We begin with the 24 torsion-free relations. It is completely straightforward to use 14 of these relations to express the elements $T_5, \ldots, T_{31}$ as words on A, $T_1 = B$, $T_3 = C$, and the result I obtained when I did this agreed with the list of words in our first table. The remaining 10 relations were then easily shown to hold in a free group F_3, so they are all redundant. Therefore all the torsion-free relations together hold in a free group, and hence in G. Now consider the three torsion relations, beginning with $(T_{17}^{-1}T_7)^3 = E$. Expanding,

$$T_{17}^{-1}T_7 = AC^{-1}AB^{-1}AB^{-1}A^{-1}BA^{-1}CA^{-1}C,$$

and this is a cyclic permutation $AC^{-1}V_1(AC^{-1})^{-1}$ of the word V_1 of relation (3.4). Hence this relation is equivalent to relation (3.4), and we turn to the relation $T_{27}^3 = E$. We have

$$T_{27} = A^{-1}W_6 = A^{-1}BA^{-1}BC^{-1}BCB^{-1}CB^{-1}AB^{-1},$$

which is a cyclic permutation of the word V_2 of (3.5). Hence $T_{27}^3 = E$ is equivalent to (3.5), and similarly $T_{31}^3 = (RT_{27}R)^3 = E$ is equivalent to (3.6). Therefore the Poincaré presentation for G is equivalent to the presentation stated in the Proposition, and all the cycle relations are indeed correct.

The last part is to verify that the sides of $\mathcal{D}$ are paired exactly as they seem to be by the presumed side-pairing transformations. Inspection of the Poincaré presentation shows that the closing trick and the torsion trick work for all cycles except cycle 16, $(T_{17}^{-1}T_7)^3 = E$. This causes somewhat more trouble, and we first note that

$$RT_7R = T_{17}^{-1}$$

is a consequence of (3.8) and the listed expressions for these elements. Hence $T_{17}^{-1}T_7 = (RT_7)^2$, and so $(RT_7)^6 = E$. Because R is an EH-isometry we have

$$I(RT_7) = I(T_7), \quad I\big((RT_7)^{-1}\big) = I\big(T_7^{-1}R\big) = I\big(RT_7^{-1}R\big) = I(T_{17}).$$

Therefore the rotation axis of RT_7 is

$$I(RT_7) \cap I\big((RT_7)^{-1}\big) = I(T_7) \cap I(T_{17}), =: l_1 \text{ say.}$$

It is now clear that T_{17}^{-1} must be an E-isometry along $T_7(l_1)$, so

$$T_7(l_1) = I\big(T_7^{-1}\big) \cap I\big(T_{17}^{-1}\big) =: l_2 \text{ say.}$$

Therefore $T_{17}(l_1) = l_2$, and these H-lines are indeed mapped on each other in the suggested manner. This completes the last of our verifications of the hypotheses of Poincaré's Theorem in the manner of Section 1. The Proposition follows immediately. □

For comparison we also ran the unit ball version of Poincaré for G, and found that the Ford domain in $\mathcal{B}^3$ is considerably more complex than in $\mathcal{U}^3$. There are 62 spherical sides, 54 edge cycles of which 4 are torsion cycles, and 154 edges. Every torsion-free edge cycle has length 3, so the closing trick works for all of them. Because of the extra complexity, the running time for $\mathcal{B}^3$ was 3.35 times longer than for $\mathcal{U}^3$.

The only remaining question about G is the identification of its orbit space $\mathcal{U}^3/G$ with some better known 3-manifold. Matthew Grayson has just proved that $\mathcal{U}^3/G$ is

homeomorphic to the knot complement $\mathbb{R}^3 - k$ where k is the pretzel knot (3, 3, 3), also known as 9_{35}. He did this by a geometric analysis of Figure 1; cf. his paper immediately following this one.

4. Bianchi Groups. In the 1890's L. Bianchi determined Ford domains for an impressive collection of Bianchi groups, cf. Vol. I of his Opere [4]. The starting point of the present investigation was [14] by R. Swan in which much of the basic theory is explained and a small portion of Bianchi's results are used to derive the Poincaré presentations of $\mathcal{G}_d$ for

$$d = -1, -2, -3, -5, -6, -7, -11, -15, -19.$$

Here we shall report on our determinations of Ford domains and their Poincaré presentations of $\mathcal{G}_d$ for all d where

$$-8 \geqslant d \geqslant -37, \qquad d = -43, -67, -163,$$

except the cases listed above that Swan considered. We shall explain our methods for dealing with Bianchi groups, state the results for one example, $d = -43$, and present a table of some rather superficial data derived from the complete results for these 30 groups. We also state the rather meager collection of observed regularities that we found in our computer output. Many of these groups are so complex that their intimate details are not fit for publication. We should perhaps confess that the Bianchi investigation was regarded as a detour from this author's primary interests, and that we did not put much effort into it except for finding the correct results. In particular, we did not complete a proof that our Ford domains are correct in 27 of the 30 cases, and the nature of the gap in our proofs will be indicated below. All these calculations were done on a CDC 7600 computer, and the hardest cases were run on unbudgeted Priority 0 time during July and August 1978.

Because a Bianchi group is obviously a discrete group of a type that admits a Ford domain for its action on $\mathcal{U}^3$, the kind of arguments of Section 1 is inappropriate for the study of these groups. We use instead the classic method, due to Bianchi himself, which is based on the ease of listing the elements of an order $\mathcal{O} = \mathbb{Z}[\omega]$ of a complex quadratic number field $F = \mathbb{Q}(\omega)$, and thereby the matrices of the corresponding Bianchi group $\mathcal{G} = \mathrm{PSL}(\mathcal{O})$. We shall assume that $\mathcal{O}$ is not the full domain of integers for $\sqrt{-1}$, $\sqrt{-3}$, because Picard and Bianchi settled these cases. Then the subgroup $\mathcal{G}_{\mathrm{EH}}$ of EH-isometries, which we now call $\mathcal{G}_\infty$, is easily seen to be generated by $A := A\{1\}$ and $A_\omega = A\{\omega\}$. Let

$$\Delta = \left\{ z \in \mathbb{C} : |\mathrm{Re}(z)| \leqslant \tfrac{1}{2},\ |\mathrm{Im}(z)| \leqslant \tfrac{1}{2}|\mathrm{Im}(\omega)| \right\}.$$

We all normalize the non-EH-side pairing transformations T for our Ford domain $\mathcal{D}$ for $\mathcal{G}$ by requiring that $cn(T^{\pm 1})$ belong to Δ. This is an application of our standard rule for normalizing groups acting on $\mathcal{U}^3$, and it certainly works but it is not optional. The search for $\mathcal{D}$ is based on searching for all $T \in \mathcal{G} - \mathcal{G}_\infty$ with $cn(T^{\pm 1}) \in \Delta$ and $rd(T) >$ constant, say h_0. The computer listed elements of $\mathcal{O}$ in order of increasing absolute value. The list begins with 1, which corresponds to

$$T_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

For the nth number, say c, on this list the first check is whether $|c| > h_0^{-1}$, for if so the search is complete. Otherwise, choose a, then d, on the list so that a/c, $-d/c$

belong to Δ and $|\operatorname{Im}(a/c)|$, $|\operatorname{Im}(d/c)| > 1/2$. Then solve for b such that $ad - bc = 1$ and see whether $b \in \mathfrak{O}$. If not, continue trying values of a and d. If $b \in \mathfrak{O}$, then

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathcal{G},$$

and T is fed to subroutine TEST to see whether it might contribute something to $\mathcal{D}$. It is obvious that such a search will produce a correct Ford domain when h_0 is small enough, and that the correct $\mathcal{D}$ will be found relatively early if the rule for choosing h_0 is rather pessimistic.

The rule for choosing h_0 for all but one, $\mathcal{G}_{-33}$, of the Bianchi groups listed above was to determine a good guess $\mathcal{D}'$ for $\mathcal{D}$ on a preliminary run and then take

$$h_0 = \text{E-height of the lowest proper vertex of } \mathcal{D}' - \text{(little bit)}$$

for the final run. We wanted to do this for −33 too, but the lowest vertex was so low (cf. Table 1) that we would have felt guilty about wasting even unbudgeted computer time on something this silly. (I recall that we stopped at $h_0 = 750^{-1/2}$, and that this took $19\frac{1}{2}$ minutes on a CDC 7600. One minute on this computer is equivalent to about one hour on the first computer to run Poincaré.) The main reason why these searches took so long is that we did not take the trouble to reprogram Poincaré to eliminate the long stretches of redundant calculations that the standard version needs for this kind of search. Perhaps if we had, the searches would have gone 2 or 3 times faster. Naturally, a correct domain is obtained when the search is stopped at height h_1, where h_1 is just less than the smallest radius of an isometric sphere needed for the true domain $\mathcal{D}$. If one relies on our subroutines VXCLN and SLUIT, which work on inconsistencies of a trial domain, the search can be stopped at height $h_2 > h_1 > h_0$ when the domain seems to be consistent. Therefore one could get Ford domains and presentations for many more Bianchi groups at the cost of some reprogramming of the system and of a willingness to accept results which are only probable. Incidently, it would have been easy for us to have modified this search to get results for congruence subgroups of Bianchi groups back in 1978, but nobody asked us to do it.

The ideal classes of $\mathfrak{O}$ are associated to ends of the orbit space $\mathcal{M} = \mathcal{M}_d\, \mathcal{U}^3/\mathcal{G}$ in the following manner. Swan [14] calls $z \in \mathbb{C}$ *singular* when for no $\lambda, \mu \in \mathfrak{O}$ such that the ideal (λ, μ) equals $\mathfrak{O}$ do we have $|\mu z + \lambda| < 1$. A singular z belongs to F, and if $z = \alpha/\beta$ where $\alpha, \beta \in \mathfrak{O}$, then z is fixed by the parabolic

$$T = \begin{bmatrix} 1 - \alpha\beta & \alpha^2 \\ -\beta^2 & 1 + \alpha\beta \end{bmatrix} \in \mathcal{G}.$$

Also z corresponds to the ideal $(\alpha, \beta) \subset \mathfrak{O}$. If a singular point z belongs to the closure $\overline{\mathcal{D}}_\infty$, then it represents one end of $\mathcal{U}^3/\mathcal{G}$. Two singular points in the same orbit of $\mathcal{G}$ correspond to ideals in the same class, and conversely. The class of principal ideals corresponds to the sole point ∞ which is fixed by the EH-translations. Further results of Humbert and Swan imply that the H-closure $\mathcal{D}$ is compact except for solid cusps reaching down to the singular points below and up to ∞ above, so that $\mathcal{U}^3/\mathcal{G}$ is $N - l$, where N is a closed orientable 3-manifold, and l is a union of $h(\mathfrak{O})$ disjoint 1-spheres in N, and $h(\mathfrak{O})$ is the class number of $\mathfrak{O}$. These assertions are due to Humbert, Swan, and Serre when $\mathfrak{O}$ is a maximal order, but

Swan's proofs can easily be extended to nonmaximal orders. Incidently, if

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad T^* = \begin{bmatrix} a & -b \\ -c & d \end{bmatrix}$$

belong to $\mathcal{G}$ and $p \in \mathcal{U}^3$, then Swan defines $T(p)$ to be our $T^*(p)$. Because the EH-rotation

$$R := \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} : (z, t) \mapsto (-z, t)$$

normalizes $\mathcal{G}$ by $RTR = T^*$, we all come to the same conclusions.

It follows from these results that when the class number $h(\mathcal{O}) = 1$ the lowest point of the E-closure $\overline{\mathcal{D}}$ is also the lowest proper vertex of $\mathcal{D}$, and so our search automatically produces the correct domain in these cases. In our sample this applies to $\mathcal{G}_d$ for $d = -43, -67, -163$. When $h(\mathcal{O}) > 1$ we have to supplement our search by an examination of the solid cusps reaching down to the (presumed) singular points of $\mathcal{D}_\infty$. These singular points show up conspicuously on our Calcomp plots as points on many isometric circles but not inside any. Swan provided a discussion of the methods for verifying that our presumed $\mathcal{D}$ is correct near each singular point, and in [5] H. Cohn gives a table, based on [13], of arithmetic data for $\mathcal{O}$ used in the verifications. Once our $\mathcal{D}$ is proved correct in the solid cusps reading down to singular points, our setup implies that it is correct everywhere.

There are other ways to verify that our candidate is correct. One can locate the singular points on our Calcomp plots and verify without difficulty that the isometric circles which seem to pass through them actually do so. Then the method of last resort, which is actually rather easy here, could be used to verify that our $\mathcal{D}$ is a Ford domain of some subgroup $\mathcal{H}$ of $\mathcal{G}$. To prove that the index $[\mathcal{G} : \mathcal{H}]$ is < 1 we could use an estimate of the H-volumes, vol($\mathcal{G}$), vol($\mathcal{H}$), of the respective orbit spaces. For a recent account of the classic lore of H-volumes see J. Milnor's contribution to [15]. On the one hand, vol($\mathcal{G}$) is determined by a formula, due to Humbert, in which the value of $\zeta(2)$ is the only ingredient whose estimation requires serious effort, where $\zeta(s)$ is the Dedekind ζ-function of $\mathcal{O}$. On the other hand, vol($\mathcal{H}$) = H-volume ($\mathcal{D}$) can be approximated with the aid of Milnor's Lobachevsky function provided that we know the dihedral angles of our polyhedron $\mathcal{D}$. But these have been computed and stored because they were needed to bound angle sums, so vol($\mathcal{H}$) can be estimated mechanically. Thus

$$[\mathcal{G} : \mathcal{H}] = \frac{\text{vol}(\mathcal{H})}{\text{vol}(\mathcal{G})}$$

can be estimated accurately enough to determine which integer it must be. Incidently, Humbert's formulae for volume have only been proved for maximal orders. Finally, we suspect that there really is no need to supplement the lowest vertex search because it always gives the right answer.

We now come to the specific results for $\mathcal{G} = \mathcal{G}_d$ where $d = -43$. We have

$$\omega = \frac{1 + \sqrt{-43}}{2}, \qquad \mathcal{O} = \mathbb{Z}[\omega].$$

The generators of $\mathcal{G}$ are A, A_ω, and the following non-EH-transformations, and again missing subscripts were used internally by the computer for the inverses of listed elements.

$$T_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} -\omega & 5 \\ 2 & -1+\omega \end{bmatrix}, \quad T_4 = \begin{bmatrix} \omega & -4 \\ 3 & -1+\omega \end{bmatrix},$$

$$T_6 = \begin{bmatrix} 1+\omega & 3-\omega \\ 3 & -1-\omega \end{bmatrix}, \quad T_7 = \begin{bmatrix} 2-\omega & 2+\omega \\ 3 & -2+\omega \end{bmatrix}, \quad T_8 = \begin{bmatrix} -\omega & -4 \\ 3 & 1-\omega \end{bmatrix}.$$

The computer arranged the 36 non-EH-edges of $\mathcal{D}$ into 14 cycles, and the Poincaré presentation for $\mathcal{G}$ is as follows.

$$T_4^{-1}A_\omega T_2 A_\omega^{-1} A T_8^{-1} T_1 = T_4 T_2^{-1} T_8 T_1 = E,$$
$$T_7 A^{-1} T_8 T_4 = T_8 A T_6 T_4 = E, A \rightleftarrows A_\omega,$$
$$(AT_1)^3 = (T_6 A^{-1} T_1)^3 = (T_7 A^{-1} T_1)^3 = T_2^3 = (A^{-1} T_2)^3 = E,$$
$$(A_\omega^{-1} T_6 A_\omega T_2)^2 = (T_7 A^{-1} T_2)^2 = T_1^2 = T_6^2 = T_7^2 = E.$$

Note that the first two cycle transformations have four non-EH-factors, and so the closing trick fails for them. Figure 2 is a copy of a Calcomp diagram for $\mathcal{D}$.

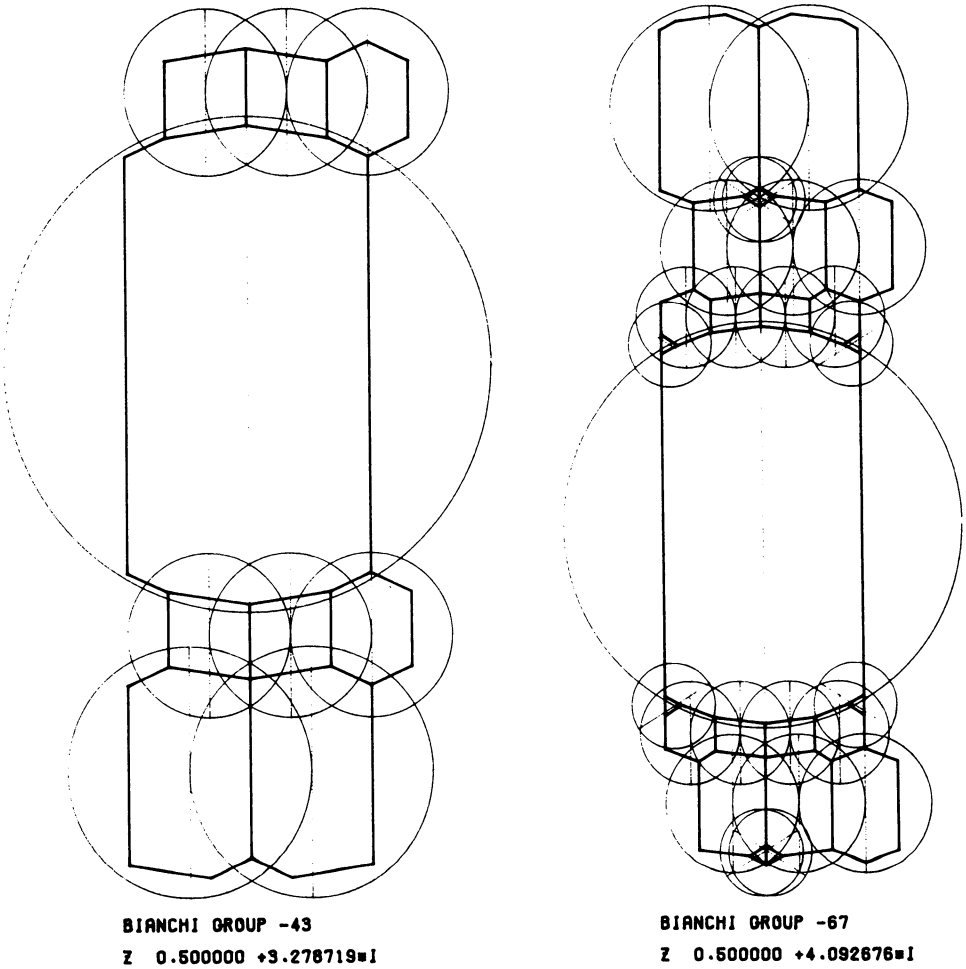

FIGURE 2
The projection of a Ford domain for the Bianchi group $\mathcal{G}_{-43}$.

The following table summarizes the computer printout for our 30 groups. The headings NSS and NEHE are, respectively, the numbers of spherical sides and of non-EH-edges of $\mathcal{D}$ for $\mathcal{G}_d$. We used our standard rule for producing Ford domains, so NSS is an invariant of d but NEHE is often larger than the least possible value for d. The headings NR and NTR are, respectively, the number of non-EH-edge cycles and the number of such cycles where the angle sum is $2\pi/r$ for $r = 2$ or 3 (the number of torsion relations in the Poincaré presentation). Note that the Poincaré presentation will always require the EH-relation $A \rightleftarrows A_\omega$ as in our example. The heading RSHLV means the reciprocal of the square of the height of the lowest proper vertex of $\mathcal{D}$. Add something like 0.1 to this to get the cutoff for $|c_{21}(T)|^2$ in our lowest vertex search. Remark that RSHLV was written in Fortran format F10.5, and we guessed the rational expression for the decimal part whenever we could. In the cases $d = -32, -33$ a continued fraction expansion of this part did not give us a fraction we could trust. The final heading, c_{21}, gives $c_{21}(T)$ in the form $m + n\omega$,

TABLE 1

d	NSS	NEHE	NR	NTR	RSHLV	c_{21}
−8	6	20	7	3	33 1/3	2ω
−9	8	29	11	9	49	2ω
−10	10	39	15	9	65 1/3	2ω
−12	10	34	13	9	65 1/3	2ω
−13	16	64	25	17	85 1/3	2ω
−14	26	84	26	7	108	2ω
−16	18	66	20	9	110 1/4	2ω
−17	32	110	36	15	133 1/3	2ω
−18	26	88	29	11	161 1/3	2ω
−20	40	117	33	7	117 3/5	2ω
−21	30	99	35	19	192	2ω
−22	44	147	49	13	225 1/3	2ω
−23	11	33	10	2	16 8/11	4
−24	34	108	32	11	161 1/3	2ω
−25	42	143	49	25	261 1/3	11
−26	80	228	73	13	300	11
−27	3	11	6	6	13 1/2	2
−28	52	168	53	17	161 1/3	2ω
−29	86	246	77	19	341 1/3	13
−30	62	195	63	21	385 1/3	13
−31	13	47	15	6	24 1/2	$4 - \omega$
−32	100	314	91	9	176.35393	$10 - \omega$
−33	70	230	74	29	1090.39080	4ω
−34	102	317	104	21	481 1/3	15
−35	7	26	8	4	40 1/3	$-1 + 2\omega$
−36	90	257	77	17	225 1/3	2ω
−37	96	314	103	29	533 1/3	17
−43	9	36	14	10	21 1/2	3
−67	25	82	28	12	33 1/2	$-3 + \omega$
−163	99	332	102	20	81 1/2	$4 - \omega$

where $\omega = \omega_d$ for the last of the listed generators. The isometric sphere for the last side pairing transformation has the least radius.

The first regularity in our output that we could not overlook is that for every one of our 30 groups there was at least one relation in the Poincaré presentation which asserted that a product $S_1 \cdots S_n = E$ in $\mathcal{G}_d$, and there were four non-EH-transformations in this product. The closing trick fails for such a cycle. But we never found a relation $(S_1 \cdots S_n)^r = E$ where $\geqslant 5$ of the S_j were non-EH-transformations. This author predicts that the closing trick is likely to fail for a group when there is no advantage in using it, and the Bianchi groups are cited as evidence for this prediction.

Perhaps some observations on our calculations of the fundamental group $\pi_1 \mathcal{M}_d$ are of more general interest. Recall that for a discrete subgroup G of PSL($\mathbb{C}$) acting on $\mathbb{H}^3$, $\pi_1 \mathcal{M}(G) \approx G/G^f =: \mathcal{F}(G)$, where G^f is the smallest normal subgroup of G containing all the elliptic elements. A presentation for $\mathcal{F}(G)$ is derived from one of G by replacing every relation $(S_1 \cdots S_n)^r = E$ by $S_1 \cdots S_r = E$. We applied this to 18 Bianchi groups and, after simplifying, obtained the following results. For the first batch of 12 groups, $\mathcal{F}(\mathcal{G}_d)$ turned out to be free, of rank $r(d)$ say, and the values of $r(d)$ are as follows.

d	-5	-6	-10	-13	-18	-21	-22	-24	-37	-43	-67	-163
$r(d)$	2	2	3	3	5	7	5	9	8	2	3	7

In addition, $\mathcal{F}(\mathcal{G}_d)$ for $-14, -17, -20, -26, -31$ is as follows.

$$d = -14, -17\colon |x_1, \ldots, x_5 \colon x_4 \rightleftarrows x_5|.$$

$$d = -23, -31\colon |x_1, x_2, x_3 \colon x_2 \rightleftarrows x_3|.$$

$$d = -20\colon |x_1, \ldots, x_7 \colon x_6 \rightleftarrows x_7|.$$

$$d = -26\colon |x_1, \ldots, x_8 \colon x_7^{-1}x_8 \rightleftarrows x_3^{-1}x_4,\ x_3x_7^{-1}x_5x_4^{-1} \rightleftarrows x_4x_7^{-1}x_3x_6|.$$

This was pencil and paper work, and we urge anyone wishing to rely on these assertions for any serious purpose to check them first. All the known $\mathcal{F}(\mathcal{G}_d)$ are HNN extensions of free groups, and we wonder if this is always true.

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