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The Determination of the Value of Rado's Noncomputable Function $\Sigma(k)$ for Four-State Turing Machines

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Abstract. The well-defined but noncomputable functions $\Sigma(k)$ and $S(k)$ given by T. Rado as the "score" and "shift number" for the k-state Turing machine "Busy Beaver Game" were previously known only for $k \leqslant 3$. The largest known lower bounds yielding the relations $\Sigma(4) \geqslant 13$ and $S(4) \geqslant 107$, reported by this author, supported the conjecture that these lower bounds are the actual particular values of the functions for $k = 4$.

The four-state case has previously been reduced to solving the blank input tape halting problem of only 5,820 individual machines. In this final stage of the $k = 4$ case, one appears to move into a heuristic level of higher order where it is necessary to treat *each machine* as representing a *distinct theorem*. The remaining set consists of two primary classes in which a machine and its tape are viewed as the representation of a growing string of cellular automata. The proof techniques, embodied in programs, are entirely heuristic, while the inductive proofs, once established by the computer, are completely rigorous and become the key to the proof of the new and original mathematical results: $\Sigma(4) = 13$ and $S(4) = 107$.

"In any case, even though skilled mathematicians and experienced programmers attempted to evaluate $\Sigma(3)$ and $S(3)$, there is no evidence that any known approach will yield the answer, even if we avail ourselves of high-speed computers and elaborate programs. As regards $\Sigma(4)$, $S(4)$, the situation seems to be entirely hopeless at present."—Tibor Rado, 1963.

1. Background and Introduction. The "Busy Beaver Game" was devised by Tibor Rado [8] for the purpose of illustrating the notion of noncomputability. Given the set of Turing machines of exactly k states which operate with the minimum alphabet of two symbols (a space and a mark or 0 and 1) one considers the problem of behavior of these machines on a tape which is initially all blank (all 0's). This is a finite set of machines, there being exactly $(4k + 1)^{2k}$ distinct machines, where, with two table entries per state, each table entry may consist of printing 0 or 1, moving right or left, branching to state $1, 2, \ldots, k$ or simply an entry to declare a halt.*

Since the input tape is blank, each machine faces one of two possible fates: either it eventually halts, or else it continues running forever. After a particular machine

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*Rado treated a halt as a branch to a state "0" within a normal entry, so that in such a case there would be $(4k + 4)^{2k}$ machines.

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halts, a certain number of 1's will be marked upon its tape. This will be the score of that machine. The object of the "game" for a given k states is to find a machine which will produce the largest possible score. This score is referred to as the "Busy Beaver Number" for k states. It defines in the "strongest possible way" an integer function $\Sigma(k)$ which Rado demonstrated to be *noncomputable*.

Related to $\Sigma(k)$ is another noncomputable function $S(k)$ called the *maximum shift number*: the greatest number of moves that a k-state machine with a blank input tape can make before halting.

Rado and others became intrigued with the actual problems of determining as mathematical fact the values of $\Sigma(k)$ and $S(k)$ for reasonably small k. (See Appendix A.) He also proposed this and related problems as important to artificial intelligence [4].

Using computer programs, Lin [5] finally determined that $\Sigma(3) = 6$ and $S(3) = 21$. (See also Lin and Rado [6].) Acceptance of that result in the ordinary manner of seeing a mathematical proof presents difficulties. For this author it was a case of making a completely independent and, as it turns out, a somewhat different attack on the three-state problem [1].

The Four-State Problem. One formidable obstacle to solving the four-state case appears to be the large number of four-state machines. From the expression $(4k + 1)^{2k}$, giving the number of all possible k-state machines, one may exclude the set of k-state machines without any halts giving

$$(4k + 1)^{2k} - (4k)^{2k}.$$

Furthermore, one can also impose the requirement (cf. Lin and Rado [6]) that no machine need be considered which does not have the initial entry (in state 1 scanning 0) to print 1, move right, and enter state 2 ("1 R 2"). This yields the expression

$$(4k + 1)^{2k-1} - (4k)^{2k-1}.$$

While a reasonable number is now produced for $k = 2$ and 3 (217 and 122,458, respectively), the situation still has a doubtful appearance for $k = 4$ (141,903,217) and seems totally impossible for $k = 5$ (282,525,287,122).

The most significant reduction (see only *after the fact*) is obtained by extending the argument for the quintuple $(1, 0, 1, R, 2)$ into a tree generation of all of the quintuples for the machines to be considered (Brady [2]). With the tree generation scheme (in current terminology a form of *back tracking*) coupled to a programmed "solution" to the halting problem, little more than a minute of computer time was required on a second generation computer, generating fewer than 4,000 three-state machines and producing only 27 *holdouts*** for direct inspection. For $k = 4$, approximately 550,000 four-state machines were generated and 5,820 holdouts ultimately remained.

Among the four-state machines, two were discovered [2] which yield the score of 13. Their tables are as follows.

**A *holdout* [8] refers to a Turing machine for which the blank input tape halting problem remains *undecided* after analysis by the computer.

(State)	(Scanned Symbol) 0	1
1	1 *R* 2	0 *R* 3
2	1 *L* 1	1 *R* 1
3	(halt)	1 *R* 4
4	1 *L* 4	0 *L* 2

Score = 13 (96 shifts)

(State)	(Scanned Symbol) 0	1
1	1 *R* 2	1 *L* 2
2	1 *L* 1	0 *L* 3
3	(halt)	1 *L* 4
4	1 *R* 4	0 *R* 1

Score = 13 (107 shifts)

(In order to achieve the scores and shift numbers claimed, the *halt* entry must be replaced by a triple which *prints a mark*, moves, and branches to a zero state following Rado's convention.)

The remaining 5,820 machines were all run for at least 500 moves with none observed to halt, and a 100 machine sample was inspected for a general picture of machine behavior. It was then *conjectured* [2] that $\Sigma(4) = 13$ and $S(4) = 107$.

The remainder of this paper describes the final reduction of the 5,820 holdouts to only 218 through the use of new heuristic computer programs especially devised for the problem. Direct inspection of each of the 218 holdouts from these programs reveals that none will ever halt, and it can be stated as known that $\Sigma(4) = 13$ and $S(4) = 107$.

2. Heuristic Classification of the Machines.

Proofs by Induction. Among the 27 "holdout" machines of three states the following two machines represent in a simple way their general classes of behavior:

	0	1
1	1 *R* 2	0 *L* 3
2	1 *L* 2	1 *R* 1
3	____	1 *L* 1

3-State Holdout No. 6
(Xmas Tree)

	0	1
1	1 *R* 2	0 *L* 3
2	0 *L* 1	0 *R* 2
3	1 *L* 1	____

3-State Holdout No. 2
(Counter)

Machine no. 6 demonstrates an oscillatory pattern as it sweeps across the written tape (Figure 1), growing at each end as it continues the extension of its pattern. S. Lin (private conversation) refers to such machines as "Christmas Trees," an appropriately graphic term which we will use throughout our discussion.

A proof by mathematical induction that the Xmas Tree pattern of machine no. 6 grows continually is not at all difficult, but the behavior is fairly obvious. Machine no. 2 demonstrates a pattern which is considerably less apparent. Its history of operation may be used to establish an inductive proof based upon the observations of its actual behavior shown in Figure 2 and the hypothesis that after $3 \cdot 2^{n-1} - 2n - 7$ moves the machine will arrive at the configuration

$$0\,\underset{\underset{3}{\wedge}}{0}\,\overbrace{1\,1\,1\,\cdots\,1\,1\,1}^{2n-1\ \textit{marks}}\,0\,0$$

having started on a blank segment of tape of length $2n + 1$ squares.

```
0000000000000  1        0001111110000  21       0011111010000  41
     * 1                     * 2                     * 1
0000010000000  2        0001111110000  22       0011101010000  42
     * 2                     * 1                     * 3
0000011000000  3        0001111100000  23       0011101010000  43
     * 2                     * 3                     * 1
0000011000000  4        0001111100000  24       0010101010000  44
     * 1                     * 1                   * 3
0000010000000  5        0001110100000  25       0010101010000  45
     * 3                     * 3                  * 1
0000010000000  6        0001110100000  26       0110101010000  46
    * 1                     * 1                   * 2
0000110000000  7        0001010100000  27       0110101010000  47
     * 2                     * 3                     * 1
0000110000000  8        0001010100000  28       0111101010000  48
     * 1                   * 1                     * 2
0000111000000  9        0011010100000  29       0111101010000  49
     * 2                     * 2                     * 1
0000111100000  10       0011010100000  30       0111111010000  50
     * 2                     * 1                     * 2
0000111100000  11       0011110100000  31       0111111010000  51
     * 1                     * 2                     * 1
0000111000000  12       0011110100000  32       0111111110000  52
     * 3                     * 1                     * 2
0000111000000  13       0011111100000  33       0111111110000  53
     * 1                     * 2                     * 1
0000101000000  14       0011111100000  34       0111111111000  54
     * 3                     * 1                     * 2
0000101000000  15       0011111110000  35       0111111111100  55
   * 1                       * 2                     * 2
0001101000000  16       0011111111000  36       0111111111100  56
     * 2                     * 2                     * 1
0001101000000  17       0011111111000  37       0111111111000  57
     * 1                     * 1                     * 3
0001111000000  18       0011111110000  38       0111111111000  58
     * 2                     * 3                     * 1
0001111000000  19       0011111110000  39       0111111101000  59
     * 1                     * 1                     * 3
0001111100000  20       0011111010000  40       0111111101000  60
       * 2                     * 3                     * 1
```

FIGURE 1. *Behavior of failure number six* (*Xmas Tree*)

After the inductive proof was devised it was observed [1] that machine no. 2 is in essence a *binary counter*. The observation was triggered by the similarity of the expression $3 \cdot 2^{n-1} - 2n - 7$ to the computed rate of growth of a simple binary counter which had been constructed as an exercise.

Studying the tape configurations in Figure 2 one can see that a "zero" is represented by the code pair (0 0), while a "one" is represented by (1 0) as the machine counts continuously, adding "one" on the right and accumulating to the left. During the carry sweep (to the left), a "one" goes through an intermediate code (1 1), finally being converted to "zero" on the return sweep to the right as seen in the 49th "move" (Figure 2):

$$(0\,0)\ (0\,1)\ (0\,0)\ (0\,1)\ \underset{\hat{}}{(1\,1)}\ (0\,0)$$

$$2$$

It appeared that what was needed for handling this problem was some sort of "mechanical induction" capability. Techniques involving symbolic and algebraic manipulation were considered, but the algebraic analysis of machines did not seem to be a sufficient approach. Algebraic characteristics did, however, turn out to yield a fruitful method for first order identification.

```
0000000000000  1        0000001000000  21       0000100011100  41
        * 1                      * 1                    * 2
0000000000100  2        0000001000100  22       0000100001100  42
        * 2                      * 2                    * 2
0000000000100  3        0000001000100  23       0000100000100  43
        * 1                      * 1                    * 2
0000000001100  4        0000001001100  24       0000100000000  44
        * 3                      * 3                    * 2
0000000001100  5        0000001001100  25       0000100000000  45
        * 1                      * 1                    * 1
0000000011100  6        0000001011100  26       0000100000100  46
        * 2                      * 2                    * 2
0000000010100  7        0000001010100  27       0000100000100  47
        * 2                      * 2                    * 1
0000000010000  8        0000001010000  28       0000100000100  48
        * 2                      * 2                    * 3
0000000010000  9        0000001010000  29       0000100001100  49
        * 1                      * 1                    * 1
0000000010100  10       0000001010100  30       0000100011100  50
        * 2                      * 2                    * 2
0000000010100  11       0000001010100  31       0000100010100  51
        * 1                      * 1                    * 2
0000000010100  12       0000001010100  32       0000100010000  52
        * 3                      * 3                    * 2
0000000011100  13       0000001011100  33       0000100010000  53
        * 1                      * 1                    * 1
0000000011100  14       0000001011100  34       0000100010100  54
        * 3                      * 3                    * 2
0000000111100  15       0000001111100  35       0000100010100  55
        * 1                      * 1                    * 1
0000001111100  16       0000001111100  36       0000100010100  56
        * 2                      * 3                    * 3
0000001011100  17       0000011111100  37       0000100011100  57
        * 2                      * 1                    * 1
0000001001100  18       0000111111100  38       0000100011100  58
        * 2                      * 2                    * 3
0000001000100  19       0000101111100  39       0000100111100  59
        * 2                      * 2                    * 1
0000001000000  20       0000100111100  40       0000101111100  60
        * 2                      * 2                    * 2
```

FIGURE 2. *Behavior of failure number two (Counter)*

Rates of Growth of the Scanned Tape. For the simple "sweep across" or Xmas Tree pattern, the rate of growth is inversely proportional to the tape excursion, y, plus a constant,

$$dy/dx = 1/(ky + c_1),$$

yielding the relation

$$x = ay^2 + by + c. \tag{1}$$

For a counter, experience has shown that a sequence of the form

$$a2^n + bn + c \tag{2}$$

identifies the characteristic extremum. In particular, for a simple binary counter (one bit code) it can be seen that the sequence identifying the start of counting on an exact power of two will have the relationship

$$S_n = 2S_{n-1} + 2.$$

Through integration of the differential equation

$$dx/(x + 2) = dy,$$

suggested by the difference equation

$$S_n - S_{n-1} = S_{n-1} + 2,$$

one obtains

$$x = a2^y - 2.$$

(The same form can be obtained using known difference equation methods.)

From our derived expressions it can be seen that the characteristic (or *necessary* feature) identification is readily discernable through a table of differences formed from the sequence of maximum excursions occurring on the left (at l_i moves) and on the right (r_i moves) during machine operation.

If the second differences are constant, the maximum excursions follow a parabolic sequence (1) which is indicative of the Xmas Tree. A table of differences formed from (2) would yield

$$D^1 = \left(a2^{n+1} + b(n+1) + c\right) - \left(a2^n + bn + c\right) = a2^n + b$$

and

$$D^2 = \left(a2^{n+1} + b\right) - \left(a2^n + b\right) = a2^n.$$

Therefore, if the second differences yield a simple geometric sequence progressing by a power of two, a binary counter is *indicated* (but not *demonstrated*) so long as the extreme on the opposite side is constant.

A Heuristic Filter. Observations of a small sample of Xmas Trees and Counters will make apparent the fact that the table of differences is applicable only to an *enveloping* sequence, since machine "gyrations" on the growing edge of the traversed region of tape can be quite complex.

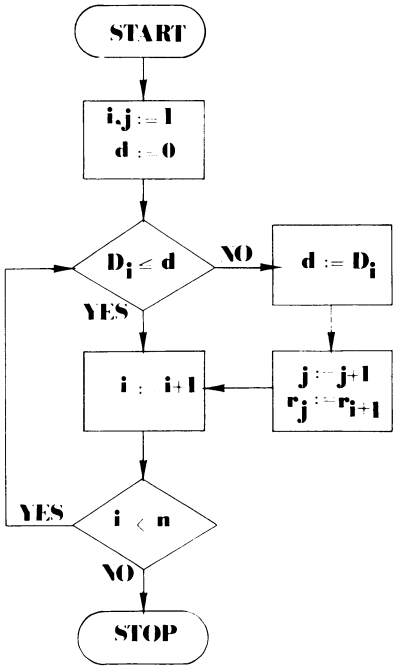


FIGURE 3. *Sequence filtering algorithm*

Various techniques for extracting the envelope were considered, including even numerical curve fitting. Ultimately, the following simple algorithm for extracting an enveloping ascending sequence was devised: Start with a sequence of maximum excursions (for instance, r_i) and form a table of first differences ($D_i = r_{i+1} - r_i$).

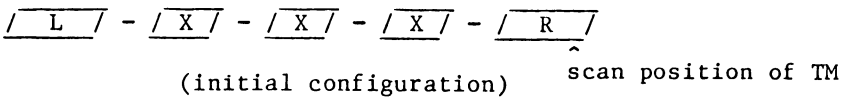
Then trace forward through the table of first differences extracting a strictly increasing sequence while discarding all the rest. The extracted sequence can then be used to form the table of first and second differences which we want to inspect. A flowchart for the algorithm is shown in Figure 3.

The filtering scheme and difference computations were embodied in a program along with some other parameter dependent heuristics and applied to the 5,820 4-state holdouts. This computer program separated the holdouts into three tentative or probable classes: XMAS (4,849); COUNTER (402); and UNKNOWN (569).

3. The Cellular Representation of Xmas Trees.

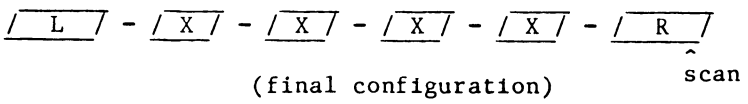
The Basic Picture. Initial attempts to handle the Xmas Trees using a tediously constructed program were unsatisfactory, so after the failure of this first program to make a significant reduction, a careful study was made of the resistant holdouts of the Xmas Tree sample. From the thought that had already gone into the Counter problem, a *cellular* picture had evolved in "hardware" terms. It then became apparent that (1) a cellular representation for Xmas Trees also had merit, and (2) the inductive pattern should be sought relative to the growing chain of *perceived* cells and not fixed absolutely to the tape of the Turing machines.

A Xmas Tree can be represented by a chain of tape segments (cells) which go through metamorphic changes in the process of reproducing themselves. At some point in the metamorphic and reproductive cycle, say, when the machine has reached a new right extremum in its tape excursion, we observe that the tape is made up of a left segment, a right segment, and a sequence of identical interior segments:



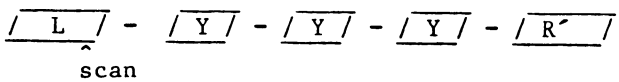
As we will learn later, it is not so important that we identify the L and R cells as it is for us to identify the X cell. (The reader may find it helpful to observe the behavior of the four-state example no. 1 in Appendix B.)

If we operate the machine until it reaches the next major right extremum (determined by the filtering process) we will see that a new X cell has been produced:



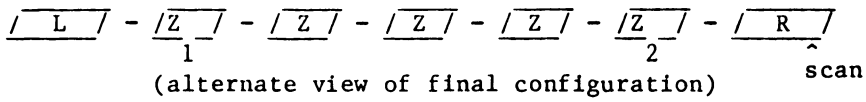
What happens in the general case is that the L and R segments act as parent cells and generate partial cells Z_1 and Z_2 which, when juxtaposed, form an X cell.

Starting with the initial configuration, as the machine proceeds to process the right segment, it eventually emerges, sweeping from right to left converting the X cells to Y cells, their metamorphic successors.



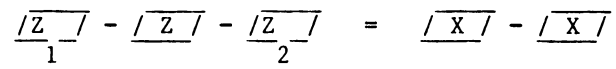
The machine then enters the left segment, takes it through its reproductive transformation and emerges from where it entered, sweeping from left to right converting

the Y cells to Z cells. The machine enters the right segment and eventually reaches a new extreme at which point we observe the chain



(alternate view of final configuration)

The Z cells, which were the metamorphic successors to the Y cells, are in fact complementary images of the original X cells, so that the cell wall boundaries simply shift if (as is usually the case)



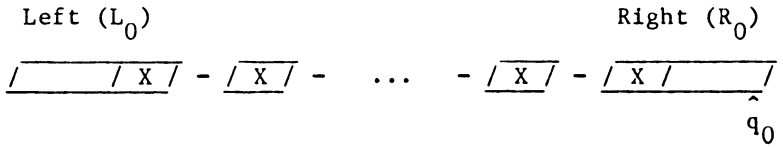
It is apparent by inductive extension (since the X and Z cells are the same size) that a chain of n Z cells,

bounded by a Z_1 cell on the left and a Z_2 cell on the right is identical to a chain of $n + 1$ X cells:

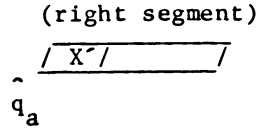
The actual situation, however, is made much more complicated by the fact that the metamorphic transformation of a cell is intertwined with the transformation of its neighboring cells. As will be seen, this interdependence is (fortunately) not inextricable.

Furthermore, the transformations of the L and R segments are frequently not simply described, but problems with their behavior are easily circumvented by incorporating several X cells into these end segments to act as buffers.

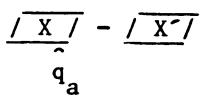
The Detailed Metamorphic Cycle. Assume, then, that at least one X cell is in the right end of the left segment, L_0, and at least one X cell is in the left end of the right segment, R_0:



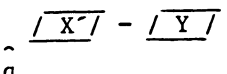
Let the controlling Turing machine be in state q_0 at the extreme right. Operate the machine until it emerges from the right segment to the left. Call the emerging state q_a:



The X cell on the left end will now be in an intermediate metamorphic stage, X'. We must next demonstrate that the machine will operate on a two cell segment

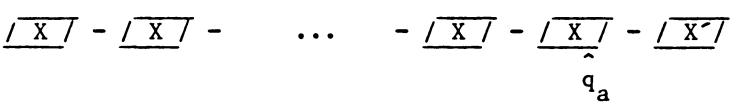


resulting in the transformation to the segment

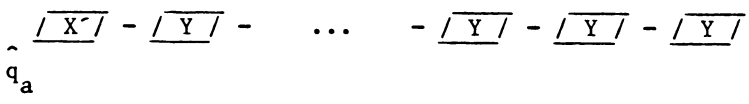


with the machine emerging on the left in the same state q_a.

If this has occurred as described, then obviously by extension we can see that any chain of X cells entered on the right in the same state q_a with an X' cell to the right of the chain

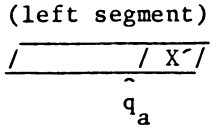


must undergo transformation into a string of Y cells with an X' cell on the left and the controlling machine emerging in state q_a:

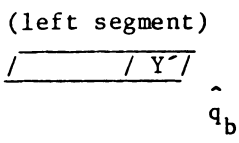


In other words, a chain of X cells of any length is transparent to the passage of the controlling machine as it travels from the right segment to the left segment.

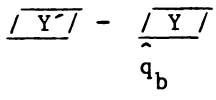
As our next step in the inductive proof, we operate on a partially changed left segment (its X cell converted to an X' cell) to the left of the X' cell in state q_a:



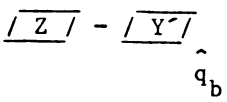
We run the machine until it emerges to the right. Call the emerging state q_b. The cell on the right end will now be in an intermediate metamorphic stage Y':



We must next show that a chain of Y cells is transparent to passage of the machine to the right. Forming a segment consisting of a Y' and a Y cell, we demonstrate that

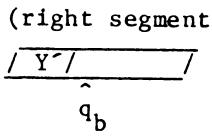


becomes

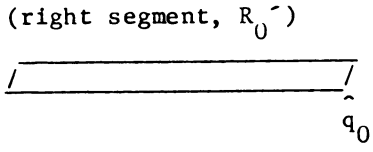


by simulating the controlling machine under the implied restrictions.

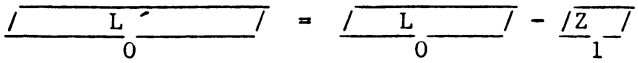
As the final simulation step in the establishment of our proposition, we operate on a partially changed right segment (the right segment as we left it but with its Y cell converted to a Y' cell) positioned to the right of its Y' cell in state q_b:



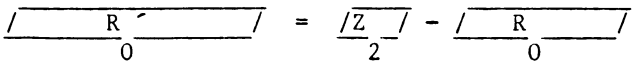
We run this configuration until the machine reaches the new extremum on the right in state q_0:



The metamorphosed left segment with its Y' cell converted to a Z cell (which we have proved will happen) we will call L_0'. The right segment above we will call R_0'. We determine that under these transformations "birth" has taken place. I.e.,

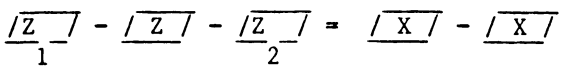


and



—the L_0' segment is identical to the original L_0 with a partial cell Z_1 on the right, while the R_0' segment is identical to the original R_0 with a partial cell Z_2 on the left.

As already mentioned, we must finally show that



The original premise is the existence of the initial configuration. Since this was already established by the feature extraction process, the inductive proof for the particular machine is then complete.

It should be pointed out that should any particular step fail there is no proof. There are also heuristically established limits placed upon any simulation to prevent the possible occurrence of a loop. Wherever we have in effect said "until anticipated condition arises" it should be assumed that we mean "until anticipated condition arises or limiting condition occurs." The occurrence of a limiting condition implies failure of the proof procedure for a given Turing machine.

Heuristic Extraction of Xmas Tree Features. Extraction of the features of a Xmas Tree is a simple process which was arrived at after several experiments.

The machine is run for a thousand moves or so, and the enveloping sequences of extreme (max. and min.) excursions are filtered out. The last extreme point (on the right) occurs at what we shall refer to as i_2 moves (shifts). The preceding filtered extreme occurs at i_1 moves.

We rerun the machine from scratch to the point i_1. We then "snip" the tape (Figure 4) in the middle of its range of excursion creating and saving the segments L_0 and R_0. We then run the machine from i_1 to i_2. We match L_0 from the left on the new tape configuration at i_2 and R_0 from the right. (If no match occurs, the test fails at this point.) The residual segment in the center of the tape after removal of L_0 and R_0 is taken to be the X cell.

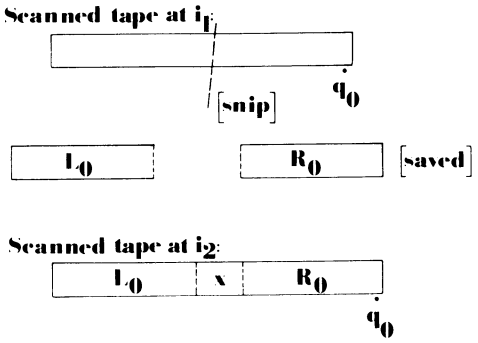


FIGURE 4. *Extraction of Xmas Tree cells*

It is important to note that while the success of the extraction process is necessary to the proof, it is not sufficient. The precise demonstration of the independence of the L_0 and R_0 transformation must be made along with the demonstration of the transparency of the chain of X cells to passage of the machine back and forth through them. These demonstrations, as already described, are then made step by step from move i_1 to move i_2.

Xmas Tree Variants. Two variants of Xmas Trees occur which were easily accomodated by minor modifications in the extraction heuristics and the mechanical proofs. These we will refer to as Leaning Trees and Trees with a Shadow. (See Appendix B.) In both of these variants, the segment R_0 gives birth to an extra X cell while the L_0 segment consumes a portion of an X cell. The Tree with a Shadow in addition remains attached to a trail of skeletons left behind.

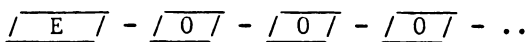
The other variant is the Xmas Tree with a double sweep, which we will refer to as the Alternating Xmas Tree. This variant was handled in a manner similar to that used for the standard Xmas Tree. The Alternating Xmas Tree becomes a generalization of the Xmas Tree, and the mechanical proof for Alternating Xmas Trees is applicable to Xmas Trees.

The minor variants analogous to the Leaning Trees and the Trees with Shadow show up in the Alternating Xmas Trees. Minor changes were made in the programs to accomodate the (alternating) Leaning Trees. Only eight "Alternating Trees with Shadow" appeared in the 218 final holdouts.

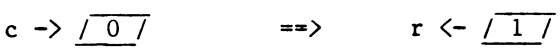
From the set of 4,829 holdouts tentatively separated into XMAS, 4,841 were proved never to halt, while 8 holdouts remained. From the set of 569 holdouts tentatively separated as UNKNOWN, 389 were proved never to halt, while 180 holdouts remained.

4. Cellular Representation of Counters.

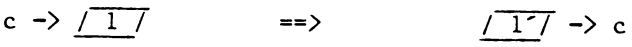
The Hardware Analog. For a Counter one can picture a semi-infinite chain of registers, initially all in the ZERO state,



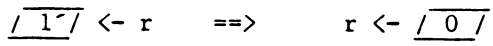
driven from an END cell or register labeled E. Carry and return signals are propagated through this chain. When a carry signal enters a cell or register in the ZERO state, the cell goes into the ONE state and sends back a return signal:



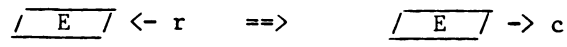
When a carry enters a cell in the ONE state, the cell goes into the transition or 1′ state and sends the carry on:



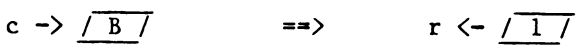
When a return signal enters a cell (in the 1′ state), the cell goes into the ZERO state and the return signal is continued back:



Finally, when the return signal is detected by the END cell, the carry is regenerated and sent into the register chain:



Another possible cell state (which does arise) is the blank or unwritten state. In such a cell, the code for zero is not all Turing machine 0's. In this case, a blank (B) cell is first converted to a ONE:



For input signals of interest we may now look at the complete state diagram of a cell with state B, 0, 1′ and 1 shown in Figure 5(a).

Heuristic Extraction of Counter Components. The initial difficulty in analyzing a counter rests completely in the recognition of its components. Cell size is taken to be the increase in tape excursion from one filtered extreme to the next. At an extreme

point we have a blank cell, B, being converted into a ONE cell:

(i_1 moves)--

$$\overline{/\ E\ /} - \overline{/\ 1'\ /} - \overline{/\ 1'\ /} - \quad \ldots \quad - \overline{/\ 1'\ /} - \overline{/\ B\ /} - \overline{/\ B\ /} - \ldots$$
$$\hat{q}_c$$

At the first relative minimum point (prior to reaching i_2, the next extreme of carry propagation) the end cell, E, is in the process of regenerating (or reflecting) a carry signal:

$$\overline{/\ E'\ /} - \overline{/\ 0\ /} - \overline{/\ 0\ /} - \ \ldots \ - \overline{/\ 0\ /} - \overline{/\ 1\ /} - \overline{/\ B\ /} - \ \ldots$$
$$q_m$$

From the tape configuration at this point we attempt to extract the code for a ONE and for a ZERO, based upon cell-size alignment and the unfiltered excursion extreme occurring between i_1 moves and this point of operation. From alignment of the configuration at i_1, we extract the code for $1'$, and tentatively for E. We augment the tentative end cell with two $1'$ cells for a margin of safety. (Once a counter is in operation, it is evident that the addition of a fixed number of counter cells to the END cell does not change its defined operation as depicted in the state diagram of Figure 5(b).)

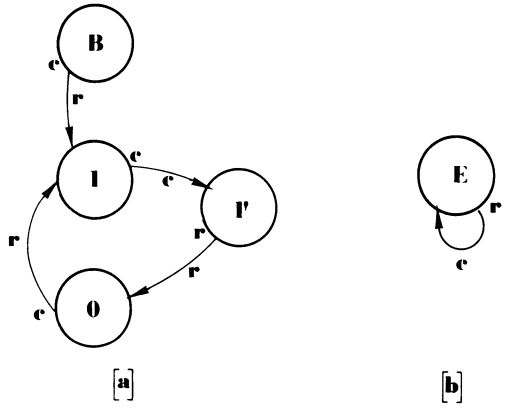


FIGURE 5. (a) and (b) *State diagram of counter cells*

With the alignment of cells now determined, we snip the chain of the i_1 configuration at a cell boundary:

(i_1 moves)--

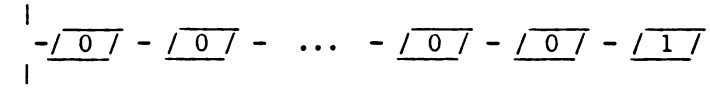
$$\overline{/\ E\ /} - \overline{/\ 1'\ /} - \ \ldots \ - \overline{/\ 1'\ /} - |\!- \overline{/\ 1'\ /} - \ \ldots \ - \overline{/\ 1'\ /} - \overline{/\ B\ /}$$
$$\text{(snip)} \qquad\qquad \hat{q}_c$$

and run the partial configuration on the right until the controlling Turing machine emerges at the cut:

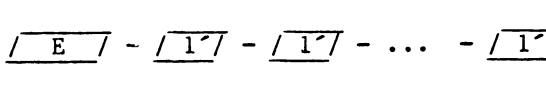
$$|\ - \overline{/\ 0\ /} - \ \ldots \ - \overline{/\ 0\ /} - \overline{/\ 1\ /}$$
$$|\hat{q}_r$$

The emerging state at the cut, q_r, is assumed to be the return signal.

As the first step in our inductive proof, we now establish the premise that a counting state exists, i.e. the right segment is indeed a chain of ZEROS with a ONE on the right:

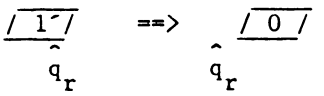


and the left segment is a chain of transitional cells (1′) with the END cell on the left:

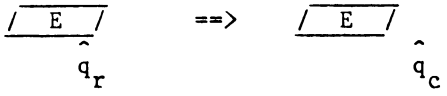


Final Steps in the Proof. We are finally left with the straightforward task of demonstrating the state conversions of our heuristically extracted cells.

First we run the Turing machine on the restricted configuration representing a 1′ cell, starting on the right end in state q_r to demonstrate that the Turing machine emerges from the left end in state q_r, and that the tape configuration is identical with the extracted code for ZERO:

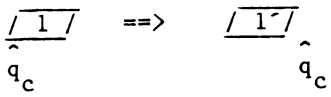


Next we demonstrate that

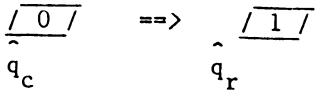


(This determines q_c, the carry signal, but we must show that E is unchanged.)

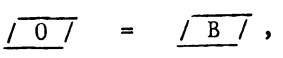
Then we demonstrate that the carry will be propagated by ONE cells:



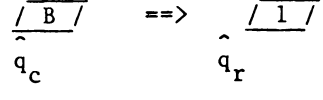
Finally, we demonstrate that ZERO reflects the carry as it changes to a ONE:



We must also determine whether or not



and if the ZERO code differs from the BLANK code, we must also demonstrate that



The Counter proving program was run on the set of 402 machines which had been classified as counters. All but 73 machines were demonstrated never to halt.

A Modification to the Program. At this point it was no surprise to observe an interdependence of counter cells, i.e. the conversion from one cell state to the next can be dependent upon the preceding cell. Such behavior is exactly what occurs with what one could term "two-shot" carry and return signals.

Another difficulty which plagues counters is their rather slow rate of growth. It was not always possible to extract the features and establish existence of a counting state in only 1,000 moves!

The new cell conversion tests were incorporated into the program. The modified counter program was run with a 2,000 move limit on the 73 Counter holdouts. It found 26 more counters leaving 47 holdouts. But, additionally, the modified counter program was run on the miscellaneous holdouts from the Xmas Tree runs. It found 17 Counters. The Counter proofs were therefore effective in eliminating 372 ma chines. *A total of* 218 *holdouts remained from all programs.*

Other Counter Behavior. Several other counter variants were observed among the holdouts, but it was decided that further programming to devise the necessary cell extraction heuristics was not worth the trouble.

5. Final Steps in the Solution of the Four-State Problem.

The Computer Reduction Runs. A total of eight programs were used directly in the process of reducing the final set of four-state holdouts. (See Figure 6.) BBFILT was used to separate heuristically the 5,820 holdouts into "Xmas Trees", "Counters," and "Unknown," while BBFXX, a modification of BBFILT separated the "Alternating Xmas Trees" from the "Unknown" set.

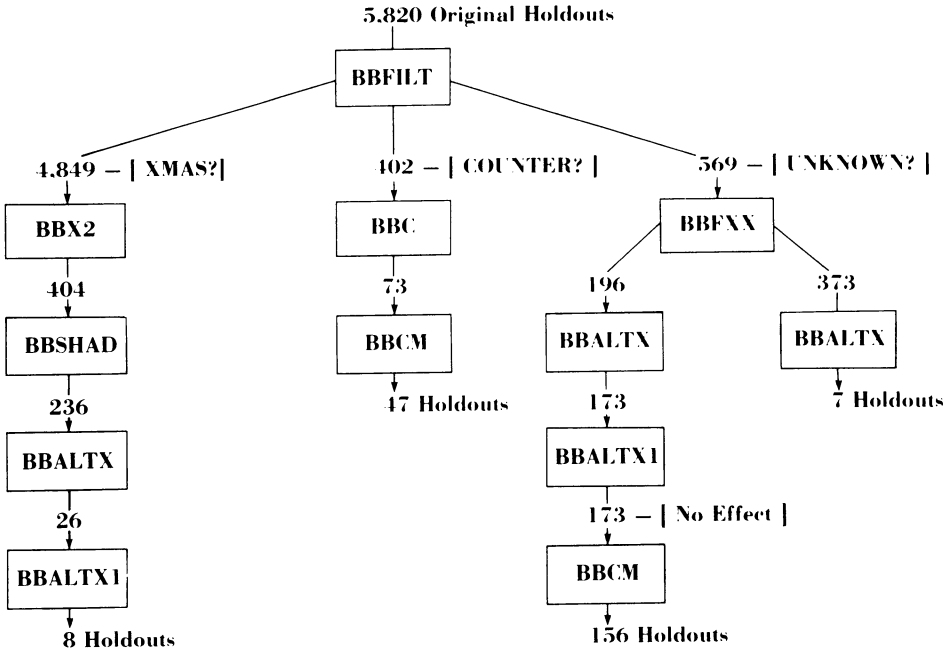


FIGURE 6. *The computer reduction runs*

BBX2 was the Xmas Tree prover which used the cellular automata approach. BBSHAD was a modification to handle "Trees with Shadow", BBALTX was an extension of BBX2 to handle "Alternating Xmas Trees", while BBALTX1 was a minor modification of BBALTX to handle double sweeps in which the extremum was reached on alternate sweeps only.

BBC was the counter prover, while BBCM was a modification of BBC to handle "two-shot" carries and some cases of cell interdependence.

More than 18 other programs were written for various housekeeping purposes, simulating and displaying machine behavior, exploring other reduction and filtering possibilities, etc. In all, at least 53 files were created and maintained for the project. Keeping track of what resembled a large scientific experiment became a major task in itself.

While not all of the exploratory activities are reproducible, the runs shown in Figure 6 can be reproduced, so that by utilizing the techniques described in this paper the proof can be corroborated.

All of the programs utilized all or part of the filtering techniques of the program BBFILT. However, it must be remembered that the filtering was a heuristic technique based upon experimental observation. The reasons why it was not always effective were not explored in every case, since the ultimate proofs were mechanized independently of the heuristic separation. Furthermore, the heuristic separation was dependent upon such parameters as the length of behavior histories (number of moves or shifts). E.g., a 1,000 move limit picked up fewer Counters than a 2,000 move limit due to anomalous behavior in the early portion of the maximum excursion sequences.

The 218 *Remaining Holdouts*. The general behavior of some of the 218 holdout machines has already been described. Three of the machines escaped detection by programs which should have proved they would never halt. One of these machines was in fact a simple "left traveling loop," but it required 422 moves to display a repetition in its sequence. Several machines showed up which are forms of ternary (base 3) and quaternary (base 4) counters.

A new generic type which may be described as "Tail-Eating Dragons" dominates the "Other" set. This type displays either a rapid growth approximating Xmas Trees or else an extremely slow growth approximately quaternary counters. Tail-Eating Dragons are almost the converse of "Trees with Shadow". Starting with a long string of characters (their "tail") these machines sweep back and forth like a Xmas Tree while nibbling a bit off their tail on each swing. Once the tail is consumed, they create a much larger tail and begin the process anew. The "rapid growth" machines add to the size of their tape on each swing, while the "slow growth" machines add to their scanned tape only when they create a new tail. Their behavior is precariously similar to that of the "champion" lower bound machines devised by Green [3].

All of the remaining holdouts were examined by means of voluminous printouts of their histories along with some program extracted features. It was determined to the author's satisfaction that none of these machines will ever stop.

6. Conclusion. It was mentioned in the introduction that these results should be independently verified. Proofs of "correctness" of the programs used are not practical. Independent verification is the only means we currently have at our disposal.

This attack on the $k = 4$ problem has produced the bonus of creating the components of an "intelligent machine" which will accept any three-state binary Turing machine and render a decision on its blank input tape halting problem without any resort to an enumeration of specific machines supplied by its human creators. I.e., all of the 27 holdouts for $k = 3$ were recognized by the new programs as either Xmas Trees or Counters and were *mechanically proved* never to halt. An effort to solve the problem for $k = 5$ might similarly produce enhancements verifying the four-state problem.

By fitting an expression of the form $(ak + b)^{ck}$ to the number of tree normal machines for $k = 2, 3, 4$ one obtains values for a, b, and c of 1.72, −0.513, and 1.80, respectively, yielding a projection of 150,000,000 for $k = 5$. Using nothing more than a small computer one could probably generate this entire set and reduce the five-state case to perhaps a few million holdouts. These five-state holdouts should present an interesting and reasonable challenge to persons interested in mechanical proof techniques.

The final steps in the solution to this problem for $k = 4$ have resulted in a successful effort in applying a form of mechanized mathematical induction. (The computer programs could have been designed to *print out individual proofs for each machine* expressed in English and mathematical prose.) There are no doubt other classes of new and interesting automata problems which could be attacked successfully by similar methods. The pursuit of such techniques is important in bringing about the day when mathematical research may, as a matter of course, produce results using automated proofs with these results being communicated, understood, and accepted.

APPENDIX A. Current Known Results. The best known results for Rado's problem insofar as the author is able to determine are the following.

k	$\Sigma(k)$	$S(k)$	Source
1	$= 1$	$= 1$	T. Rado et al.
2	$= 4$	$= 6$	T. Rado et al.
3	$= 6$	$= 21$	S. Lin
4	$= 13$	$= 107$	A. Brady
5	$\geqslant 112$	$\geqslant 7{,}707$	D. Lynn
6	$\geqslant 117$	$\geqslant 13{,}488$	D. Lynn
7	$\geqslant 22{,}961$		M. Green
8	$\geqslant 3 \cdot (7 \cdot 3^{92} - 1)/2$		M. Green
...			
n	(cf. [3])		M. Green

The results for $k = 5$ and $k = 6$ have not been published. The results for $k = 5$ have been communicated to the author by Donald S. Lynn and are an extension of work beyond that reported in [7]. The results for $k = 6$ have been generated by the author using the 5-state machines discovered by Lynn.

APPENDIX B. Machine Examples. Since the generic machine types discussed are relatively rare among all four-state machines, some examples are given here which may be used to illustrate the discussion in the paper.

Among the proved set:

1. Xmas Tree–

	0	1
1	1*R*2	0*L*3
2	1*L*1	0*R*4
3	0*L*2	0*L*2
4	1*R*1	---

2. Leaning Tree–

	0	1
1	1*R*2	0*L*3
2	0*L*1	1*R*4
3	1*L*1	1*L*2
4	0*R*1	---

3. Shadow Tree–

	0	1
1	1*R*2	1*L*1
2	0*L*1	0*R*3
3	1*R*4	1*R*3
4	1*L*1	---

4. Alternating Tree–

	0	1
1	1*R*2	0*L*3
2	1*L*1	0*R*3
3	1*L*4	0*R*1
4	1*L*1	---

5. Counter (3-square cell)–

	0	1
1	1*R*2	1*L*3
2	0*L*1	0*R*2
3	1*R*1	1*L*4
4	1*L*1	---

6. Counter (Nonblank zero)–

	0	1
1	1*R*2	1*L*3
2	0*L*1	1*R*4
3	1*R*1	1*L*1
4	---	0*R*2

Among the holdouts:

7. Ternary Counter–

	0	1
1	1*R*2	0*L*3
2	1*L*1	1*R*1
3	0*R*1	0*L*4
4	---	1*L*3

8. Tailing-eating Dragon (fast growth)–

	0	1
1	1*R*2	0*R*4
2	1*R*3	---
3	0*L*1	1*R*1
4	1*L*3	1*R*4

9. Tailing-eating Dragon (slow growth)–

	0	1
1	1*R*2	1*L*1
2	1*R*3	0*R*4
3	1*L*1	---
4	0*L*1	1*R*4

Examination of the behavior of these machines without the aid of a computer is somewhat tedious. A simple Turing machine simulator written in a machine independent form of BASIC is available from the author upon request.

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