

## Three Summation Criteria for Fermat's Last Theorem

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**Abstract.** This paper extends the search for solutions of the congruences

$$\sum_1^{[p/6]} \frac{1}{i} \equiv 0, \quad \sum_1^{[p/6]} \frac{1}{i^2} \equiv 0 \quad \text{and} \quad \sum_{[p/6]+1}^{[p/5]} \frac{1}{i} \equiv 0 \pmod{p}$$

to the limit  $p < 600000$ . The only solutions found were  $p = 61$  in the first case, in the second  $p = 205129$ , and in the third case  $p = 109$  and  $p = 491$ .

If the first case of Fermat's Last Theorem fails for the exponent  $p$ , then

$$\sum_1^{[p/6]} \frac{1}{i} \equiv 0, \quad \sum_1^{[p/6]} \frac{1}{i^2} \equiv 0 \quad \text{and} \quad \sum_{[p/6]+1}^{[p/5]} \frac{1}{i} \equiv 0 \pmod{p}.$$

The first criterion results from theorems of Wieferich and Mirimanoff and the congruences of Lerch [1].

The second criterion results from a theorem of Vandiver and the lemma of Schwindt [2].

H. S. Vandiver and Emma Lehmer proved the third criterion [3].

For all primes  $< 600000$  only  $p = 61$  satisfies the first congruence,  $p = 205129$  the second, and  $p = 109$  and  $p = 491$  the third congruence. The equation

$$i \cdot x + p \cdot y = 1$$

is equivalent to the congruence

$$1/i \equiv x \pmod{p}.$$

Euclid's algorithm can calculate  $x$  and  $y$ , such that

$$i \cdot x + p \cdot y = \gcd(i, p)$$

at the same time the greatest common divisor  $\gcd(i, p) = 1$  is being calculated; see Knuth [4].

The order of this calculation is  $p \cdot \log p$ . The calculation-time for  $p = 205129$  and the first and second criterion is 29 seconds on the computer "DEC 10". For  $p = 600011$  it is 88 seconds.

Wilfrid Keller\* has written the program for Euclid's algorithm, and he calculated for the criterion of Emma Lehmer the following data.  $D$  is the average number of steps for  $x$  and  $t$  is the total time in seconds of the calculation  $\sum 1/i \equiv \sum x \pmod{p}$  on the computer "TR 440".

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$p$	$D$	$D/\log p$	$t$	$t \cdot 10^5 / (p \cdot \log p)$
10 007	6.734	0.731	0.34	0.369
20 011	7.324	0.739	0.64	0.323
30 011	7.668	0.744	0.95	0.307
40 009	7.894	0.745	1.39	0.328
50 021	8.065	0.745	1.67	0.309
100 003	8.695	0.755	3.48	0.302
500 009	10.035	0.765	19.63	0.299
1000 003	10.645	0.771	40.66	0.294
5000 011	11.988	0.772	232.01	0.301

The time for the first and second criterion is five times as much. The program was run at night over half a year.

For a more cost effective criterion see Lehmer [5]. Since Lehmer's criterion [5] goes to  $6 \cdot 10^9$ , while here we go only to  $6 \cdot 10^5$ , it is clear that the foregoing gives no new information about Fermat's Last Theorem. But since the three criteria here are independent, and of interest, it seems desirable to record some data concerning them.

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