Sets of $n$ Squares of Which Any $n - 1$
Have Their Sum Square

By Jean Lagrange

Abstract. A systematic method is given for calculating sets of $n$ squares of which any $n - 1$
have their sum square. A particular method is developed for $n = 4$. Tables give the smallest
solution for each $n \leq 8$ and other small solutions for $n \leq 5$.

1. Introduction. We give numerical solutions in positive integers of the equations

$$x_1^2 + y_1^2 = \cdots = x_n^2 + y_n^2 = x_1^2 + \cdots + x_n^2 \quad (n \geq 3),$$

with $x_i \neq x_j$ for $i \neq j$. The cases $n = 3, 4$ have been studied by many authors;
references are given in [1, Chapter XIX].

For general $n$, Gill [2] gave in 1848 a method for finding solutions of (1), but his
method, based on complicated trigonometrical calculations, is impractical for finding
actual solutions for $n \geq 5$.

We give a simple method for finding explicit solutions for $n \geq 5$.

2. Method. We study the more general equations

$$\alpha x_1^2 + y_1^2 = \cdots = \alpha x_n^2 + y_n^2 = \beta (x_1^2 + \cdots + x_n^2),$$

where $\alpha$ and $\beta$ are given integers. From a known solution $(x_i, y_i)$ we construct
another solution $(x'_i, y'_i)$. Setting

$$S = \sum_{i=1}^{n} x_i^2, \quad P = \sum_{i=1}^{n} x_i y_i,$$

we seek $\lambda, \mu$ such that

$$\begin{cases}
    x'_i = \lambda S x_i - \mu P y_i, \\
    y'_i = \alpha \mu P x_i + \lambda S y_i,
\end{cases}$$

is another solution. We easily find

$$\alpha x'_i^2 + y'_i^2 = \beta S(\lambda^2 S^2 + \alpha \mu^2 P^2),$$

$$\sum_{i=1}^{n} x_i^2 = S\left[\lambda^2 S^2 + (\mu^2(n \beta - \alpha) - 2 \lambda \mu) P^2\right],$$

whence $2\lambda = \mu(n \beta - 2 \alpha)$. The solution sought is

$$\begin{cases}
    x'_i = (n \beta - 2 \alpha) S x_i - 2 P y_i, \\
    y'_i = 2 \alpha P x_i + (n \beta - 2 \alpha) S y_i.
\end{cases}$$
Iteration of the formulae (3) leads back to the original solution. However, we obtain a different solution if we first change the sign of one or more of the \( x_i \). We can thus construct solutions of the equations (2) provided that we know a particular solution, which may be trivial. For the equations (1) the formulae (3) become

\[
\begin{align*}
x_i' &= (n-2)Sx_i - 2Py_i, \\
y_i' &= 2Px_i + (n-2)Sy_i,
\end{align*}
\]

and we have a trivial solution

\[x_1 = \cdots = x_{n-2} = 0, \quad x_{n-1} = a, \quad x_n = b,\]

where \( a \) and \( b \) are integers satisfying \( a^2 + b^2 = c^2 \).

3. Small Values of \( n \).

(a) \( n = 3 \). The solution \( 0, a, b, \) with \( a^2 + b^2 = c^2 \), is not wholly trivial, as it satisfies \( x_i \neq x_j \) for \( i \neq j \), but it is of little interest. An application of the formulae (4) gives

\[
\begin{align*}
x_1 &= 4abc, \quad x_2 = a(c^2 - 4b^2), \quad x_3 = b(c^2 - 4a^2), \\
y_1 &= c^3, \quad y_2 = b(c^2 + 4a^2), \quad y_3 = a(c^2 + 4b^2).
\end{align*}
\]

We thus obtain the Euler cuboid (rectangular parallelepiped with integer edges \( x_1, x_2, x_3 \) and integer face diagonals \( y_1, y_2, y_3 \); see [4], for example). From \( a = 3, b = 4, c = 5 \) we obtain the solution

\[44, \quad 117, \quad 240.\]

(b) \( n = 4 \). The same method gives the “semitrivial” solution

\[
\begin{align*}
x_1 &= x_2 = 2abc, \quad x_3 = a(b^2 - a^2), \quad x_4 = b(a^2 - b^2), \\
y_1 &= y_2 = c^3, \quad y_3 = b(2a^2 + c^2), \quad y_4 = a(2b^2 + c^2).
\end{align*}
\]

Changing the sign of \( x_2 \) (to ensure a new solution) and \( x_4 \) (to simplify), we apply (4) to obtain

\[
\begin{align*}
x_1 &= 2abc(4b^4 - 3c^4), \quad x_2 = 2abc(4a^4 - 3c^4), \\
x_3 &= a(b^2 - a^2)(4a^4 - 3c^4), \quad x_4 = b(b^2 - a^2)(4b^4 - 3c^4).
\end{align*}
\]

From \( a = 3, b = 4, c = 5 \) we obtain the solution

\[23828, \quad 32571, \quad 102120, \quad 186120.\]

(c) \( n = 5 \). We give only a numerical solution. Beginning with a trivial solution having \( x_1 = x_2 = x_3 = x_4 \), we apply the formulae (4) to

\[
\begin{align*}
x_1 &= x_2 = -x_3 = -x_4 = 4, \quad x_5 = 1, \\
y_1 &= y_2 = y_3 = y_4 = 7, \quad y_5 = 8.
\end{align*}
\]

This gives

\[
\begin{align*}
x_1 &= x_2 = 668, \quad x_3 = x_4 = 892, \quad x_5 = 67, \\
y_1 &= y_2 = 1429, \quad y_3 = y_4 = 1301, \quad y_5 = 1576.
\end{align*}
\]

Changing the sign of \( x_2 \) and \( x_4 \) and applying (4) again, we obtain the solution

\[1673 15281, \quad 46847 01124, \quad 52882 64996, \quad 63838 46756, \quad 69333 47524.\]

(d) \( n = 6 \). We apply (4) to the trivial solution

\[
\begin{align*}
x_1 &= x_2 = x_3 = x_4 = 0, \quad x_5 = 3, \quad x_6 = 4, \\
y_1 &= y_2 = y_3 = y_4 = 5, \quad y_5 = 4, \quad y_6 = 3.
\end{align*}
\]
SQUARES WITH SQUARE SUMS

and obtain

\[ x_1 = x_2 = x_3 = x_4 = 60, \quad x_5 = 64, \quad x_6 = 64, \]
\[ y_1 = y_2 = y_3 = y_4 = 125, \quad y_5 = 136, \quad y_6 = 123. \]

Changing the sign of \(x_3\) and \(x_4\) and applying (4) again, we obtain

\[ x_1 = x_2 = 56440, \quad x_3 = x_4 = 35640, \quad x_5 = 32187, \quad x_6 = 38884, \]
\[ y_1 = y_2 = 91085, \quad y_3 = y_4 = 101165, \quad y_5 = 102316, \quad y_6 = 99963. \]

Change of sign of \(x_3\) and \(x_4\) and a third application of (4) gives the solution

\[ 3039928895652, \quad 3205366606047, \quad 3341350001384, \]
\[ 3520435290636, \quad 4996634759436, \quad 5429263880052. \]

4. \(n = 4\) Reconsidered. Tebay [9] gives the simple solution

\[ x_1 = (s^2 - 1)(s^2 - 9)(s^2 + 3), \quad x_3 = 4s(s + 1)(s - 3)(s^2 + 3), \]
\[ x_2 = 4s(s - 1)(s + 3)(s^2 + 3), \quad x_4 = 2s(s^2 - 1)(s^2 - 9). \]

With changes of sign and sequence, \(s = 2\) gives the solution 60, 105, 168, 280. He obtains this parametric solution by imposing special conditions, the first being \(x_1x_2 + x_2x_3 + x_3x_4 + x_4x_1 = 0\) (with change of sign of \(x_3\)).

Martin [6] examines Tebay’s method and corrects some mistakes. He remarks that Euler had given an equivalent solution without derivation [1, p. 503]. We now give a method for constructing numerous solutions for \(n = 4\), the foregoing parametric solution appearing as a special case. Consider the equation

\[ u_1^4 + u_2^4 + u_3^4 + u_4^4 = 2(u_1^2u_2^2 + u_1^2u_3^2 + u_1^2u_4^2 + u_2^2u_3^2 + u_2^2u_4^2 + u_3^2u_4^2), \]

which we abbreviate as

\[ (5) \quad \sum u_i^4 = 2 \sum u_i^2u_j^2. \]

Numerical solutions of this equation are easily found by computer search. The following equations are equivalent:

\[ (6) \quad 4(u_1^2u_2^2 + u_3^2u_4^2) = (u_2^2 + u_3^2 + u_4^2 - u_1^2)^2, \]
\[ (7) \quad 4(u_1^2u_2^2 + u_3^2u_4^2) = (u_1^2 + u_2^2 - u_3^2 - u_4^2)^2, \]
\[ (8) \quad (\sum u_i^2)^2 = 4 \sum u_i^2u_j^2, \]
\[ (9) \quad (u_1^2 + u_2^2 - u_3^2 - u_4^2)(u_1^2 + u_3^2 - u_2^2 - u_4^2)(u_1^2 + u_4^2 - u_2^2 - u_3^2) = 8 \sum u_i^2u_j^2u_k^2. \]

Set

\[ x_1 = u_2u_3u_4, \quad x_2 = u_1u_3u_4, \quad x_3 = u_1u_2u_4, \quad x_4 = u_1u_2u_3. \]

Then Eq. (6) shows that we have a solution of the equations (1). This solution has some interesting properties.

Setting

\[ A^2 = x_1^2x_2^2 + x_3^2x_4^2, \quad B^2 = x_1^2x_3^2 + x_2^2x_4^2, \quad C^2 = x_1^2x_4^2 + x_2^2x_3^2, \]

we see from (7) that \(A, B, C\) are integers. Setting \(E^2 = A^2 + B^2 + C^2\), we see from (8) that \(E\) is an integer. Finally, Eq. (9) shows that

\[ S = x_1^2 + x_2^2 + x_3^2 + x_4^2 = ABC/x_1x_2x_3x_4. \]
These relations are homogeneous and so are valid whether or not the solution 
\( x_1, x_2, x_3, x_4 \) is primitive. The following result is valid only for a primitive solution. Set
\[
D = \gcd(x_1 x_2 x_3, x_1 x_2 x_4, x_1 x_3 x_4, x_2 x_3 x_4),
\]
\[
\Delta = \gcd(A, B, C).
\]
Then we have
\[
x_1 x_2 x_3 x_4 = D^2 / \Delta,
\]
as is easily verified by calculating the \( p \)-adic values of \( D, \Delta, x_1 x_2 x_3 x_4 \). For \( p \) prime we may suppose that
\[
v_p(u_1) = 0, \quad v_p(u_2) = \alpha, \quad v_p(u_3) = \beta, \quad v_p(u_4) = \gamma,
\]
with \( 0 \leq \alpha \leq \beta \leq \gamma \). For the corresponding primitive solution we then have
\[
v_p(x_1) = \gamma, \quad v_p(x_2) = \gamma - \alpha, \quad v_p(x_3) = \gamma - \beta, \quad v_p(x_4) = 0,
\]
and we easily obtain
\[
v_p(D) = 2 \gamma - \alpha - \beta, \quad v_p(\Delta) = \gamma - \alpha - \beta,
\]
\[
v_p(x_1 x_2 x_3 x_4) = 3 \gamma - \alpha - \beta,
\]
from which the result follows.

A parametric solution to Eq. (5) is obtained by the following method. The identity
\[
(p + q + r)(p - q - r)(q - r - p)(r - p - q) = p^4 + q^4 + r^4 - 2(q^2 r^2 + r^2 p^2 + p^2 q^2)
\]
shows that
\[
(10) \quad p + q + r = 0 \quad \text{implies} \quad p^4 + q^4 + r^4 = 2(q^2 r^2 + r^2 p^2 + p^2 q^2).
\]
We rewrite (5) in the form
\[
u_4^4 - 2u_2^2(u_1^2 + u_2^2 + u_3^2) + u_4^6 + u_1^4 + u_2^4 + u_3^4 - 2(u_2^2 u_3^2 + u_3^2 u_1^2 + u_1^2 u_2^2) = 0.
\]
Setting \( u_1 + u_2 + u_3 = 0 \), we have from (10)
\[
u_4^2 = 2(u_1^2 + u_2^2 + u_3^2).
\]
To make \( u_4 \) rational, we set
\[
u_1 = v_2^2 - v_3^2, \quad u_2 = v_1^2 - v_3^2, \quad u_3 = v_1^2 - v_2^2 \quad \text{with} \quad v_1 + v_2 + v_3 = 0.
\]
In effect we have from (10)
\[
2(u_1^2 + u_2^2 + u_3^2) = (v_1^2 + v_2^2 + v_3^2)^2,
\]
whence \( u_4 = v_1^2 + v_2^2 + v_3^2 \). We thus obtain
\[
x_1 = (v_3^2 - v_1^2)(v_1^2 - v_2^2)(v_1^2 + v_2^2 + v_3^2),
\]
\[
x_2 = (v_2^2 - v_1^2)(v_1^2 - v_3^2)(v_1^2 + v_2^2 + v_3^2),
\]
\[
x_3 = (v_2^2 - v_3^2)(v_3^2 - v_1^2)(v_1^2 + v_2^2 + v_3^2),
\]
\[
x_4 = (v_2^2 - v_3^2)(v_3^2 - v_2^2)(v_1^2 - v_2^2),
\]
with \( v_1 + v_2 + v_3 = 0 \). This is equivalent to Tebay’s solution, which is obtained by setting \( v_2 = 2 \) (abandoning homogeneity) and \( v_1 = s - 1 \), whence \( v_3 = -(s + 1) \).
We note that Euler made several studies of (5) [1, p. 661]; however, there is no mention of the relation between Eqs. (1) and (5).

5. Tables. In Table 1 we give the smallest solution (that with minimum $S$) for $3 \leq n \leq 8$, and in Tables 2–4 we give all solutions for $3 \leq n \leq 5$ having $S \leq 10^9$. For $n = 3$ tables have been given by Lal and Blundon [3], Leech [5] and Spohn [8]. The present computations were done on the IBM 370 computer at C.I.R.C.E. Each $S$ is expressed as the sum of two squares $x_i^2 + y_i^2$ in all possible ways by the method of Nicolas [7]. We retain only those $S$ which are expressible in at least $n$ ways; we then have to test whether any $n$ of these satisfy

$$\sum_{i=1}^{n} x_i^2 = S.$$ 

It may be remarked that it is never necessary to test whether an integer is a perfect square.

### Table 1

The smallest solutions

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### Table 2

$n = 3$

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Table 3

\[ n = 4 \]

\[ \begin{array}{llllllllllllllll}
 & x_1 & x_2 & x_3 & x_4 & S & u_1 & u_2 & u_3 & u_4 \\
1 & 60 & 105 & 168 & 280 & 121249 & 3 & 5 & 8 & 14 \\
2 & 420 & 728 & 1365 & 1560 & 5003209 & 7 & 8 & 15 & 26 \\
3 & 385 & 792 & 840 & 1980 & 5401489 & 14 & 33 & 35 & 72 \\
4 & 672 & 1120 & 1980 & 3465 & 17623629 & 32 & 56 & 99 & 165 \\
5 & 585 & 1008 & 1456 & 5460 & 33289825 & 12 & 45 & 65 & 112 \\
6 & 840 & 1520 & 1995 & 6384 & 47751481 & 16 & 16 & 21 & 38 \\
7 & 880 & 1155 & 5040 & 5544 & 58245961 & 10 & 11 & 48 & 63 \\
8 & 624 & 2625 & 3220 & 6432 & 59019025 & & & & \\
9 & 1848 & 3575 & 4620 & 7800 & 98380129 & 77 & 130 & 168 & 325 \\
10 & 2508 & 5544 & 5985 & 8360 & 142735825 & 63 & 88 & 95 & 210 \\
11 & 2295 & 3808 & 7344 & 10080 & 175308625 & 51 & 70 & 135 & 224 \\
12 & 1232 & 8316 & 9141 & 10368 & 261726985 & & & & \\
13 & 3276 & 5005 & 11880 & 16632 & 453540025 & 65 & 91 & 216 & 330 \\
14 & 2040 & 2520 & 11781 & 26180 & 834696361 & 18 & 40 & 187 & 231 \\
15 & 4620 & 8184 & 11935 & 26040 & 908848081 & 11 & 24 & 35 & 62 \\
\end{array} \]

Where a solution can be obtained by the method of Section 4, the values of \( u_i \) are given.

Table 4

\[ n = 5 \]

\[ \begin{array}{llllllllllllll}
 & x_1 & x_2 & x_3 & x_4 & x_5 & S \\
1 & 28 & 64 & 259 & 392 & 680 & 688025 \\
2 & 1112 & 1225 & 1876 & 3184 & 5768 & 49664225 \\
3 & 2105 & 2648 & 2980 & 3736 & 4720 & 56559425 \\
4 & 203 & 2240 & 3920 & 4240 & 6104 & 75661625 \\
5 & 696 & 1200 & 3475 & 4980 & 6360 & 79250041 \\
6 & 56 & 208 & 1400 & 4060 & 9065 & 100664225 \\
7 & 557 & 1747 & 4141 & 5219 & 8285 & 116389325 \\
8 & 427 & 3164 & 3980 & 6220 & 7420 & 119778425 \\
9 & 1183 & 1300 & 2240 & 7280 & 8080 & 126391889 \\
10 & 1095 & 3063 & 4119 & 5527 & 10329 & 164783125 \\
11 & 1952 & 2360 & 5020 & 6089 & 10520 & 182326625 \\
12 & 595 & 3549 & 5235 & 9555 & 10893 & 250310125 \\
13 & 2328 & 5824 & 7368 & 9975 & 14196 & 394653025 \\
14 & 2207 & 4417 & 5215 & 12479 & 14161 & 407836325 \\
15 & 483 & 5328 & 6356 & 15000 & 17304 & 593448025 \\
16 & 49 & 2152 & 5600 & 16076 & 18088 & 621607025 \\
17 & 3799 & 9560 & 11384 & 13732 & 16112 & 683585825 \\
18 & 2425 & 3020 & 8596 & 19628 & 20020 & 874951025 \\
\end{array} \]

Remark. In the solutions 7, 10, 12 and 14, all the \( x_i \) are odd.

6. Concluding Remarks. (a) Examination of the tables suggests that there may be simple parametric solutions for \( n \geq 5 \), but we have not found them by the present method.

(b) There exist values of \( \alpha, \beta \) for which Eq. (2) has trivial solutions; these can then be transformed into nontrivial solutions. This is the case when we replace the sums of \( n - 1 \) squares by their arithmetic means.

(c) I shall return later to the case of \( n = 3 \) with general \( \alpha, \beta \). Several of the systems of equations studied in [1, Chapter XIX], are effectively of this type. They are, however, treated by methods specific to each problem; we can now treat them by a uniform method.
Acknowledgement. I am indebted to Professor J. Leech for the English version of this paper and for supplying photocopies of [6] and [9].

Département de Mathématiques
Faculté des Sciences
Université de Reims
B.P. 347
51062 Reims, Cedex, France

6. A. Martin, "Find four square numbers such that the sum of every three of them shall be a square," Math. Quest. Educ. Times, v. 24, 1913, pp. 81–82.
9. S. Tebay, "Find four square numbers such that the sum of every three of them shall be a square," Math. Quest. Educ. Times, v. 68, 1898, pp. 103–104.