Sets of \( n \) Squares of Which Any \( n - 1 \)
Have Their Sum Square

By Jean Lagrange

Abstract. A systematic method is given for calculating sets of \( n \) squares of which any \( n - 1 \)
have their sum square. A particular method is developed for \( n = 4 \). Tables give the smallest
solution for each \( n \leq 8 \) and other small solutions for \( n \leq 5 \).

1. Introduction. We give numerical solutions in positive integers of the equations
\[
\sum_{i=1}^{n} x_i^2 + y_i^2 = \cdots = x_n^2 + y_n^2 = x_1^2 + \cdots + x_n^2 \quad (n \geq 3),
\]
with \( x_i \neq x_j \) for \( i \neq j \). The cases \( n = 3,4 \) have been studied by many authors;
references are given in [1, Chapter XIX].

For general \( n \), Gill [2] gave in 1848 a method for finding solutions of (1), but his
method, based on complicated trigonometrical calculations, is impractical for finding
actual solutions for \( n \geq 5 \).

We give a simple method for finding explicit solutions for \( n \geq 5 \).

2. Method. We study the more general equations
\[
\alpha x_1^2 + y_1^2 = \cdots = \alpha x_n^2 + y_n^2 = \beta(x_1^2 + \cdots + x_n^2),
\]
where \( \alpha \) and \( \beta \) are given integers. From a known solution \((x_i, y_i)\) we construct
another solution \((x'_i, y'_i)\). Setting
\[
S = \sum_{i=1}^{n} x_i^2, \quad P = \sum_{i=1}^{n} x_i y_i,
\]
we seek \( \lambda, \mu \) such that
\[
\begin{align*}
  x'_i &= \lambda S x_i - \mu P y_i, \\
  y'_i &= \alpha \mu P x_i + \lambda S y_i,
\end{align*}
\]
is another solution. We easily find
\[
\alpha x'_i^2 + y'_i^2 = \beta S(\lambda^2 S^2 + \alpha \mu^2 P^2),
\]
\[
\sum_{i=1}^{n} x_i'^2 = S[\lambda^2 S^2 + (\mu^2(n\beta - \alpha) - 2\lambda \mu)P^2],
\]
whence \( 2\lambda = \mu(n\beta - 2\alpha) \). The solution sought is
\[
\begin{align*}
  x'_i &= (n\beta - 2\alpha)S x_i - 2P y_i, \\
  y'_i &= 2\alpha P x_i + (n\beta - 2\alpha)S y_i.
\end{align*}
\]
Iteration of the formulae (3) leads back to the original solution. However, we obtain a different solution if we first change the sign of one or more of the $x_i$. We can thus construct solutions of the equations (2) provided that we know a particular solution, which may be trivial. For the equations (1) the formulae (3) become

\[
\begin{align*}
x'_i &= (n - 2)Sx_i - 2Py_i, \\
y'_i &= 2Px_i + (n - 2)Sy_i,
\end{align*}
\]

and we have a trivial solution

\[
x_1 = \cdots = x_{n-2} = 0, \quad x_{n-1} = a, \quad x_n = b,
\]

where $a$ and $b$ are integers satisfying $a^2 + b^2 = c^2$.

3. **Small Values of $n$.**

(a) $n = 3$. The solution $0, a, b$, with $a^2 + b^2 = c^2$, is not wholly trivial, as it satisfies $x_i \neq x_j$ for $i \neq j$, but it is of little interest. An application of the formulae (4) gives

\[
\begin{align*}
x_1 &= 4abc, \quad x_2 = a(c^2 - 4b^2), \quad x_3 = b(c^2 - 4a^2), \\
y_1 &= c, \quad y_2 = b(c^2 + 4a^2), \quad y_3 = a(c^2 + 4b^2).
\end{align*}
\]

We thus obtain the Euler cuboid (rectangular parallelepiped with integer edges $x_1, x_2, x_3$ and integer face diagonals $y_1, y_2, y_3$; see [4], for example). From $a = 3, b = 4, c = 5$ we obtain the solution

\[
44, \quad 117, \quad 240.
\]

(b) $n = 4$. The same method gives the “semitrivial” solution

\[
\begin{align*}
x_1 &= x_2 = 2abc, \quad x_3 = a(b^2 - a^2), \quad x_4 = b(a^2 - b^2), \\
y_1 &= y_2 = c, \quad y_3 = b(2a^2 + c^2), \quad y_4 = a(2b^2 + c^2).
\end{align*}
\]

Changing the sign of $x_2$ (to ensure a new solution) and $x_4$ (to simplify), we apply (4) to obtain

\[
\begin{align*}
x_1 &= 2abc(4b^4 - 3c^4), \quad x_2 = 2abc(4a^4 - 3c^4), \\
x_3 &= a(b^2 - a^2)(4a^4 - 3c^4), \quad x_4 = b(b^2 - a^2)(4b^4 - 3c^4).
\end{align*}
\]

From $a = 3, b = 4, c = 5$ we obtain the solution

\[
23828, \quad 32571, \quad 102120, \quad 186120.
\]

(c) $n = 5$. We give only a numerical solution. Beginning with a trivial solution having $x_1 = x_2 = x_3 = x_4$, we apply the formulae (4) to

\[
\begin{align*}
x_1 &= x_2 = -x_3 = -x_4 = 4, \quad x_5 = 1, \\
y_1 &= y_2 = y_3 = y_4 = 7, \quad y_5 = 8.
\end{align*}
\]

This gives

\[
\begin{align*}
x_1 &= x_2 = 668, \quad x_3 = x_4 = 892, \quad x_5 = 67, \\
y_1 &= y_2 = 1429, \quad y_3 = y_4 = 1301, \quad y_5 = 1576.
\end{align*}
\]

Changing the sign of $x_2$ and $x_4$ and applying (4) again, we obtain the solution

\[
1673 \quad 15281, \quad 46847 \quad 01124, \quad 52882 \quad 64996, \quad 63838 \quad 46756, \quad 69333 \quad 47524.
\]

(d) $n = 6$. We apply (4) to the trivial solution

\[
\begin{align*}
x_1 &= x_2 = x_3 = x_4 = 0, \quad x_5 = 3, \quad x_6 = 4, \\
y_1 &= y_2 = y_3 = y_4 = 5, \quad y_5 = 4, \quad y_6 = 3.
\end{align*}
\]
and obtain
\[ x_1 = x_2 = x_3 = x_4 = 60, \quad x_5 = 64, \quad x_6 = 64, \]
\[ y_1 = y_2 = y_3 = y_4 = 125, \quad y_5 = 136, \quad y_6 = 123. \]

Changing the sign of \( x_3 \) and \( x_4 \) and applying (4) again, we obtain
\[ x_1 = x_2 = 56440, \quad x_3 = x_4 = 35640, \quad x_5 = 32187, \quad x_6 = 38884, \]
\[ y_1 = y_2 = 91085, \quad y_3 = y_4 = 101165, \quad y_5 = 102316, \quad y_6 = 99963. \]

Change of sign of \( x_3 \) and \( x_4 \) and a third application of (4) gives the solution
\[ 303 \ 99288 \ 95652, \quad 320 \ 53666 \ 06047, \quad 334 \ 13500 \ 01384, \]
\[ 352 \ 04352 \ 90636, \quad 499 \ 66347 \ 59436, \quad 542 \ 92638 \ 80052. \]

4. \( n = 4 \) Reconsidered. Tebay [9] gives the simple solution
\[ x_1 = (s^2 - 1)(s^2 - 9)(s^2 + 3), \quad x_3 = 4s(s + 1)(s - 3)(s^2 + 3), \]
\[ x_2 = 4s(s - 1)(s + 3)(s^2 + 3), \quad x_4 = 2s(s^2 - 1)(s^2 - 9). \]

With changes of sign and sequence, \( s = 2 \) gives the solution 60, 105, 168, 280. He obtains this parametric solution by imposing special conditions, the first being \( x_1 x_2 + x_3 x_4 + x_5 x_6 = 0 \) (with change of sign of \( x_3 \)).

Martin [6] examines Tebay’s method and corrects some mistakes. He remarks that Euler had given an equivalent solution without derivation [1, p. 503]. We now give a method for constructing numerous solutions for \( n = 4 \), the foregoing parametric solution appearing as a special case. Consider the equation
\[ u_1^4 + u_2^4 + u_3^4 + u_4^4 = 2(u_1^2 u_2^2 + u_1^2 u_3^2 + u_1^2 u_4^2 + u_2^2 u_3^2 + u_2^2 u_4^2 + u_3^2 u_4^2), \]
which we abbreviate as
\[ \sum u_i^4 = 2 \sum u_i^2 u_j^2. \]

Numerical solutions of this equation are easily found by computer search. The following equations are equivalent:
\[ 4(u_3^2 u_4^2 + u_3^2 u_2^2 + u_2^2 u_3^2) = (u_2^2 + u_3^2 + u_4^2 - u_1^2)^2, \]
\[ 4(u_1^2 u_2^2 + u_1^2 u_4^2) = (u_1^2 + u_2^2 - u_3^2 - u_4^2)^2, \]
\[ (\sum u_i^2)^2 = 4 \sum u_i^2 u_j^2, \]
\[ (u_1^2 + u_2^2 - u_3^2 - u_4^2)(u_1^2 + u_3^2 - u_4^2 - u_2^2)(u_1^2 + u_4^2 - u_2^2 - u_3^2) = 8 \sum u_i^2 u_j^2 u_k^2. \]

Set
\[ x_1 = u_2 u_3 u_4, \quad x_2 = u_1 u_3 u_4, \quad x_3 = u_1 u_2 u_4, \quad x_4 = u_1 u_2 u_3. \]

Then Eq. (6) shows that we have a solution of the equations (1). This solution has some interesting properties.

Setting
\[ A^2 = x_1^2 x_2^2 + x_3^2 x_4^2, \quad B^2 = x_1^2 x_3^2 + x_2^2 x_4^2, \quad C^2 = x_1^2 x_4^2 + x_2^2 x_3^2, \]
we see from (7) that \( A, B, C \) are integers. Setting \( E^2 = A^2 + B^2 + C^2 \), we see from (8) that \( E \) is an integer. Finally, Eq. (9) shows that
\[ S = x_1^2 + x_2^2 + x_3^2 + x_4^2 = ABC/x_1 x_2 x_3 x_4. \]
These relations are homogeneous and so are valid whether or not the solution $x_1, x_2, x_3, x_4$ is primitive. The following result is valid only for a primitive solution. Set

$$D = \gcd(x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_2x_3x_4),$$

$$\Delta = \gcd(A, B, C).$$

Then we have

$$x_1x_2x_3x_4 = D^2/\Delta,$$

as is easily verified by calculating the $p$-adic values of $D$, $\Delta$, $x_1x_2x_3x_4$. For $p$ prime we may suppose that

$$v_p(u_1) = 0, \quad v_p(u_2) = \alpha, \quad v_p(u_3) = \beta, \quad v_p(u_4) = \gamma,$$

with $0 \leq \alpha \leq \beta \leq \gamma$. For the corresponding primitive solution we then have

$$v_p(x_1) = \gamma, \quad v_p(x_2) = \gamma - \alpha, \quad v_p(x_3) = \gamma - \beta, \quad v_p(x_4) = 0,$$

and we easily obtain

$$v_p(A) = 2\gamma - \alpha - \beta, \quad v_p(D) = \gamma - \alpha - \beta,$$

from which the result follows.

A parametric solution to Eq. (5) is obtained by the following method. The identity

$$(p + q + r)(p - q - r)(q - r - p)(r - p - q) = p^4 + q^4 + r^4 - 2(q^2r^2 + r^2p^2 + p^2q^2)$$

shows that

$$p + q + r = 0 \quad \text{implies} \quad p^4 + q^4 + r^4 = 2(q^2r^2 + r^2p^2 + p^2q^2).$$

We rewrite (5) in the form

$$u_4^4 - 2u_2^4(u_1^2 + u_2^2 + u_3^2) + u_1^4 + u_2^4 + u_3^4 - 2(u_1^2u_3^3 + u_2^2u_4^3 + u_4^2u_1^3) = 0.$$ 

Setting $u_1 + u_2 + u_3 = 0$, we have from (10)

$$u_4^2 = 2(u_1^2 + u_2^2 + u_3^2).$$

To make $u_4$ rational, we set

$$u_1 = v_2^2 - v_3^2, \quad u_2 = v_3^2 - v_1^2, \quad u_3 = v_1^2 - v_2^2$$

with $v_1 + v_2 + v_3 = 0$.

In effect we have from (10)

$$2(u_1^2 + u_2^2 + u_3^2) = (v_1^2 + v_2^2 + v_3^2)^2,$$

whence $u_4 = v_1^2 + v_2^2 + v_3^2$. We thus obtain

$$x_1 = (v_2^2 - v_1^2)(v_1^2 - v_2^2)(v_1^2 + v_2^2 + v_3^2),$$

$$x_2 = (v_1^2 - v_2^2)(v_2^2 - v_3^2)(v_1^2 + v_2^2 + v_3^2),$$

$$x_3 = (v_2^2 - v_3^2)(v_3^2 - v_1^2)(v_1^2 + v_2^2 + v_3^2),$$

$$x_4 = (v_3^2 - v_1^2)(v_1^2 - v_2^2)(v_1^2 - v_3^2),$$

with $v_1 + v_2 + v_3 = 0$. This is equivalent to Tebay's solution, which is obtained by setting $v_2 = 2$ (abandoning homogeneity) and $v_1 = s - 1$, whence $v_3 = -(s + 1)$. 

We note that Euler made several studies of (5) [1, p. 661]; however, there is no mention of the relation between Eqs. (1) and (5).

5. TABLES. In Table 1 we give the smallest solution (that with minimum $S$) for $3 \leq n \leq 8$, and in Tables 2–4 we give all solutions for $3 \leq n \leq 5$ having $S \leq 10^9$. For $n = 3$ tables have been given by Lal and Blundon [3], Leech [5] and Spohn [8]. The present computations were done on the IBM 370 computer at C.I.R.C.E. Each $S$ is expressed as the sum of two squares $x_i^2 + y_i^2$ in all possible ways by the method of Nicolas [7]. We retain only those $S$ which are expressible in at least $n$ ways; we then have to test whether any $n$ of these satisfy

$$\sum_{i=1}^{n} x_i^2 = S.$$

It may be remarked that it is never necessary to test whether an integer is a perfect square.

**Table 1**

The smallest solutions

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**Table 2**

$n = 3$

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Table 3
\( n = 4 \)  
\[
\begin{array}{cccc|c|ccc}
\hline
x_1 & x_2 & x_3 & x_4 & S & u_1 & u_2 & u_3 & u_4 \\
\hline
1 & 60 & 105 & 168 & 280 & 121249 & 3 & 5 & 8 & 14 \\
2 & 420 & 728 & 1365 & 1560 & 5003209 & 7 & 8 & 15 & 26 \\
3 & 385 & 792 & 840 & 1980 & 5401489 & 14 & 33 & 35 & 72 \\
4 & 672 & 1120 & 1980 & 3465 & 17632609 & 32 & 56 & 99 & 165 \\
5 & 585 & 1008 & 1456 & 5460 & 33289825 & 12 & 45 & 65 & 112 \\
6 & 840 & 1520 & 1995 & 6384 & 47751481 & 5 & 16 & 21 & 38 \\
7 & 880 & 1155 & 5040 & 5544 & 58245961 & 10 & 11 & 48 & 63 \\
8 & 624 & 2625 & 3220 & 6432 & 59019025 & & & & \\
9 & 1848 & 3575 & 4620 & 7800 & 98380129 & 77 & 130 & 168 & 325 \\
10 & 2508 & 5544 & 5985 & 8360 & 142735825 & 63 & 88 & 95 & 210 \\
11 & 2295 & 3808 & 7344 & 10080 & 175308625 & 51 & 70 & 135 & 224 \\
12 & 1232 & 8316 & 9141 & 10368 & 261726985 & & & & \\
13 & 3276 & 5005 & 11880 & 16632 & 453540025 & 65 & 91 & 216 & 330 \\
14 & 2040 & 2520 & 11781 & 26180 & 834696361 & 18 & 40 & 187 & 231 \\
15 & 4620 & 8184 & 11935 & 26040 & 908848081 & 11 & 24 & 35 & 62 \\
\hline
\end{array}
\]

Where a solution can be obtained by the method of Section 4, the values of \( u_i \) are given.

Table 4
\( n = 5 \)  
\[
\begin{array}{cccc|c|ccc}
\hline
x_1 & x_2 & x_3 & x_4 & x_5 & S \\
\hline
1 & 28 & 64 & 259 & 392 & 680 & 688025 \\
2 & 1112 & 1225 & 1876 & 3184 & 5768 & 49664225 \\
3 & 2105 & 2648 & 2980 & 3736 & 4720 & 56559425 \\
4 & 203 & 2240 & 3920 & 4240 & 6104 & 75661625 \\
5 & 696 & 1200 & 3475 & 4980 & 6360 & 79250041 \\
6 & 56 & 208 & 1400 & 4060 & 9065 & 100664225 \\
7 & 557 & 1747 & 4141 & 5219 & 8285 & 116389325 \\
8 & 427 & 3164 & 3980 & 6220 & 7420 & 119778425 \\
9 & 1183 & 1300 & 2240 & 7280 & 8080 & 126391889 \\
10 & 1095 & 3063 & 4119 & 5527 & 10329 & 164783125 \\
11 & 1952 & 2360 & 5020 & 6089 & 10520 & 182326625 \\
12 & 595 & 3549 & 5235 & 9555 & 10893 & 250310125 \\
13 & 2328 & 5824 & 7368 & 9975 & 14196 & 394653025 \\
14 & 2207 & 4417 & 5215 & 12479 & 14161 & 407836325 \\
15 & 483 & 5328 & 6356 & 15000 & 17304 & 593448025 \\
16 & 49 & 2152 & 5600 & 16076 & 18088 & 621607025 \\
17 & 3799 & 9560 & 11384 & 13732 & 16112 & 683585825 \\
18 & 2425 & 3020 & 8596 & 19628 & 20020 & 874951025 \\
\hline
\end{array}
\]

Remark. In the solutions 7, 10, 12 and 14, all the \( x_i \) are odd.

6. Concluding Remarks. (a) Examination of the tables suggests that there may be simple parametric solutions for \( n \geq 5 \), but we have not found them by the present method.

(b) There exist values of \( \alpha, \beta \) for which Eq. (2) has trivial solutions; these can then be transformed into nontrivial solutions. This is the case when we replace the sums of \( n - 1 \) squares by their arithmetic means.

(c) I shall return later to the case of \( n = 3 \) with general \( \alpha, \beta \). Several of the systems of equations studied in [1, Chapter XIX], are effectively of this type. They are, however, treated by methods specific to each problem; we can now treat them by a uniform method.
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6. A. Martin, “Find four square numbers such that the sum of every three of them shall be a square,” Math. Quest. Educ. Times, v. 24, 1913, pp. 81–82.
9. S. Tebay, “Find four square numbers such that the sum of every three of them shall be a square,” Math. Quest. Educ. Times, v. 68, 1898, pp. 103–104.