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On the Sharpness of Certain Local Estimates for $\mathring{H}^1$ Projections into Finite Element Spaces: Influence of a Reentrant Corner*

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Abstract. In a plane polygonal domain with a reentrant corner, consider a homogeneous Dirichlet problem for Poisson's equation $-\Delta u = f$ with f smooth and the corresponding Galerkin finite element solutions in a family of piecewise polynomial spaces based on quasi-uniform (uniformly regular) triangulations with the diameter of each element comparable to h, $0 < h \leqslant 1$. Assuming that u has a singularity of the type $|x - v_M|^\beta$ at the vertex v_M of maximal angle π/β, we show: (i) For any subdomain A and any s, the error measured in $H^{-s}(A)$ is not better than $O(h^{2\beta})$. (ii) On annular strips of points of distance of order d from v_M, the pointwise error is not better than $O(h^{2\beta}d^{-\beta})$.

1. Context and Results. Let Ω be a polygonal bounded simply connected domain in the plane and consider the Dirichlet problem

$$(1.1)\qquad \begin{aligned} -\Delta u &= f \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $f \in \mathcal{C}^\infty(\bar{\Omega})$. Let $\alpha_1 \leqslant \alpha_2 \leqslant \cdots \leqslant \alpha_{M-1} < \alpha_M$ denote the interior angles at the corners. Assume that there is only one vertex, v_M, of maximal angle and that the angle is reentrant, $\alpha_M > \pi$. For simplicity in notation set

$$\alpha := \alpha_M, \quad \bar{\alpha} := \alpha_{M-1}, \quad \beta := \pi/\alpha, \quad \bar{\beta} := \pi/\bar{\alpha}.$$

Then $\beta < 1$ and $\beta < \bar{\beta}$.

As is well known, cf. Grisvard [4], Kellogg [5] or Kondrat'ev [6], the solution of (1.1) can be expressed in terms of polar coordinates r, θ centered at v_M and with the positive θ-axis along one leg as

$$(1.2)\qquad u = at_\beta + w; \qquad t_\beta(r,\theta) = \omega_0(r,\theta)r^\beta \sin(\beta\theta),$$
$$w = o(r^\beta) \quad \text{as } r \to 0,$$

where a is a constant and $\omega_0 \in \mathcal{C}^\infty(\bar{\Omega})$ with $\omega_0 \equiv 1$ for $r \leqslant r_0$, $\omega_0 \equiv 0$ for $r \geqslant 2r_0$. We assume that $2r_0$ is less than the length of the shortest leg emanating from v_M so that $t_\beta = 0$ on $\partial\Omega$.

Let S_h, $0 < h \leqslant 1$, be finite element subspaces of $\mathring{H}^1(\Omega)$ such that, for $v \in H^\gamma \cap \mathring{H}^1$ with $\gamma > 1$,

$$(1.3)\qquad \min_{\chi \in S_h} \|v - \chi\|_{\mathring{H}^1(\Omega)} \leqslant Ch^{\gamma'-1}\|v\|_{H^\gamma(\Omega)}, \qquad \gamma' = \min(\gamma, R),$$

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where C does not depend on h or v and where $R \geqslant 2$; cf. Ciarlet [2]. Here and below we use standard notation for the L_2-based Sobolev spaces H^γ and $\mathring{H}^\gamma$; cf. Adams [1]. $H^{-\gamma}(A)$ for $\gamma > 0$ shall refer to the dual space of $\mathring{H}^\gamma(A)$ with respect to the pivot space L_2, i.e., for $w \in L_2(A)$,

$$\|w\|_{H^{-\gamma}(A)} := \sup_{\substack{v \in C_0^\infty(A) \\ v \neq 0}} \frac{\int_A vw}{\|v\|_{\mathring{H}^\gamma(A)}}.$$

The $\mathring{H}^1$ projection of u into S_h, $P_h u$, is defined by

$$(1.4) \qquad (u - P_h u, \chi)_{H^1(\Omega)} := \int_\Omega \nabla(u - P_h u) \cdot \nabla\chi = 0 \quad \text{for all } \chi \in S_h,$$

i.e., $P_h u$ is the Galerkin finite element solution of (1.1). (The effect of numerical integration is not considered in this note.)

On interior subdomains A of Ω away from the corners the solution u is smooth and hence, for commonly used finite element spaces, approximable to "high" order, typically $O(h^R)$ in $L_\infty(A)$. Numerical observations show that, for finite element spaces without special arrangements to treat corner singularities, the error in $u - P_h u$ on A is generally not of this "high" order. This is often referred to as a "pollution" effect from the corners.

For $A \subseteq \Omega$ and with the error measured in a negative norm $H^{-s}(A)$, a standard duality argument using (1.3) gives immediately an upper bound $O(h^{2\beta-\varepsilon})$ for the error, for any $\varepsilon > 0$. ($\varepsilon = 0$ can be taken in most cases.) If $a \neq 0$ in (1.2), it has long been felt that the error is not better than $O(h^{2\beta})$ for general finite element spaces; this is sometimes also referred to as "pollution", even when $A = \Omega$.

As for previous proofs that "pollution" (in the second sense) occurs, we cite the following two works.

(i) An example by Babuška and Bramble with Ω the L-shaped domain, $\beta = 2/3$, and regular uniform meshes. Given $\varepsilon > 0$, for each h there exists u_h such that

$$\|u_h - P_h u_h\|_{L_2(\Omega)} \geqslant c_\varepsilon h^{4/3-\varepsilon}\|u_h\|_{H^{s+1}(\Omega)},$$

where $c_\varepsilon > 0$ is independent of h. This example was given in [10, Section 7, Example 4] but a full proof has never appeared.

(ii) A result of Dobrowolski [3, Theorem 7.1]. He considered a family of "unrefined" meshes in the sense that if $a \neq 0$ in (1.2), then there exists $c > 0$ such that

$$(1.5) \qquad \min_{\chi \in S_h} \|u - \chi\|_{H^1(\Omega)} \geqslant ch^\beta,$$

and he showed that then

$$\|u - P_h u\|_{L_2(\Omega)} \geqslant ch^{2\beta}.$$

For piecewise polynomial spaces on a family of triangulations of Ω, (1.5) would hold if each mesh contained merely one element τ_h at v_M with its largest inscribed disc of radius $\geqslant ch$, c positive and independent of h. To see this, note that by a simple scaling argument we have already

$$\min_{\substack{\chi \text{ polynomial} \\ \text{of degree} \leqslant R-1}} \|r^\beta \sin(\beta\theta) - \chi\|_{H^1(\tau_h)} \geqslant ch^\beta.$$

Thus, (1.5) would follow for $u = at_\beta$. Since, by [4], [5], [6], $w := u - at_\beta \in H^{1+\sigma-\varepsilon}(\Omega)$, $\sigma = \min(2\beta, \bar{\beta}) > \beta$, we have from (1.3) that $\|w - P_h w\|_{H^1} \leqslant Ch^{\sigma'-\varepsilon} = h^\beta o(1)$ for ε small enough, where $\sigma' = \min(\sigma, R-1) > \beta$. Thus (1.5) would follow for a general u with $a \neq 0$.

In this context we mention the interesting structure results of Nitsche [8]. They seem less successful in explaining the present "pollution" from corners than in investigating the corresponding "pollution" from boundary singularities in one-dimensional singular Sturm-Liouville problems [9]; cf. also Schreiber [13, Section 6.2].

Our first result in this note generalizes the results of Babuška, Bramble and Dobrowolski mentioned above.

THEOREM 1.1. *Assume* (1.3), (1.5) *and that* $a \neq 0$ *in* (1.2). *Let* $A \subseteq \Omega$ *be any subdomain of* Ω *and* s *any nonnegative number. There exist positive constants* c *and* h_0 *such that, for* $h \leqslant h_0$,

$$\|u - P_h u\|_{H^{-s}(A)} \geqslant ch^{2\beta}.$$

The proof will be furnished in Section 2.

We shall next describe the second result of this note. We shall have to be more precise about various properties of the finite element spaces. For simplicity, cf. Remark 1.1, consider edge-to-edge triangulations parametrized by h, $0 < h \leqslant 1$, of Ω into disjoint triangular elements τ_i^h, $i = 1, \dots, I_h$, and let S_h consist of functions χ such that $\chi \in \mathcal{C}^0(\bar{\Omega})$, $\chi = 0$ on $\partial\Omega$ and $\chi|_{\tau_i^h}$ is a polynomial of total degree $R - 1 \geqslant 1$. Let the family of meshes be quasi-uniform (a.k.a. uniformly regular), i.e., let there exist positive constants c and C independent of i and h such that, with ρ_i^h denoting the radius of the largest inscribed disc of τ_i^h,

$$ch \leqslant \rho_i^h \leqslant \operatorname{diam}(\tau_i^h) \leqslant Ch \quad \text{for all } i, h.$$

For brevity, let us call such a family S_h a "quasi-uniform Lagrange" family.

In Schatz and Wahlbin [11] it was shown that, for x close to v_M, for any $\varepsilon > 0$,

$$|(u - P_h u)(x)| \leqslant C_\varepsilon h^{2\beta-\varepsilon}|x - v_M|^{-\beta}. \tag{1.6}$$

This estimate was derived as a mix of two effects: local approximability of the solution and the "pollution" influence. In fact, "pollution" dominates in this estimate as can be seen via the following argument.

One knows from [4], [5], [6] that $|D^\alpha u(r,\theta)| \leqslant Cr^{\beta-|\alpha|}$ as $r \to 0$. Thus, with

$$A_d = \{x : d \leqslant |x - v_M| \leqslant 2d\} \cap \Omega, \tag{1.7}$$

we have, for $d \geqslant h^{1-\delta}$, $\delta > 0$, but d not too large (the remaining corners should be well away),

$$\min_{\chi \in S_h} \|u - \chi\|_{L_\infty(A_d)} \leqslant Ch^R d^{\beta-R} = Ch^{2\beta} d^{-\beta}\left(\frac{h}{d}\right)^{R-2\beta} = h^{2\beta} d^{-\beta} o(1).$$

The result analogous to (1.6) for the five-point difference scheme and various classical ways of imposing the boundary conditions was given in Laasonen [7].

Our second result establishes the sharpness of (1.6).

THEOREM 1.2. *Let S_h be a quasi-uniform Lagrange family. Assume that $a \neq 0$ in* (1.2).

Then, for any $\delta > 0$, there exist positive constants, c, d_0 and h_0 such that with A_d as in (1.7) *and $h^{1-\delta} \leqslant d \leqslant d_0$, $h \leqslant h_0$,*

$$\|u - P_h u\|_{L_\infty(A_d)} \geqslant ch^{2\beta}d^{-\beta}.$$

Again, the proof will be given in Section 2.

Remark 1.1. Theorem 1.2 can be established for more general finite element spaces than quasi-uniform Lagrange families. In fact, it holds under the general assumptions of [11, Section 2] if furthermore (1.5) is assumed. The proof is the same, but we do not wish to repeat the rather lengthy hypotheses of [11] in this note.

2. Proofs. Before giving the proofs of Theorems 1.1 and 1.2 we shall collect some more precise information about the problem (1.1) and the decomposition (1.2). The exact results can be found in [4], [5], [6]. For motivation one notes that away from vertices the solution u of (1.1) is smooth, whereas at a vertex v_i of angle α_i one has, in polar coordinates centered at that vertex and with $\beta_i = \pi/\alpha_i$, $\gamma_i \in \mathbf{Z}^+ \cup \{0\}$,

$$u(r,\theta) = a_i r^{\beta_i}\sin(\beta_i\theta) + b_i r^{2\beta_i}\sin(2\beta_i\theta)(\ln r)^{\gamma_i} + \text{smoother terms.}$$

The exact results are as follows.

For any $\varepsilon > 0$ there exists a constant C_ε such that

$$\|u\|_{H^{\beta+1-\varepsilon}(\Omega)} \leqslant C_\varepsilon \|f\|_{H^{\beta-1}(\Omega)}. \tag{2.1}$$

Let Ω_M be a neighborhood of v_M avoiding all other vertices. Then for any $\varepsilon > 0$, with w as in (1.2), we have in fact that

(2.2.i) $\quad |D^\alpha w(r,\theta)| \leqslant C_{\alpha,\varepsilon} r^{2\beta - |\alpha| - \varepsilon}$ as $r \to 0$,

(2.2.ii) $\quad w \in H^{1+2\beta-\varepsilon}(\Omega_M)$,

(2.2.iii) $\quad u \in H^{1+\bar\beta-\varepsilon}(\Omega \setminus \Omega_M)$,

(2.2.iv) $\quad w \in H^{1+\sigma-\varepsilon}(\Omega), \qquad \sigma = \min(2\beta, \bar\beta) > \beta.$

Proof of Theorem 1.1. The heart of the matter is the short calculation (2.4), once a suitable u_0 has been found. The rest of the proof is routine.

We first show that it suffices to treat a particular $u_0 = a_0 t_\beta + \cdots$ with $a_0 \neq 0$. For this let

$$\bar w := a a_0^{-1} u_0 - u.$$

Then, by (1.2) and (2.2.iv), $\bar w \in H^{1+\sigma-\varepsilon/2}(\Omega)$, with $\sigma > \beta$. Using (2.1) and (1.3), we have by the standard duality argument, writing $\bar E := \bar w - P_h \bar w$,

$$\|\bar E\|_{H^{-s}(\Omega)} \leqslant Ch^{\beta-\varepsilon/2}\|\bar E\|_{H^1(\Omega)} \leqslant Ch^{\beta+\sigma'-\varepsilon},$$

where $\sigma' = \min(\sigma, R-1) > \beta$. Thus, for ε small enough,

$$\|\bar E\|_{H^{-s}(\Omega)} = h^{2\beta}o(1) \quad \text{as } h \to 0.$$

Our specific u_0 is constructed as follows. Let $x_0 \in \operatorname{Int} A$, and let $A_0 \subset\subset A$ be an annulus centered at x_0. If $A_0 = B_1 \setminus B_0$, where $B_0 \subset\subset B_1 \subset\subset A$ are concentric discs, let $\omega \in \mathcal{C}^\infty(\bar{\Omega})$ with

$$\omega = \begin{cases} 1 & \text{outside } B_1, \\ 0 & \text{inside } B_0. \end{cases}$$

Let further $G_0(x)$ be the Green's function for (1.1) with singularity at x_0, and set

$$u_0 := \omega G_0.$$

Then $u_0 \in \mathcal{C}^\infty(\operatorname{Int}\Omega)$, $u_0 = 0$ on $\partial\Omega$, and

$$\operatorname{supp}(\Delta u_0) \subseteq A_0 \subset\subset A. \tag{2.3}$$

Also, $a_0 \neq 0$, as can be seen by classical means from the fact that G_0 is harmonic and positive in a neighborhood of v_M. For completeness, we give the argument. Use a conformal map z^β to locally straighten the boundary. The transformed function $\tilde{u}_0$ is then harmonic and vanishes on a piece of the real axis; hence it is smooth and harmonic in a neighborhood of the origin by Schwarz' reflection principle and Weyl's lemma (or, by Schauder estimates). Thus, in new polar coordinates ρ, ϕ,

$$\tilde{u}_0 = \sum_1^\infty A_i \rho^i \sin(i\phi)$$

for ρ small enough; again, since $\tilde{u}_0$ is harmonic and vanishes on a piece of the real axis. Since $\tilde{u}_0$ is positive for ρ small, A_1 is positive by the orthogonality of $\sin(i\phi)$ on $[0,\pi]$. Conclude by transforming back to original coordinates; A_1 corresponds to a_0.

We now have by (1.5), (1.4), Green's formula (that its use is permitted is easily checked) and by (2.3), setting $E_0 := u_0 - P_h u_0$,

$$\begin{aligned} ch^{2\beta} &\leqslant \|E_0\|^2_{\dot{H}^1(\Omega)} = (E_0, u_0 - P_h u_0)_{\dot{H}^1(\Omega)} = (E_0, u_0)_{\dot{H}^1(\Omega)} = -\int E_0(\Delta u_0) \\ &\leqslant \|E_0\|_{H^{-s}(A)} \|\Delta u_0\|_{\mathring{H}^s(A)}. \end{aligned} \tag{2.4}$$

Thus,

$$\|E_0\|_{H^{-s}(A)} \geqslant \frac{c}{\|\Delta u_0\|_{\mathring{H}^s(A)}} h^{2\beta},$$

which proves Theorem 1.1.

Proof of Theorem 1.2. The heart of the matter is (2.6) below, corresponding to (2.4) in the proof of Theorem 1.1. The rest of the proof consists of nontrivial technicalities.

Note that (1.3), (1.5) and all results of [11] hold for a quasi-uniform Lagrange family.

We shall first consider specific u_d, depending on d. Let $\omega_d \in \mathcal{C}^\infty(\bar{\Omega})$ with

$$\omega_d(x) = \begin{cases} 1 & \text{for } |x - v_M| \leqslant d, \\ 0 & \text{for } |x - v_M| \geqslant 2d. \end{cases}$$

Thus, $\operatorname{supp}(\nabla\omega_d) \subseteq A_d$, and we may assume by a scaling argument that

$$\|\omega_d\|_{\mathcal{C}^k} \leqslant C_k d^{-k}, \qquad C_k \text{ independent of } d. \tag{2.5}$$

Set now

$$u_d := \omega_d r^\beta \sin(\beta\theta).$$

Assume that $2d$ is less than the length of the shortest leg emanating from v_M so that $u_d = 0$ on $\partial\Omega$. Since $r^\beta \sin(\beta\theta)$ is harmonic, we find that $\operatorname{supp}(\Delta u_d) \subseteq A_d$.

Let $E_d := u_d - P_h u_d$. We note that, with $c > 0$ independent of d and h,

$$\|E_d\|_{H^1(\Omega)} \geqslant ch^\beta.$$

This follows since, assuming that $\operatorname{supp}\omega_d \subseteq \{w\colon \omega_0 \equiv 1\}$ with ω_0 as in (1.2), we have $u_d = t_\beta + y_d$, where $y_d := (\omega_d - 1)t_\beta$. Here, by (1.5),

$$\|t_\beta - P_h t_\beta\|_{H^1} \geqslant ch^\beta.$$

By (1.3) and a simple calculation of $\|y_d\|_{H^2}$ (using (2.5)), we find that, for $d \geqslant h^{1-\delta}$,

$$\|y_d - P_h y_d\|_{H^1} \leqslant Ch\|y_d\|_{H^2} \leqslant Chd^{\beta-1} = Ch^\beta\left(\frac{h}{d}\right)^{1-\beta} = h^\beta o(1).$$

As in (2.4), we now have

$$ch^{2\beta} \leqslant -\int E_d \Delta u_d \leqslant \|E_d\|_{L_\infty(A_d)}\|\Delta u\|_{L_1(A_d)}. \tag{2.6}$$

Using (2.5), it is easily calculated that

$$\|\Delta u_d\|_{L_1(A_d)} \leqslant Cd^\beta.$$

Hence,

$$\|E_d\|_{L_\infty(A_d)} \geqslant ch^{2\beta}d^{-\beta}.$$

It remains to verify the same result for any u with $a \neq 0$. This is now slightly more complicated than in the proof of Theorem 1.1 since the exact dependence on d needs to be accounted for and our model functions u_d depend on d. With

$$\overline{w}_d := au_d - u, \qquad \overline{E}_d := \overline{w}_d - P_h\overline{w}_d$$

we have to show that

$$\|\overline{E}_d\|_{L_\infty(A_d)} = h^{2\beta}d^{-\beta}o(1) \quad \text{as } h \to 0, \text{ for } h^{1-\delta} \leqslant d \leqslant d_0. \tag{2.7}$$

Let ω_0 be the cutoff function in (1.2). We may assume that $\operatorname{supp}\omega_d \subseteq \{x\colon \omega_0 \equiv 1\}$. Thus,

$$\overline{w}_d = a\omega_d t_\beta - u, \qquad u = at_\beta + w,$$

and so, with w as in (1.2) and (2.2),

$$\begin{aligned} \overline{w}_d &= \omega_0\overline{w}_d + (1-\omega_0)\overline{w}_d = \omega_0\big(a\omega_d t_\beta - u\big) - (1-\omega_0)u \\ &= \omega_0\big(-w - a(1-\omega_d)t_\beta\big) - (1-\omega_0)u \\ &= -\omega_0 w - a\omega_0(1-\omega_d)t_\beta - (1-\omega_0)u \equiv \overline{w}_d^1 + \overline{w}_d^2 + \overline{w}_d^3. \end{aligned} \tag{2.8}$$

We next quote a result from [11, Theorem 3.2] on local maximum norm estimates. For any $\varepsilon > 0$, there exists C_ε such that

$$\begin{aligned} \|\overline{E}_d\|_{L_\infty(A_d)} \leqslant C_\varepsilon h^{-\varepsilon}\Big\{ &\min_{\chi \in S_h}\Big(\|\overline{w}_d - \chi\|_{L_\infty(A'_d)} + h\|\overline{w}_d - \chi\|_{W^1_\infty(A'_d)}\Big) \\ &+ d^{-1}\|\overline{E}_d\|_{L_2(A'_d)}\Big\}, \end{aligned} \tag{2.9}$$

where $A'_d = \{x\colon d/2 \leqslant |x - v_M| \leqslant 4d\} \cap \Omega$. (In the quasi-uniform context, cf. [12, Theorem 7.1] for a simpler proof.)

By well-known approximation results, cf. [2], using (2.8), (2.2.i) and a simple direct calculation on $\bar{w}_d^2$, we have, with $A''_d = (A'_d + h) \cap \Omega$,

$$\min_{\chi \in S_h} \left(\|\bar{w}_d - \chi\|_{L_\infty(A'_d)} + h\|\bar{w}_d - \chi\|_{W^1_\infty(A'_d)} \right) \tag{2.10}$$
$$\leqslant Ch^2 \|\bar{w}_d\|_{\mathcal{C}^2(A''_d)} = Ch^2 \|\bar{w}_d^1 + \bar{w}_d^2\|_{\mathcal{C}^2(A''_d)}$$
$$\leqslant Ch^2 \{d^{2\beta - 2 - \varepsilon} + d^{\beta - 2}\} \leqslant Ch^2 d^{\beta - 2}.$$

A slight variation of the usual duality argument, in order to account for the precise dependence on d, gives (see [11, Lemma 5.1] for details)

$$\frac{1}{d}\|\bar{E}_d\|_{L_2(A'_d)} \leqslant Ch^{\beta - \varepsilon} d^{-\beta} \|\bar{E}_d\|_{H^1(\Omega)}. \tag{2.11}$$

Next, by (1.3) and (2.2.ii),

$$\|\bar{w}_d^1 - P_h \bar{w}_d^1\|_{H^1(\Omega)} \leqslant Ch^{2\beta - \varepsilon} \|\bar{w}_d^1\|_{H^{1 + 2\beta - \varepsilon}(\Omega)} \leqslant Ch^{2\beta - \varepsilon}. \tag{2.12}$$

Further, by (1.3) and a simple calculation of $\|\bar{w}_d^2\|_{H^2}$,

$$\|\bar{w}_d^2 - P_h \bar{w}_d^2\|_{H^1(\Omega)} \leqslant Ch \|\bar{w}_d^2\|_{H^2(\Omega)} \leqslant Chd^{\beta - 1}. \tag{2.13}$$

Also, by (1.3) and (2.2.iii), we have, with $\beta' = \min(\bar{\beta}, R - 1) > \beta$,

$$\|\bar{w}_d^3 - P_h \bar{w}_d^3\|_{H^1(\Omega)} \leqslant Ch^{\beta' - \varepsilon} \|\bar{w}_d^3\|_{H^{1 + \bar{\beta} - \varepsilon}(\Omega)} \leqslant Ch^{\beta' - \varepsilon}. \tag{2.14}$$

Inserting (2.12)–(2.14) into (2.11) gives

$$\frac{1}{d}\|\bar{E}_d\|_{L_2(A'_d)} \leqslant C\{h^{3\beta - \varepsilon} d^{-\beta} + h^{1 + \beta - \varepsilon} d^{-1} + h^{\beta' + \beta - 2\varepsilon} d^{-\beta}\}.$$

Using this and (2.10) in (2.9),

$$\|\bar{E}_d\|_{L_\infty(A_d)} \leqslant Ch^{-\varepsilon}\{h^2 d^{\beta - 2} + h^{3\beta - 2\varepsilon} d^{-\beta} + h^{1 + \beta - \varepsilon} d^{-1} + h^{\beta' + \beta - 2\varepsilon} d^{-\beta}\}$$
$$= Ch^{2\beta} d^{-\beta} \left\{ \left(\frac{h}{d}\right)^{2 - 2\beta} h^{-\varepsilon} + h^{\beta - 3\varepsilon} + \left(\frac{h}{d}\right)^{1 - \beta} h^{-2\varepsilon} + h^{\beta' - \beta - 3\varepsilon} \right\}.$$

Since $h/d \leqslant h^\delta$ and $\beta' > \beta$, we obtain the desired result (2.7) for ε small enough.

This completes the proof of Theorem 1.2.

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