

The techniques used are based on the methods described in [1] and [2]. In [3], it was pointed out that the method in [2] would fail on a certain class of matrices, and a method was given extending the collection of solvable problems to cover this class. The class in question can be described in terms of the interlacing property

$$(2) \quad \lambda_i \leq \mu_i \leq \lambda_{i+1}, \quad i = 1, \dots, n - 1.$$

If the inequalities are strict, then we say J satisfies the *strict interlacing property*. This last property is necessary for the algorithm in [2] but not for the one in [3]. In addition, unlike [2], the method in [3] does not require that the order of the system, n , be even. However the method in [3] starts with substantially different initial data and hence does not directly solve the problem posed in [2].

In this method, we need to assume only that the μ_i be simple. This is equivalent to assuming that the off-diagonal elements of \bar{J} be nonzero [8, p. 300]. In order to ensure that a solution exists in the real numbers, we also need to assume that the λ_i satisfy (2), and that β lies in an interval $(0, \beta_{\max})$, which will be defined below.

2. Preliminaries. We need to define the following two bordered diagonal matrices A, A^- [4] which appear as intermediate results in the course of the computation, and which are similar to J and J^- , respectively, by the same transformation:

$$(3a) \quad A = \begin{bmatrix} 1 & 0 \\ 0 & P^T \end{bmatrix} J \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix}, \quad A^- = \begin{bmatrix} 1 & 0 \\ 0 & P^T \end{bmatrix} J^- \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix},$$

where P is the orthogonal matrix of eigenvalues of the submatrix \bar{J} . Hence A and J have the same characteristic polynomial, and so do A^- and J^- .

The matrices A, A^- , can be written in the form:

$$(3b) \quad A = \begin{bmatrix} a_{11} & \mathbf{c}^T \\ \mathbf{c} & M \end{bmatrix}, \quad A^- = \begin{bmatrix} a_{11} & (\mathbf{c}^-)^T \\ \mathbf{c}^- & M \end{bmatrix},$$

where $M = \text{diag}(\mu_1, \dots, \mu_{n-1}) = P^T \bar{J} P$, and the eigenvalues of A and A^- are λ_i and λ_i^- , respectively. We shall later see how to define A^- using only β instead of the λ_i^- .

In the following, we will denote the i th row of P as \mathbf{p}_i^T , that is $P^T = [\mathbf{p}_1, \dots, \mathbf{p}_{n-1}]$.

It is obvious that

$$(4a) \quad a_{11} = \text{trace}(A) - \text{trace}(M) = \sum_1^n \lambda_i - \sum_1^{n-1} \mu_i.$$

By expanding the determinant, the characteristic polynomial of A may be written as

$$(4b) \quad \det(\lambda I - A) = (\lambda - a_{11}) \prod_{j=1}^{n-1} (\lambda - \mu_j) - \sum_{k=1}^{n-1} c_k^2 \left(\prod_{\substack{j=1, \\ j \neq k}}^{n-1} (\lambda - \mu_j) \right).$$

Setting $\lambda = \mu_1, \dots, \mu_{n-1}$ and solving the $n - 1$ equations for the c_i yields

$$(4c) \quad c_i^2 = - \frac{\prod_{j=1}^n (\mu_i - \lambda_j)}{\prod_{j=1; j \neq i}^{n-1} (\mu_i - \mu_j)} \geq 0,$$

thus completely defining A up to the choice of signs of the c_i .

$$n = 6, \beta = 1$$

initial λ_i	initial μ_i	c_i	c_i^-
0	0.26794919	0.57735027	0
1.00000000	1.00000000	0	1.00000000
1.00000000	2.00000000	1.15470054	0
3.00000000	3.00000000	0	1.00000000
3.00000000	3.73205081	0.57735027	0
4.00000000	-	-	-

The following example illustrates the fact that the b_i need not be positive. This example is simply the J^- , defined by (1b), corresponding to the J used in the above case $n = 4$.

$$n = 4, \beta = -1$$

initial λ_i	initial μ_i	c_i	c_i^-
0.58578644	0.58578644	0	1.00000000
0.58578644	2.00000000	1.41421356	0
3.41421356	3.41421356	0	1.00000000
3.41421356	-	-	-

The above cases all have remarkable property that for all $i, i = 1, \dots, n - 1$, either $c_i = 0$ or $c_i^- = 0$, so that the reconstructed matrix J is essentially unique (up to signs of the off-diagonal elements). In the following example, we show a more general case, in which the answer is not unique. The results were obtained by starting with one set of eigenvalue data and cycling through all possible sign combinations for the c_i, c_i^- . The initial data and the c vectors were:

$$n = 4, \beta = .25$$

initial λ_i	initial μ_i	c_i	c_i^-
0	0.58578644	1.00000000	0.86602540
2.00000000	2.00000000	0	0.70710678
2.00000000	3.41421356	1.00000000	0.86602540
4.00000000	-	-	-

The 4 distinct matrices that were computed are given below. For brevity, we show each distinct matrix only once, since each one actually appeared 8 times. Since there were no degeneracies encountered in this example, and by observing that any matrix satisfying the initial conditions (given by λ_i, μ_i, β) must satisfy (4a)–(4d) and (5c), it follows that the following are all the possible distinct answers for the given initial data above.

a_i	b_i
2.00000000	1.36602540
2.00000000	1.36602540
2.00000000	0.36602540
2.00000000	0.36602540

a_i	b_i
2.00000000	0.36602540
2.00000000	0.36602540
2.00000000	1.36602540
2.00000000	1.36602540
a_i	b_i
2.00000000	1.00000000
0.77525513	0.50000000
2.00000000	0.50000000
3.22474487	1.00000000
a_i	b_i
2.00000000	1.00000000
3.22474487	0.50000000
2.00000000	0.50000000
0.77525513	1.00000000

As a final example, we tried the example 3 of Table 2 on page 1218 of [3], in which the matrices in question were reconstructed only with large errors. The matrices are defined as follows, for $n = 5, 10, 15, 20, 25, 30$:

$$a_i = \frac{i}{n} - 2, \quad i = 1, \dots, n-1,$$

$$b_i = 1 - \frac{i}{n}, \quad i = 1, \dots, n-2,$$

$$b_n = b_{n-1} = 1,$$

$$a_n = 0.$$

The algorithm was suitably modified to account for the fact that the submatrix \bar{J} in this example was obtained by deleting the last row and column rather than the first.

Unfortunately, there are too many combinations of signs for the c_i, c_i^- to compute all the possible matrices. For $n = 10$ alone there are 262144 different choices of signs. However, in each case we are able to compare the eigenvalues of the reconstructed matrix with the initial eigenvalues. Since the eigenvalues for symmetric matrices are always well-conditioned, the discrepancy between the two sets of eigenvalues is a good indicator of the error in the reconstructed matrices.

n	Norm of discrepancy
5	.364539663e - 15
10	.558570184e - 15
15	.130290552e - 14
20	.191718261e - 14
25	.304003744e - 14
30	.340721065e - 14

Here, the norm of discrepancy is the square root of the sum of the squares of the differences between the two sets of eigenvalues.

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