

## Some Inequalities for Elementary Mean Values

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**Abstract.** Upper and lower bounds for the difference between the arithmetic and harmonic means of  $n$  positive numbers are obtained in terms of  $n$  and the largest and smallest of the numbers. Also, results of S. H. Tung [2], are used to obtain upper and lower bounds for the elementary mean values  $M_p$  of Hardy, Littlewood, and Pólya.

1. In 1975, S. H. Tung proved the following theorem [2]:

Let  $0 < b = x_1 \leq x_2 \leq \dots \leq x_n = B$ . Let  $A$  and  $G$  be the arithmetic and geometric means, respectively, of  $x_1, \dots, x_n$ . Then

$$n^{-1}(B^{1/2} - b^{1/2})^2 \leq A - G \leq g(b, B),$$

where  $g(b, B) = cb + (1 - c)B - b^c B^{1-c}$ , and

$$c = \frac{\log[(b/B - b) \log B/b]}{\log B/b}.$$

We will derive somewhat similar bounds for the difference between the arithmetic and the harmonic means of  $n$  positive numbers.

2. In [1, Chapter 2] Hardy, Littlewood, and Pólya discussed the elementary mean values, which are defined as follows:

Let  $x_1, x_2, \dots, x_n$  be positive numbers, and let  $p$  be a real number. Then  $M_p(x_1, \dots, x_n)$  is defined as  $[n^{-1} \sum_{k=1}^n x_k^p]^{1/p}$ , if  $p \neq 0$ ;  $M_0(x_1, \dots, x_n)$  is defined as  $(\prod_{k=1}^n x_k)^{1/n}$ . We denote  $M_1$ , the arithmetic mean, by  $A$ ;  $M_0$ , the geometric mean, by  $G$ ; and  $M_{-1}$ , the harmonic mean, by  $H$ . Since  $M_p(kx_1, \dots, kx_n) = kM_p(x_1, \dots, x_n)$  for all  $p$  and for all  $k > 0$ , we may, without loss of generality, assume  $x_1 = 1$ .

**THEOREM 1.** Let  $1 = x_1 \leq x_2 \leq \dots \leq x_n = B$ . Then

$$\frac{(B - 1)^2}{n(B + 1)} \leq A(1, \dots, B) - H(1, \dots, B) \leq (B^{1/2} - 1)^2.$$

*Proof.* For each  $k$ ,  $2 \leq k \leq n$ , let

$$A_k = A(x_1, x_2, \dots, x_{k-1}, x_n) \quad \text{and} \quad H_k = H(x_1, x_2, \dots, x_{k-1}, x_n).$$

Fix  $x_1, x_2, \dots, x_{n-2}, x_n$ , and let  $x_{n-1} = x$  vary in  $[1, B]$ . Let

$$D(x) = A_n - H_n = \frac{(n-1)A_{n-1} + x}{n} - \frac{nxH_{n-1}}{(n-1)x + H_{n-1}}.$$

Computation of  $D'(x)$  shows that  $x = H_{n-1}$  is its only positive zero, and standard methods of analysis show that a minimum for  $D(x)$  is attained at  $x = H_{n-1}$ .

Received November 2, 1982; revised February 11, 1983.  
 1980 *Mathematics Subject Classification.* Primary 26D20.

Therefore,

$$A_n - H_n \geq D(H_{n-1}) = n^{-1}(n-1)(A_{n-1} - H_{n-1}).$$

This process may be repeated, giving

$$\begin{aligned} A_n - H_n &\geq \frac{n-1}{n}(A_{n-1} - H_{n-1}) \geq \frac{n-2}{n}(A_{n-2} - H_{n-2}) \\ &\geq \cdots \geq \frac{2}{n}(A_2 - H_2) = \frac{(B-1)^2}{n(B+1)}. \end{aligned}$$

The maximum of  $D(x)$  must occur at an endpoint, 1 or  $B$ , as each of the variables  $x_2, x_3, \dots, x_{n-1}$  in turn varies from 1 to  $B$ . So

$$A_n - H_n \leq \frac{nB - (B-1)k}{n} - \frac{nB}{(B-1)k + n} = F(k),$$

for some  $k$ ,  $0 \leq k \leq n$ . The maximum of  $F(x)$  on  $[0, n]$  will, then, be an upper bound for  $A_n - H_n$ . Again, computation of  $F'(x)$  and standard methods of analysis show that a maximum is attained for  $x = n(B^{1/2} - 1)/(B - 1)$ . Hence

$$A_n - H_n \leq F\{n(B^{1/2} - 1)/(B - 1)\} = (B^{1/2} - 1)^2.$$

This completes the proof of Theorem 1.

Upper and lower bounds for  $G_n - H_n$  may be obtained using Theorem 1 and Tung's Theorem, since

$$G_n - H_n = (A_n - H_n) - (A_n - G_n).$$

3. Tung's Theorem may be used to obtain upper and lower bounds for the elementary mean values  $M_p$ , by using the relation

$$M_p(x_1, \dots, x_n) = \{A(x_1^p, \dots, x_n^p)\}^{1/p}.$$

(See [1].)

**THEOREM 2.** Let  $1 = x_1 \leq x_2 \leq \cdots \leq x_n = B$ , and let  $p > 0$ . Then

$$\left[ n^{-1}(B^{p/2} - 1)^2 + G^p \right]^{1/p} \leq M_p(1, \dots, B) \leq [g(1, B^p) + G^p]^{1/p},$$

where  $G = G(1, \dots, B)$ , and  $g$  is the function defined in Tung's Theorem.

**THEOREM 3.** Let  $1 = x_1 \leq x_2 \leq \cdots \leq x_n = B$ , and let  $p < 0$ . Then

$$\left[ g(B^p, 1) + G^p \right]^{1/p} \leq M_p(1, \dots, B) \leq \left[ n^{-1}(1 - B^{p/2})^2 + G^p \right]^{1/p},$$

where  $G = G(1, \dots, B)$  and  $g$  is the function of Tung's Theorem.

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1. G. H. HARDY, J. E. LITTLEWOOD & G. POLYA, *Inequalities*, Cambridge Univ. Press, Cambridge, 1952.

2. S. H. TUNG, "On lower and upper bounds of the difference between the arithmetic and the geometric mean," *Math. Comp.*, v. 29, 1975, pp. 834-836.