A Series Expansion for the First Positive Zero of the Bessel Functions

By R. Piessens

Abstract. It is shown that the first positive zero \( j_{\nu,1} \) of the Bessel function \( J_\nu(x) \) is given by

\[
 j_{\nu,1} = 2(\nu + 1)^{1/2} \left[ 1 + \frac{(\nu + 1)}{4} - \frac{7(\nu + 1)^2}{96} + \frac{49(\nu + 1)^3}{1152} - \frac{8363(\nu + 1)^4}{276480} + \ldots \right]
\]

for \(-1 < \nu < 0\).

1. It is well known that, when \( \nu \) is real and \( \nu > -1 \), the Bessel function \( J_\nu(x) \) has an infinite number of zeros and that all zeros are real (Watson [9]). We denote the \( s \)th positive zero of \( J_\nu(x) \) by \( j_{\nu,s} \).

Several approximations, asymptotic expansions or bounds for the zeros of Bessel functions exist (see [1], [2], [4], [6], [7], [9]). Especially McMahon’s expansion for large zeros (see Abramowitz and Stegun [1]), Olver’s asymptotic expansion for large orders and Olver’s uniform asymptotic expansions (see Olver [6]) are interesting formulas, but, unfortunately, they are not applicable when \( s \) and \( \nu \) are small. The purpose of this note is to give a series expansion for \( j_{\nu,1} \) when \(-1 < \nu < 0\).

2. Cayley [3] noticed that Graeffe’s method for solving a polynomial equation can be applied for the efficient computation of

\[
 \sum_{s=1}^{\infty} j_{\nu,s}^{-2r} = \sigma_\nu^{(r)}, \quad r = 1, 2, \ldots.
\]

An upper bound for \( j_{\nu,1} \) is given by Chambers [4]:

\[
 j_{\nu,1} < (\nu + 1)^{1/2} \left[ (\nu + 2)^{1/2} + 1 \right].
\]

Further it is known that, when \( k > 1 \),

\[
 \lim_{\nu \to -1} j_{\nu,k} = j_{1,k-1} > 0.
\]

Thus, the first term in the left side of (1) is dominant when \( \nu \approx -1 \), so that

\[
 j_{\nu,1} = 2(\nu + 1)^{1/2} \phi_\nu(\nu) + o((\nu + 1)^{r-1}), \quad \nu \to -1,
\]

where

\[
 \phi_\nu(\nu) = \left[ \frac{1}{2^r(\nu + 1)^r \sigma_\nu^{(r)}} \right]^{1/2r}
\]

is analytic at \( \nu = -1 \).

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By approximating \( \phi_r(v) \) by a Taylor polynomial, we obtain

\[
\phi_{r,1} = 2(v + 1)^{1/2} \sum_{k=0}^{r-1} C_k (v + 1)^k + O\left( (v + 1)^{r-1} \right), \quad v \to -1,
\]

where

\[
C_k = \frac{1}{k!} \frac{d^k \phi_r(v)}{dv^k} \bigg|_{v=-1}
\]

is independent of \( r \).

When \( r \to \infty \), (6) becomes a series expansion for \( \phi_{r,1} \), which, because of the presence of a branchpoint of \( \phi_r(v) \) at \( v = -2 \), converges only in the interval \(-1 < v < 0\).

Using REDUCE, which is a computer language for formula manipulation [5], we have computed \( C_k \), \( k = 0, 1, 2, 3, 4 \), using (7), where \( r = 8 \) and

\[
\phi_r(v) = \left( \frac{(v + 2)^4(v + 3)^2(v + 4)^2(v + 5)(v + 6)(v + 7)(v + 8)}{429v^5 + 7640v^4 + 53752v^3 + 185430v^2 + 311387v + 202738} \right)^{1/16}.
\]

The result is

\[
\phi_{r,1} = 2(v + 1)^{1/2} \left[ 1 + \frac{(v + 1)}{4} - \frac{7(v + 1)^2}{96} + \frac{49(v + 1)^3}{1152} - \frac{8363(v + 1)^4}{276480} + \ldots \right].
\]

In Table 1, we compare the exact values of \( \phi_{r,1} \) with the approximate values given by (9), for \( v = -3/4, -2/3, -1/2, -1/3, -1/4 \) and also for \( v = 0 \) (although we were not able to prove the convergences of the expansion for \( v = 0 \)).

<table>
<thead>
<tr>
<th>( v )</th>
<th>exact value</th>
<th>approximation (9)</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3/4</td>
<td>1.058508</td>
<td>1.058489</td>
</tr>
<tr>
<td>-2/3</td>
<td>1.243046</td>
<td>1.242958</td>
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<tr>
<td>-1/2</td>
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<td>1.570056</td>
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<td>2.000273</td>
</tr>
<tr>
<td>0</td>
<td>2.404826</td>
<td>2.378740</td>
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</tbody>
</table>

3. An interesting application of (9) is the estimation of the smallest zero of Laguerre- and Gegenbauer-polynomials [8]. For example, the smallest zero \( \xi_n \) of the Laguerre polynomials \( L_n^{(\alpha)}(x) \) is approximated by (see Tricomi [8])

\[
\xi_n = j_{a,1} \left[ 1 + \frac{2(\alpha^2 - 1) + j_{a,1}^2}{48k_n^2} \right],
\]

where \( k_n = n + (\alpha + 1)/2 \). In Table 2, this approximation, where \( j_{a,1} \) is replaced by (9), is compared with the exact value of \( \xi_n \).
### Table 2

*Values of the smallest zero of $I_n^{(a)}(x)$*

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$n$</th>
<th>exact</th>
<th>approximation (10)</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.089679</td>
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<td>0.297530</td>
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<td>0.093308</td>
<td>0.091294</td>
</tr>
</tbody>
</table>

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