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Diophantine Equations in Partitions

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Abstract. Given positive integers $r_1, r_2, r_3, \ldots, r_j$ such that

$$r_1 < r_2 < r_3 < \cdots < r_j < m; \qquad m > 1;$$

we find the number $P(n, m; R)$ of partitions of a given positive integer n into parts belonging to the set R of residue classes

$$r_1 \pmod{m}, \quad r_2 \pmod{m}, \ldots, r_j \pmod{m}.$$

This leads to an identity which is more general though less elegant then the well-known Rogers-Ramanujan identities.

1. Notation. In what follows, small letters other than x denote nonnegative integers; $|x| < 1$; the parts in a partition are deemed to be arranged in a nonascending order unless otherwise clear from the context; $p(u, v)$ denotes the number of partitions of u into at most v parts, with $p(0, v) = 1$; square brackets denote the greatest integer function; $m > 1$; and we write

$$X(a_i) \quad \text{for} \quad 1/\{(1-x)(1-x^2)\cdots(1-x^{a_i})\} \quad \text{with } X(0) = 1.$$

2. The Problem. Given $m > 1$, and integers $r_1, r_2, r_3, \ldots, r_j$ such that

$$0 < r_1 < r_2 < \cdots < r_j < m;$$

we have to find the number $P(n, m; R)$ of partitions of a given number n into parts belonging to the set R of residue classes

$$r_1 \pmod{m}, \quad r_2 \pmod{m}, \ldots, r_j \pmod{m}$$

if any such exist.

Let us first consider the set of those partitions of n which have

$$\begin{gathered} a_1 \text{ parts each congruent to } r_1 \pmod{m}, \\ a_2 \text{ parts each congruent to } r_2 \pmod{m}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_j \text{ parts each congruent to } r_j \pmod{m}. \end{gathered}$$

For n to have such a partition, it is necessary that

$$n = r_1 a_1 + r_2 a_2 + \cdots + r_j a_j + Cm \tag{1}$$

for some integer $C \geqslant 0$.

Any partition belonging to our set can be considered to have j sections—the first consisting of a_1 r_1's, the second of a_2 r_2's;..., and the jth of a_j r_j's.

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To get a partition of n of the desired type, we must distribute the C m's between the j sections in all possible ways. Let c_1 m's be allotted to the first section, c_2 to the second section, ..., and c_j to the jth section.

Since the elements of each section are all alike, the c_i m's assigned to the ith section are partitioned into at most a_i parts which are then tagged on to the elements of the section in order. Thus the number of partitions to which the allotment leads is given by

$$p(c_1, a_1) \cdot p(c_2, a_2) \cdot p(c_3, a_3) \cdots p(c_j, a_j). \tag{2}$$

Letting $c_1, c_2, \ldots, c_j$ run through all the $\binom{C+j-1}{j-1}$ solutions of the Diophantine equation

$$c_1 + c_2 + c_3 + \cdots + c_j = C \tag{3}$$

in nonnegative integers, we can not only find the number of partitions in our set but can also write them out. We do this in the following example with

$$m = 11; \quad r_1 = 2, r_2 = 6, r_3 = 8, r_4 = 10;$$

$$a_1 = 5, \quad a_2 = 2, a_3 = 1, a_4 = 1; \quad \text{and} \quad C = 3;$$

which implies that $n = 73$.

The Diophantine equation

$$c_1 + c_2 + c_3 + c_4 = 3$$

has 20 solutions. We present them in the following table along with the partitions to which they give rise and their number.

	c_1	c_2	c_3	c_4	Partitions	Number
1.	3	0	0	0	35, 2, 2, 2, 2: 6, 6: 8: 10:	
					24,13, 2, 2, 2: 6, 6: 8: 10:	
					13,13,13, 2, 2: 6, 6: 8: 10:	3
2.	0	3	0	0	2, 2, 2, 2, 2:39, 6: 8: 10:	
					2, 2, 2, 2, 2:28,17: 8: 10:	2
3.	0	0	3	0	2, 2, 2, 2, 2: 6, 6: 41: 10:	1
4.	0	0	0	3	2, 2, 2, 2, 2: 6, 6: 8: 43:	1
5.	2	1	0	0	24, 2, 2, 2, 2:17, 6: 8: 10:	
					13,13, 2, 2, 2:17, 6: 8: 10:	2
6.	2	0	1	0	24, 2, 2, 2, 2: 6, 6: 19: 10:	
					13,13, 2, 2, 2: 6, 6: 19: 10:	2
7.	2	0	0	1	24, 2, 2, 2, 2: 6, 6: 8: 21:	
					13,13, 2, 2, 2: 6, 6: 8: 21:	2
8.	1	2	0	0	13, 2, 2, 2, 2:28, 6: 8: 10:	
					13, 2, 2, 2, 2:17,17: 8: 10:	2
9.	0	2	1	0	2, 2, 2, 2, 2:28, 6: 19: 10:	
					2, 2, 2, 2, 2:17,17: 19: 10:	2
10.	0	2	0	1	2, 2, 2, 2, 2:28, 6: 8: 21:	
					2, 2, 2, 2, 2:17,17: 8: 21:	2
11.	0	0	2	1	2, 2, 2, 2, 2: 6, 6: 30: 21:	1
12.	0	1	2	0	2, 2, 2, 2, 2:17, 6: 30: 10:	1
13.	1	0	2	0	13, 2, 2, 2, 2: 6, 6: 30: 10:	1
14.	0	0	1	2	2, 2, 2, 2, 2: 6, 6: 19: 32:	1
15.	0	1	0	2	2, 2, 2, 2, 2:17, 6: 8: 32:	1
16.	1	0	0	2	13, 2, 2, 2, 2: 6, 6: 8: 32:	1
17.	1	1	1	0	13, 2, 2, 2, 2:17, 6: 19: 10:	1
18.	1	1	0	1	13, 2, 2, 2, 2:17, 6: 8: 21:	1
19.	1	0	1	1	13, 2, 2, 2, 2: 6, 6: 19: 21:	1
20.	0	1	1	1	2, 2, 2, 2, 2:17, 6: 19: 21.	1

Thus the required number of partitions in the set is 29. A formula for the number of partitions in the set is obtained as follows. We note that $p(c_i, a_i)$ is the coefficient of x^{c_i} in the expansioin of $X(a_i)$. Hence

$$\sum_c p(c_1, a_1) \cdot p(c_2, a_2) \cdot \cdots \cdot p(c_j, a_j),$$

where c's run over all the solutions of the Diophantine equation (3), is the coefficient of x^C in the expansion (in ascending powers of x) of the product

$$X(a_1) \cdot X(a_2) \cdot X(a_3) \cdot \cdots \cdot X(a_j).$$

This is the same as the coefficient of x^C in

$$X(b_1) \cdot X(b_2) \cdot X(b_3) \cdot \cdots \cdot X(b_j), \quad \text{where} \tag{4}$$

$$b_i = \min(C, a_i), \qquad i = 1, 2, 3, \ldots, j. \tag{5}$$

In our example, it is the coefficient of x^3 in

$$X(3) \cdot X(2) \cdot X(1) \cdot X(1).$$

We leave it to the reader to verify that the coefficient is 29.

3. The Formula for $P(n, m; R)$. We have seen that

$$r_1 a_1 + r_2 a_2 + r_3 a_3 + \cdots + r_j a_j = n - Cm. \tag{6}$$

In this let C take in succession the values $0, 1, 2, \ldots, [n/m]$. For each of these values, regarding (6) as a Diophantine equation in a's, find all the solutions of (6) and the contribution of each such solution to $P(n, m; R)$. Then we get

$$\begin{aligned} P(n, m; R) &= \text{the sum of these contributions;} \\ &= \sum_{C=0}^{[n/m]} \text{coefficient of } x^C \text{ in } X(a_1) X(a_2) \cdots X(a_j), \end{aligned} \tag{7}$$

where each $a_i > C$ can be replaced by C.

The following examples will show how the calculations can best be presented.

Example 1. Let $m = 5$, $r_1 = 2$, $r_2 = 3$ and $n = 25$.

Our presentation will be as follows:

C	a1	a2	b1	b2	Contribution to P(25,5;2,3)
0	2	7	0	0	1
	5	5	0	0	1
	8	3	0	0	1
	11	1	0	0	1
1	1	6	1	1	2
	4	4	1	1	2
	7	2	1	1	2
	10	0	1	0	1
2	0	5	0	2	2
	3	3	2	2	5
	6	1	2	1	4
3	2	2	2	2	8
	5	0	3	0	3
4	1	1	1	1	5
5	0	0	0	0	0
				P(25,5;2,3) =	38

By the second Rogers-Ramanujan identity, we will have

$$\begin{array}{lcccccccccc} P(25,5;2,3) & = & p(23,1) & + & p(19,2) & + & p(13,3) & + & p(5,4), & & \\ & = & 1 & + & 10 & + & 21 & + & 6 & = & 38. \end{array}$$

Before we consider our next example, let it be recalled that the number of solutions of (6) is the coefficient of x^{n-Cm} in

$$\{(1-x^{r_1})(1-x^{r_2})(1-x^{r_3})\cdots(1-x^{r_l})\}^{-1}. \tag{8}$$

We leave it to the reader to verify this in the above example.

Example 2. Let $m = 7$; $r = 1$, $r = 2$, $r = 4$; $n = 25$. In this case (8) gives the following information:

C:	0	1	2	3
Number of solutions:	49	30	12	4.

Our calculations are a little more elaborate this time.

C	a_1	a_2	a_3	b_1	b_2	b_3	P
0	1	0	6	0	0	0	1
	5	0	5	0	0	0	1
	3	1	5	0	0	0	1
	1	2	5	0	0	0	1
	9	0	4	0	0	0	1
	7	1	4	0	0	0	1
	5	2	4	0	0	0	1
	3	3	4	0	0	0	1
	1	4	4	0	0	0	1
	13	0	3	0	0	0	1
	11	1	3	0	0	0	1
	9	2	3	0	0	0	1
	7	3	3	0	0	0	1
	5	4	3	0	0	0	1
	3	5	3	0	0	0	1
	1	6	3	0	0	0	1
	17	0	2	0	0	0	1
	15	1	2	0	0	0	1
	13	2	2	0	0	0	1
	11	3	2	0	0	0	1
	9	4	2	0	0	0	1
	7	5	2	0	0	0	1
	5	6	2	0	0	0	1
	3	7	2	0	0	0	1
	1	8	2	0	0	0	1
	21	0	1	0	0	0	1
	19	1	1	0	0	0	1
	17	2	1	0	0	0	1
	15	3	1	0	0	0	1
	13	4	1	0	0	0	1

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C	a_1	a_2	a_3	b_1	b_2	b_3	P
1	2	0	4	1	0	1	2
	0	1	4	0	1	1	2
	6	0	3	1	0	1	2
	4	1	3	1	1	1	3
	2	2	3	1	1	1	3
	0	3	3	0	1	1	2
	10	0	2	1	0	1	2
	8	1	2	1	1	1	3
	6	2	2	1	1	1	3
	4	3	2	1	1	1	3
	2	4	2	1	1	1	3
	0	5	2	0	1	1	2
	14	0	1	1	0	1	2
	12	1	1	1	1	1	3
	10	2	1	1	1	1	3
	8	3	1	1	1	1	3
	6	4	1	1	1	1	3
	4	5	1	1	1	1	3
	2	6	1	1	1	1	3
	0	7	1	0	1	1	2
	18	0	0	1	0	0	1
	16	1	0	1	1	0	2
	14	2	0	1	1	0	2
	12	3	0	1	1	0	2
	10	4	0	1	1	0	2
	8	5	0	1	1	0	2
	6	6	0	1	1	0	2
	4	7	0	1	1	0	2
	2	8	0	1	1	0	2
	0	9	0	0	1	0	1

C	a_1	a_2	a_3	b_1	b_2	b_3	P
	11	5	1	0	0	0	1
	9	6	1	0	0	0	1
	7	7	1	0	0	0	1
	5	8	1	0	0	0	1
	3	9	1	0	0	0	1
	1	10	1	0	0	0	1
	25	0	0	0	0	0	1
	23	1	0	0	0	0	1
	21	2	0	0	0	0	1
	19	3	0	0	0	0	1
	17	4	0	0	0	0	1
	15	5	0	0	0	0	1
	13	6	0	0	0	0	1
	11	7	0	0	0	0	1
	9	8	0	0	0	0	1
	7	9	0	0	0	0	1
	5	10	0	0	0	0	1
	3	11	0	0	0	0	1
	1	12	0	0	0	0	1

C	a_1	a_2	a_3	b_1	b_2	b_3	P
2	3	0	2	2	0	2	5
	1	1	2	1	1	2	7
	7	0	1	2	0	1	4
	5	1	1	2	1	1	7
	3	2	1	2	2	1	8
	1	3	1	1	2	1	7
	11	0	0	2	0	0	2
	9	1	0	2	1	0	4
	7	2	0	2	2	0	5
	5	3	0	2	2	0	5
	3	4	0	2	2	0	5
	1	5	0	1	2,	0	4
3	0	0	1	0	0	1	1
	4	0	0	3	0	0	3
	2	1	0	2	1	0	6
	0	2	0	0	2	0	2
	P(25,7:1,2,4)						= 194

To check our result, we make use of the well-known fact that $P(n, m; R)$ is the coefficient of x^n in the expansion of

$$\prod_{q=0}^{[n/m]} \left\{ \prod_{i=1}^{j} (1 - x^{r_i + qm}) \right\}^{-1}. \tag{9}$$

In our example (9) is

$$\{(1-x)(1-x^2)(1-x^4)\cdot(1-x^8)(1-x^9)(1-x^{11})$$
$$\cdot(1-x^{15})(1-x^{16})(1-x^{18})\cdot(1-x^{22})(1-x^{23})(1-x^{25})\}^{-1}.$$

Expanding this, we obtained the following table of coefficients of x^n, $0 \leqslant n \leqslant 25$.

n	0	1	2	3	4	5	6	7	8	9
0	1	1	2	2	4	4	6	6	10	11
1	15	17	23	26	32	37	47	54	66	76
2	93	105	126	143	172	194				

Incidentally, equating the results in (7) and (9), we get an identity which is more general but not as elegant as the well-known Rogers-Ramanujan identities.

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