

To get a partition of n of the desired type, we must distribute the C m 's between the j sections in all possible ways. Let c_1 m 's be allotted to the first section, c_2 to the second section, . . . , and c_j to the j th section.

Since the elements of each section are all alike, the c_i m 's assigned to the i th section are partitioned into at most a_i parts which are then tagged on to the elements of the section in order. Thus the number of partitions to which the allotment leads is given by

$$(2) \quad p(c_1, a_1) \cdot p(c_2, a_2) \cdot p(c_3, a_3) \cdots p(c_j, a_j).$$

Letting c_1, c_2, \dots, c_j run through all the $\binom{C+j-1}{j-1}$ solutions of the Diophantine equation

$$(3) \quad c_1 + c_2 + c_3 + \cdots + c_j = C$$

in nonnegative integers, we can not only find the number of partitions in our set but can also write them out. We do this in the following example with

$$m = 11; \quad r_1 = 2, r_2 = 6, r_3 = 8, r_4 = 10;$$

$$a_1 = 5, \quad a_2 = 2, a_3 = 1, a_4 = 1; \quad \text{and} \quad C = 3;$$

which implies that $n = 73$.

The Diophantine equation

$$c_1 + c_2 + c_3 + c_4 = 3$$

has 20 solutions. We present them in the following table along with the partitions to which they give rise and their number.

	c_1	c_2	c_3	c_4	Partitions	Number
1.	3	0	0	0	35, 2, 2, 2, 2; 6, 6; 8; 10; 24, 13, 2, 2, 2; 6, 6; 8; 10; 13, 13, 13, 2, 2, 2; 6, 6; 8; 10;	3
2.	0	3	0	0	2, 2, 2, 2, 2; 39, 6; 8; 10; 2, 2, 2, 2, 2; 28, 17; 8; 10;	2
3.	0	0	3	0	2, 2, 2, 2, 2; 6, 6; 41; 10;	1
4.	0	0	0	3	2, 2, 2, 2, 2; 6, 6; 8; 43;	1
5.	2	1	0	0	24, 2, 2, 2, 2; 17, 6; 8; 10; 13, 13, 2, 2, 2; 17, 6; 8; 10;	2
6.	2	0	1	0	24, 2, 2, 2, 2; 6, 6; 19; 10; 13, 13, 2, 2, 2; 6, 6; 19; 10;	2
7.	2	0	0	1	24, 2, 2, 2, 2; 6, 6; 8; 21; 13, 13, 2, 2, 2; 6, 6; 8; 21;	2
8.	1	2	0	0	13, 2, 2, 2, 2; 28, 6; 8; 10; 13, 2, 2, 2, 2; 17, 17; 8; 10;	2
9.	0	2	1	0	2, 2, 2, 2, 2; 28, 6; 19; 10; 2, 2, 2, 2, 2; 17, 17; 19; 10;	2
10.	0	2	0	1	2, 2, 2, 2, 2; 28, 6; 8; 21; 2, 2, 2, 2, 2; 17, 17; 8; 21;	2
11.	0	0	2	1	2, 2, 2, 2, 2; 6, 6; 30; 21;	1
12.	0	1	2	0	2, 2, 2, 2, 2; 17, 6; 30; 10;	1
13.	1	0	2	0	13, 2, 2, 2, 2; 6, 6; 30; 10;	1
14.	0	0	1	2	2, 2, 2, 2, 2; 6, 6; 19; 32;	1
15.	0	1	0	2	2, 2, 2, 2, 2; 17, 6; 8; 32;	1
16.	1	0	0	2	13, 2, 2, 2, 2; 6, 6; 8; 32;	1
17.	1	1	1	0	13, 2, 2, 2, 2; 17, 6; 19; 10;	1
18.	1	1	0	1	13, 2, 2, 2, 2; 17, 6; 8; 21;	1
19.	1	0	1	1	13, 2, 2, 2, 2; 6, 6; 19; 21;	1
20.	0	1	1	1	2, 2, 2, 2, 2; 17, 6; 19; 21;	1

Thus the required number of partitions in the set is 29. A formula for the number of partitions in the set is obtained as follows. We note that $p(c_i, a_i)$ is the coefficient of x^{c_i} in the expansion of $X(a_i)$. Hence

$$\sum_c p(c_1, a_1) \cdot p(c_2, a_2) \cdot \cdots \cdot p(c_j, a_j),$$

where c 's run over all the solutions of the Diophantine equation (3), is the coefficient of x^C in the expansion (in ascending powers of x) of the product

$$X(a_1) \cdot X(a_2) \cdot X(a_3) \cdot \cdots \cdot X(a_j).$$

This is the same as the coefficient of x^C in

(4) $X(b_1) \cdot X(b_2) \cdot X(b_3) \cdot \cdots \cdot X(b_j),$ where

(5) $b_i = \min(C, a_i), \quad i = 1, 2, 3, \dots, j.$

In our example, it is the coefficient of x^3 in

$$X(3) \cdot X(2) \cdot X(1) \cdot X(1).$$

We leave it to the reader to verify that the coefficient is 29.

3. The Formula for $P(n, m; R)$. We have seen that

(6) $r_1 a_1 + r_2 a_2 + r_3 a_3 + \cdots + r_j a_j = n - Cm.$

In this let C take in succession the values $0, 1, 2, \dots, [n/m]$. For each of these values, regarding (6) as a Diophantine equation in a 's, find all the solutions of (6) and the contribution of each such solution to $P(n, m; R)$. Then we get

(7) $P(n, m; R) =$ the sum of these contributions;
 $= \sum_{C=0}^{[n/m]} \text{coefficient of } x^C \text{ in } X(a_1) X(a_2) \cdots X(a_j),$

where each $a_i > C$ can be replaced by C .

The following examples will show how the calculations can best be presented.

Example 1. Let $m = 5, r_1 = 2, r_2 = 3$ and $n = 25$.

Our presentation will be as follows:

C	a ₁	a ₂	b ₁	b ₂	Contribution to P(25, 5; 2, 3)
0	2	7	0	0	1
	5	5	0	0	1
	8	3	0	0	1
	11	1	0	0	1
1	1	6	1	1	2
	4	4	1	1	2
	7	2	1	1	2
	10	0	1	0	1
2	0	5	0	2	2
	3	3	2	2	5
	6	1	2	1	4
3	2	2	2	2	8
	5	0	3	0	3
4	1	1	1	1	5
5	0	0	0	0	0
					P(25, 5; 2, 3) = <u>38</u>

By the second Rogers-Ramanujan identity, we will have

$$\begin{aligned}
 P(25, 5; 2, 3) &= p(23, 1) + p(19, 2) + p(13, 3) + p(5, 4), \\
 &= 1 + 10 + 21 + 6 = 38.
 \end{aligned}$$

Before we consider our next example, let it be recalled that the number of solutions of (6) is the coefficient of x^{n-Cm} in

$$(8) \quad \{(1 - x^{r_1})(1 - x^{r_2})(1 - x^{r_3}) \cdots (1 - x^{r_s})\}^{-1}.$$

We leave it to the reader to verify this in the above example.

Example 2. Let $m = 7$; $r = 1, r = 2, r = 4$; $n = 25$. In this case (8) gives the following information:

C:	0	1	2	3
Number of solutions:	49	30	12	4.

Our calculations are a little more elaborate this time.

C	a ₁	a ₂	a ₃	b ₁	b ₂	b ₃	P	C	a ₁	a ₂	a ₃	b ₁	b ₂	b ₃	P
0	1	0	6	0	0	0	1	1	2	0	4	1	0	1	2
	5	0	5	0	0	0	1	0	1	4	4	0	1	1	2
	3	1	5	0	0	0	1	6	0	3	3	1	0	1	2
	1	2	5	0	0	0	1	4	1	3	3	1	1	1	3
	9	0	4	0	0	0	1	2	2	3	3	1	1	1	3
	7	1	4	0	0	0	1	0	3	3	3	0	1	1	2
	5	2	4	0	0	0	1	10	0	2	2	1	0	1	2
	3	3	4	0	0	0	1	8	1	2	2	1	1	1	3
	1	4	4	0	0	0	1	6	2	2	2	1	1	1	3
	13	0	3	0	0	0	1	4	3	2	2	1	1	1	3
	11	1	3	0	0	0	1	2	4	2	2	1	1	1	3
	9	2	3	0	0	0	1	0	5	2	2	0	1	1	2
	7	3	3	0	0	0	1	14	0	1	1	1	0	1	2
	5	4	3	0	0	0	1	12	1	1	1	1	1	1	3
	3	5	3	0	0	0	1	10	2	1	1	1	1	1	3
	1	6	3	0	0	0	1	8	3	1	1	1	1	1	3
	17	0	2	0	0	0	1	6	4	1	1	1	1	1	3
	15	1	2	0	0	0	1	4	5	1	1	1	1	1	3
	13	2	2	0	0	0	1	2	6	1	1	1	1	1	3
	11	3	2	0	0	0	1	0	7	1	1	0	1	1	2
	9	4	2	0	0	0	1	18	0	0	1	0	0	0	1
	7	5	2	0	0	0	1	16	1	0	1	1	0	0	2
	5	6	2	0	0	0	1	14	2	0	1	1	0	0	2
	3	7	2	0	0	0	1	12	3	0	1	1	0	0	2
	1	8	2	0	0	0	1	10	4	0	1	1	0	0	2
	21	0	1	0	0	0	1	8	5	0	1	1	0	0	2
	19	1	1	0	0	0	1	6	6	0	1	1	0	0	2
	17	2	1	0	0	0	1	4	7	0	1	1	0	0	2
	15	3	1	0	0	0	1	2	8	0	1	1	0	0	2
	13	4	1	0	0	0	1	0	9	0	0	1	0	0	1

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C	a ₁	a ₂	a ₃	b ₁	b ₂	b ₃	P	C	a ₁	a ₂	a ₃	b ₁	b ₂	b ₃	P
11	5	1	0	0	0	0	1	2	3	0	2	2	0	2	5
9	6	1	0	0	0	0	1	1	1	2	1	1	1	2	7
7	7	1	0	0	0	0	1	7	0	1	2	0	1	1	4
5	8	1	0	0	0	0	1	5	1	1	2	1	1	1	7
3	9	1	0	0	0	0	1	3	2	1	2	2	1	1	8
1	10	1	0	0	0	0	1	1	3	1	1	2	1	1	7
25	0	0	0	0	0	0	1	11	0	0	2	0	0	0	2
23	1	0	0	0	0	0	1	9	1	0	2	1	0	0	4
21	2	0	0	0	0	0	1	7	2	0	2	2	0	0	5
19	3	0	0	0	0	0	1	5	3	0	2	2	0	0	5
17	4	0	0	0	0	0	1	3	4	0	2	2	0	0	5
15	5	0	0	0	0	0	1	1	5	0	1	2	0	0	4
13	6	0	0	0	0	0	1	-----							
11	7	0	0	0	0	0	1	3	0	1	0	0	1	1	1
9	8	0	0	0	0	0	1	4	0	0	3	0	0	0	3
7	9	0	0	0	0	0	1	2	1	0	2	1	0	0	6
5	10	0	0	0	0	0	1	0	2	0	0	2	0	0	2
3	11	0	0	0	0	0	1	-----							
1	12	0	0	0	0	0	1	P(25, 7; 1, 2, 4) = 194							

To check our result, we make use of the well-known fact that $P(n, m; R)$ is the coefficient of x^n in the expansion of

$$(9) \quad \prod_{q=0}^{\lfloor n/m \rfloor} \left\{ \prod_{i=1}^j (1 - x^{r_i + qm}) \right\}^{-1}$$

In our example (9) is

$$\begin{aligned} & \{(1 - x)(1 - x^2)(1 - x^4) \cdot (1 - x^8)(1 - x^9)(1 - x^{11}) \\ & \cdot (1 - x^{15})(1 - x^{16})(1 - x^{18}) \cdot (1 - x^{22})(1 - x^{23})(1 - x^{25})\}^{-1} \end{aligned}$$

Expanding this, we obtained the following table of coefficients of x^n , $0 \leq n \leq 25$.

n	0	1	2	3	4	5	6	7	8	9
0	1	1	2	2	4	4	6	6	10	11
1	15	17	23	26	32	37	47	54	66	76
2	93	105	126	143	172	194				

Incidentally, equating the results in (7) and (9), we get an identity which is more general but not as elegant as the well-known Rogers-Ramanujan identities.

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