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On the Diophantine Equation $\sum X_i = \prod X_i$

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Abstract. The diophantine equation $X_1 + \cdots + X_k = X_1 \cdots X_k$ has at least one solution in positive integers for $k \geqslant 2$. The set of integers k for which this is the only solution are investigated; in particular, this set is conjectured to be a known finite sequence.

The equation $f_k(\mathbf{X}) = X_1 + X_2 + \cdots + X_k - X_1 X_2 \cdots X_k = 0$ has the solution, for $k \geqslant 2$, given by $X_1 = 2$, $X_2 = k$, $X_3 = X_4 = \cdots = X_k = 1$. Schinzel showed that there are no other solutions in positive integers, apart from permutations of this given solution, for $k = 6$ and $k = 24$. Misiurewicz [2] states that $k = 2, 3, 4, 6, 24, 144, 174, 444$ are the only values of $k < 1000$ for which $f_k(\mathbf{X}) = 0$ has essentially one solution, as above. But the number 144 in this list (given in both [2] and [1, D24]) is probably a misprint for 114, for with this correction Misiurewicz's assertion is then correct (evidently 144 will not do because of the extra solution $1^{141} \cdot 2 \cdot 4 \cdot 21 = 168 = (141) \cdot 1 + 2 + 4 + 21$). We report here on some further calculations with this equation.

Proposition 1. *The equation $f_k(\mathbf{X}) = 0$, for $k \geqslant 4$, has only one solution in positive integers, apart from permutations, if and only if the following conditions hold*:

(1) *$k - 1$ is a prime number.*

(2) *Let s, n be any integers, if any, with $3 \leqslant s \leqslant \log_2 k + 1$, $2^{s-2} \leqslant n \leqslant (k^{1/s} + 1)^{s-2}$ and n being a product $x_1 \cdots x_{s-2}$ of $s - 2$ integers $x_i \geqslant 2$. Put $t = x_1 + \cdots + x_{s-2}$. Then no factor of $N = (k - s + t)n + 1$ is congruent to -1 modulo n except possibly for $n - 1$ and $N/n - 1$.*

Proof. Let $f_k(\mathbf{x}) = 0$ be a solution in positive integers; we may suppose that $x_1, \ldots, x_s$ are precisely those integers among $x_1, \ldots, x_k$ which do not equal 1. Thus $x_1 \cdots x_s = k - s + x_1 + \cdots + x_s$. Since $x_i \geqslant 2$ for all $i \leqslant s$, it follows that $2^s \leqslant k + s$. It is then elementary to show that for $k \geqslant 4$ we have $s \leqslant \log_2 k + 1$.

The case $s = 1$ is easily ruled out, so we next consider the case $s = 2$. A solution $f_k(\mathbf{x}) = 0$ with $s = 2$ gives $x_1(x_2 - 1) = k - 2 + x_2$. If this is distinct from the given solution we have that $x_2 - 1$ does not equal $k - 1$ and is a proper factor of $k - 2 + x_2$. It follows that $x_2 - 1$ is a proper factor of $k - 1$. Thus no other solution exists if and only if $k - 1$ is a prime number.

Suppose now $s \geqslant 3$. Given integers $x_1, \ldots, x_{s-2} \geqslant 2$, put $n = x_1 x_2 \cdots x_{s-2}$ and $t = x_1 + x_2 + \cdots + x_{s-2}$. Then there are integers $x_{s-1}, x_s \geqslant 2$ with $f_k(\mathbf{x}) = k - s + t + x_{s-1} + x_{s-2} - n x_{s-1} x_{s-2} = 0$ if and only if $f = x_{s-1} n - 1$ is a factor of,

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and not equal to, $k - s + t + x_{s-1}$. Put $N = (k - s + t)n + 1$. We deduce x_{s-1}, $x_s \geqslant 2$ exist as required if and only if N has a factor $f \neq n - 1$ and $\neq N/(n-1)$ with f congruent to -1 modulo n. It remains to bound n.

Let $x_1, \ldots, x_s$ be a solution, as above, with the x_i arranged in increasing magnitude. Consider the problem of maximizing n subject to $f_k(\mathbf{x}) = 0$, $x_i \geqslant 2$, and the other stated constraints on x_i except that we now allow nonintegral values. Since n, as a function of $x_1, \ldots, x_s$, has no critical values in the given region, it must take its extreme values for extreme values of the variables x_i. It is easy to see that for the maximum value of n we must have all the x_i, $i \leqslant s$, being equal. Thus the maximum value of n is x^{s-2} where x is the positive root of the equation $x^s = k - s + sx$. Plainly, $k^{1/s} \leqslant x \leqslant k^{1/s} + 1$; thus $n \leqslant (k^{1/s} + 1)^{s-2}$ as required.

COROLLARY 1. *Suppose $f_k(\mathbf{X}) = 0$ has only 1 essential solution in positive integers. Then*:

(1) $k - 1$ *and* $2k - 1$ *are prime numbers.*

(2) $6|k$ *if* $k > 4$.

(3) $4k + 1$ *and* $4k + 5$ *are sums of two squares for* $k > 4$.

Proof. (1) Take $s = 3$, $n = 2$ in the proposition. (2) follows from (1). (3) Take $n = 4$, $s = 3$ or $n = 2.2$, $s = 4$ in the proposition and apply Fermat's criterion, noting from (2) that neither $4k + 1$ nor $4k + 5$ is divisible by 3 for $k > 4$.

Proposition 1 can be used to give an algorithm for testing if an integer k has the required property; the number of steps required is at most $O(k^{3/2+\varepsilon})$, for all $\varepsilon > 0$. Using this algorithm, a PET 4032 microprocessor was used to test suitable values of k; this revealed the discrepancy in Misiurewicz's list, though it is easy to check by hand using Proposition 1 that $k = 114$ should be in the list. No other values of $k < 11{,}000$ were found for which $f_k(\mathbf{X}) = 0$ has one solution, this computation taking 40 minutes of computing time. We thus end with:

CONJECTURE. *The only values of k for which $f_k(\mathbf{X}) = 0$ has one solution are* $k = 2, 3, 4, 6, 24, 114, 174, 444$.

Added in Proof. With a different program, the conjecture has now been verified for all $k \leqslant 50{,}000$.

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