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Explicit Estimates for $\theta(x;\,3,l)$ and $\psi(x;\,3,l)$

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Abstract. Let $\theta(x;\,3,l)$ be the sum of the logarithms of the primes not exceeding x that are congruent to l modulo 3, where l is 1 or 2. By the prime number theorem for arithmetic progressions, $\theta(x;\,3,l) \sim x/2$ as $x \to \infty$. Using information concerning zeros of Dirichlet L-functions, we prove explicit numerical bounds for $\theta(x;\,3,l)$ of the form $|\theta(x;\,3,l) - x/2| < \varepsilon x$, $x \geqslant x_0(\varepsilon)$.

5. Introduction. For positive integers k and l, define

$$\theta(x;k,l) = \sum_{\substack{p \leqslant x \\ p \equiv l \,(\mathrm{mod}\, k)}} \log p, \qquad \psi(x;k,l) = \sum_{\substack{p^\alpha \leqslant x \\ p^\alpha \equiv l \,(\mathrm{mod}\, k)}} \log p,$$

where the sums extend over primes p and prime powers p^α, respectively. In a previous paper [4] we derived estimates $\psi(x;\,k,l)$ and $\theta(x;\,k,l)$ when $k \geqslant 10$. We now consider the case $k = 3$, $l = 1$ or 2. For convenience we shall continue the numbering of sections and equations from [4]. Unless otherwise noted we shall also use the same notation.

In the case $k \geqslant 10$ our estimates were primarily based on an explicit zero-free region for Dirichlet L-functions $L(s,\chi)$ and an estimate for $N(T,\chi)$, the number of zeros of $L(s,\chi)$ in $\sigma > 0$, $|t| \leqslant T$. In the case of a fixed modulus such as $k = 3$, we can make use of certain computational information concerning the zeros of the $\varphi(k)$ L-functions formed with characters modulo k. For example, if the generalized Riemann hypothesis were known to hold for zeros up to a height H ($H \gg k\log k$), then by the methods of Section 3 we could prove that

$$\left|\psi(x;\,k,l) - \frac{x}{\varphi(k)}\right| \ll x^{1/2}\log^2 kH + x\left(\frac{\log kH}{\varphi(k)H}\right)^{1/2}.$$

In particular, if the generalized Riemann hypothesis holds, we can take $H = x$ to obtain

$$\left|\psi(x;\,k,l) - \frac{x}{\varphi(k)}\right| \ll x^{1/2}\log^2 kx;$$

a well-known result.

In the case $k = 3$ there are two characters; a principal character χ_0 and a real nonprincipal character χ_1. The zeros of $L(s,\chi_0)$ have been studied extensively, since

$$L(s,\chi_0) = (1 - 3^{-s})\zeta(s). \tag{5.1}$$

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In particular, calculations of Brent [1] showed that if $A = 32585736.4$, then $N(A,\chi_0) = 150000000$ and all of the zeros $\rho = \beta + i\gamma$ with $|\gamma| \leqslant A$ have $\beta = \frac{1}{2}$.

Much less is known about the zeros of $L(s,\chi_1)$, but Davies [2], [6] calculated the first 2500 zeros in the critical strip to within 10^{-6}. Spira [8] also calculated the first 6 zeros to within 10^{-17}. The calculations of Davies showed that if $H_1 = 2571.388782$, then $N(H_1,\chi_1) = 5000$, and all of the zeros with $|\gamma| \leqslant H_1$ have $\beta = \frac{1}{2}$.

Using this information about the zeros and the methods of Section 3, we are able to tabulate values of L and ε such that

$$\left|\psi(x;3,l) - \frac{x}{2}\right| < \varepsilon x, \qquad x \geqslant e^L, \tag{5.2}$$

for $l = 1$ or 2. As an application of these results, we prove the following

THEOREM 5.1. *The maximum value of $\theta(x;\ 3,2)/x$ occurs at $x = 1619$, and furthermore*

$$\theta(x;3,2) < .50933118x, \qquad x > 0.$$

THEOREM 5.2. *If $x > 0$, then*

$$\theta(x;3,1) < .5040354x.$$

THEOREM 5.3. *If $x \geqslant x_0$, then $\theta(x;\ 3,l) \geqslant cx$, where x_0 and c are given by the following tables.*

$l = 1$

x_0	c
151	.40722
727	.45116
17377	.49026
91807	.49585

$l = 2$

x_0	c
47	.40755
233	.45398
3761	.49042
21317	.49595

6. A Zero-Free Region for $L(s,\chi_1)$. In this section we prove a result which is a slight improvement of Theorem 1 of [3].

THEOREM 6.1. *If $R = 9.645908801$, then $L(s,\chi_1) \neq 0$ for $|t| \geqslant 2000$ and $\sigma > 1 - 1/(R\log(|t|/4))$.*

The proof is essentially the same as that of Theorem 1 of [3], so we indicate only the necessary changes. Lemma 2 of [3] can be replaced by

LEMMA 2′. *If $1 < \sigma < 1.055$, $t \geqslant 2000$, $1 \leqslant m \leqslant 4$, $s = \sigma + imt$, $\sigma_1 = (1 + \sqrt{1 + 4\sigma^2})/2$, $s_1 = \sigma_1 + imt$, and $\kappa = (5 - \sqrt{5})/10$, then*

$$\frac{1}{2}\operatorname{Re}\left[\frac{\Gamma'}{\Gamma}\left(\frac{s+a}{2}\right) - \frac{1}{\sqrt{5}}\frac{\Gamma'}{\Gamma}\left(\frac{s_1+a}{2}\right)\right] < \kappa\log\left(\frac{mt}{2}\right) + \frac{1}{1000m}.$$

The proof of Lemma 2′ is the same as for Lemma 2, except that we use

$$\left|\operatorname{Re}\int_0^\infty \frac{u - [u] - \frac{1}{2}}{(u + (z+a)/2)^2}\,du\right| < \frac{1}{2}\int_0^\infty \frac{du}{|u + (z+a)/2|^2} < \frac{\pi}{2y}.$$

In place of (9) of [3] we then obtain

$$\frac{1}{2}\operatorname{Re}\left[\frac{\Gamma'}{\Gamma}\left(\frac{s+a}{2}\right)-\frac{1}{\sqrt{5}}\frac{\Gamma'}{\Gamma}\left(\frac{s_1+a}{2}\right)\right] < \kappa\log\frac{mt}{2}+\frac{(\sigma+a)^2}{4m^2t^2}+\frac{\pi}{4mt}+\frac{\pi}{4\sqrt{5}\,mt}$$
$$< \kappa\log\frac{mt}{2}+\frac{1}{1000m}.$$

If $\sigma < 1.055$ and $t \geqslant 2000$, the proofs of Lemmas 5, 6, and 7 yield

$$f(\gamma,\chi_1) < \kappa\log\frac{3\gamma}{2\pi}+\frac{1}{1000}-\frac{1}{\sigma-\beta}, \tag{6.1}$$

$$f(2\gamma,\chi_0) < \kappa\log\frac{\gamma}{\pi}+\frac{1}{2000}+T(3,1), \tag{6.2}$$

$$f(3\gamma,\chi_1) < \kappa\log\frac{9\gamma}{2\pi}+\frac{1}{3000}, \tag{6.3}$$

$$f(4\gamma,\chi_0) < \kappa\log\frac{2\gamma}{\pi}+\frac{1}{4000}+T(3,1). \tag{6.4}$$

For $\sigma < 1.055$, the proof of Lemma 3 yields

$$f(0,\chi_0) < \frac{1}{\sigma-1} - .9838 - s(3) < \frac{1}{\sigma-1} - 1.3862.$$

It follows from (6.1), (6.2), (6.3), (6.4) and (27) of [3] that

$$\frac{a_1}{\sigma-\beta} < \frac{a_0}{\sigma-1}+\kappa(a_1+a_2+a_3+a_4)\log\frac{\gamma}{\pi} - 1.3862a_0$$
$$+\sum_{m=1}^{4} a_m\left(\kappa\log\frac{m}{2}+\frac{1}{1000m}\right)+\kappa(a_1+a_3)\log 3+(a_2+\sigma_4)T(3,1)$$
$$< \frac{a_0}{\sigma-1}+\kappa(a_1+a_2+a_3+a_4)\log\frac{\gamma}{4},$$

since $T(3,1) < .65$. Theorem 6.1 then follows by choosing $\sigma = 1 + .33901/\log(\gamma/4)$.

7. Estimates for $N(T,\chi_0)$ and $N(T,\chi_1)$. In this section we improve slightly the results of Section 2 for the characters χ_0 and χ_1. Henceforth we shall write $N_i(T)$ for $N(T,\chi_i)$, $i = 0, 1$. From (2.17) we obtain immediately

$$|N_0(T) - F_0(T)| < R_0(T), \qquad T \geqslant 7436, \tag{7.1}$$

where

$$F_0(T) = \frac{T}{\pi}\log\frac{T}{2\pi e}, \qquad R_0(T) = B_1\log T + B_2,$$
$$B_1 = .49144 \quad\text{and}\quad B_2 = 4.926.$$

For the zeros of $L(s,\chi_1)$ we prove the following result.

THEOREM 7.1. *If* $T \geqslant 100$, *then*

$$|N_1(T) - F_1(T)| < R_1(T), \tag{7.2}$$

where

$$F_1(T) = \frac{T}{\pi}\log\frac{3T}{2\pi e}, \qquad R_1(T) = C_1\log T + C_2,$$
$$C_1 = \frac{2}{\pi\log 2} \quad\textit{and}\quad C_2 = 4.7928.$$

Note that this notation differs from that of Section 2. The proof of Theorem 7.1 is the same as that of Theorem 2.1, except for the following changes. In (2.8) we have

$$\operatorname{Im}\log\Gamma\left(\frac{3}{4}+i\frac{T}{2}\right)=\frac{T}{2}\log\frac{T}{2e}+\frac{T}{4}\log\left(1+\frac{9}{4T^2}\right)$$
$$+\frac{1}{4}\tan^{-1}\left(\frac{2T}{3}\right)+\frac{\theta}{3T}.$$

From

$$0<\frac{T}{4}\log\left(1+\frac{9}{4T^2}\right)+\frac{1}{4}\tan^{-1}\left(\frac{2T}{3}\right)<\frac{9}{16T}+\frac{\pi}{8}$$

we obtain

$$N_1(T)=F_1(T)+.255710\theta+\frac{1}{\pi}\Delta_{\mathscr{C}}\arg L(s,\chi_1)$$

in place of (2.9). Equation (2.12) is replaced by

$$\frac{1}{2\pi}\int_{\pi/2}^{3\pi/2}\log|f(\sigma_0+(1+2\eta)e^{i\theta})|\,d\theta$$
$$<\frac{N}{2\pi}(1+2\eta)\log(.4872T)+\frac{N}{2}\log\zeta(1+\eta).$$

Using this, we obtain (7.2) with

$$C_1=\frac{1+2\eta}{\pi\log 2},$$
$$C_2=\frac{4\log\zeta(1+\eta)}{\log 2}-\frac{2\log\zeta(2+2\eta)}{\log 2}$$
$$+\frac{2}{\pi}\log\zeta\left(\frac{3}{2}+2\eta\right)-.0745-.66043\eta.$$

If $\eta=\frac{1}{2}$, this yields the result.

8. Estimation of $\psi(x;3,l)$ and $\theta(x;3,l)$. Let H_1 and A be as in Section 5, and define $H_0=7436.76651$. Our starting point is the following.

THEOREM 8.1. *If m is a positive integer, $0<\delta<(x-2)/mx$, and l is 1 or 2, then*

$$(8.1)\quad \frac{1}{x}\left|\psi(x;3,l)-\frac{x}{2}\right|$$
$$<\frac{1}{2}\left(1+\frac{m\delta}{2}\right)x^{-1/2}\left(\sum_{\substack{\rho\in Z(\chi_0)\\|\gamma|\le H_0}}\frac{1}{|\rho|}+\sum_{\substack{\rho\in Z(\chi_1)\\|\gamma|\le H_1}}\frac{1}{|\rho|}\right)$$
$$+\frac{1}{2}A_m(\delta)\left(\sum_{\substack{\rho\in Z(\chi_0)\\H_0<|\gamma|\le A}}\frac{x^{-1/2}}{|\gamma|^{m+1}}+\sum_{\substack{\rho\in Z(\chi_0)\\A<|\gamma|}}\frac{x^{\beta-1}}{|\gamma|^{m+1}}+\sum_{\substack{\rho\in Z(\chi_1)\\H_1<|\gamma|}}\frac{x^{\beta-1}}{|\gamma|^{m+1}}\right)$$
$$+\frac{1}{x}\left[\frac{1}{2}\log\left(\frac{4x^2}{2x-1}\right)+1.39305\right]+\frac{m\delta}{4}.$$

Proof. As in the proof of Theorem 3.6, we obtain

$$(8.2)\quad |E_m(x,\pm\delta x)|$$
$$\leqslant \frac{(\delta x)^m}{2}\left(1+\frac{m\delta}{2}\right)\left[\sum_{\substack{\rho\in z(\chi_0)\\|\gamma|\leqslant H_0}}\frac{x^\beta}{|\rho|}+\sum_{\substack{\rho\in z(\chi_1)\\|\gamma|\leqslant H_1}}\frac{x^\beta}{|\rho|}\right]$$
$$+\frac{1}{2}(\delta x)^m A_m(\delta)\left[\sum_{\substack{\rho\in z(\chi_0)\\H_0<|\gamma|}}\frac{x^\beta}{|\rho(\rho+1)\cdots(\rho+m)|}\right.$$
$$\left.+\sum_{\substack{\rho\in z(\chi_1)\\H_1<|\gamma|}}\frac{x^\beta}{|\rho(\rho+1)\cdots(\rho+m)|}\right]$$
$$+\frac{1}{2}x^m\left|\sum_{\substack{\rho\in z(\chi_0)\\\beta=0}}\frac{x^\rho}{\rho}\int_0^{\pm\delta}\cdots\int_0^{\pm\delta}(1+y_1+\cdots+y_m)^\rho\,dy_1\cdots dy_m\right|$$
$$+\left|\int_0^{\pm\delta x}\cdots\int_0^{\pm\delta x}f(x+y_1+\cdots+y_m)\,dy_1\cdots dy_m\right|.$$

Note that χ_1 is primitive, and the only zeros in $z(\chi_0)$ with $\beta=0$ occur at $\rho=2\pi in/\log 3$, $0<|n|<\infty$. Hence we can write the contribution from the purely imaginary zeros as

$$\frac{x^m}{2}\left|\int_0^{\pm\delta}\cdots\int_0^{\pm\delta}\sum_{\substack{0<|n|<\infty\\\rho=2\pi in/\log 3}}\frac{[x(1+y_1+\cdots+y_m)]^\rho}{\rho}\,dy_1\cdots dy_m\right|.$$

The interchange of summation and integration is justified by the fact that the partial sums are uniformly bounded. The integrand may be written as

$$\frac{\log 3}{\pi}\sum_{n=1}^{\infty}\frac{\sin(nz)}{n},$$

where

$$z=\frac{2\pi}{\log 3}\log[x(1+y_1+\cdots+y_m)].$$

In particular the integrand is bounded by $(\log 3)/2$ in absolute value. Hence the contribution from the imaginary zeros in (8.2) does not exceed

$$(8.3)\qquad \frac{x^m}{2}\int_0^{\delta}\cdots\int_0^{\delta}\frac{\log 3}{2}\,dy_1\cdots dy_m=\frac{(\delta x)^m\log 3}{4}.$$

From (3.10) we obtain

$$(8.4)\qquad |f(y)|\leqslant\frac{1}{2}\log\left(\frac{y^2}{y-1}\right)+|d_1+d_2|,$$

since $d_2=-\frac{1}{2}$ from (3.5). Furthermore we have

$$(8.5)\qquad d_1+d_2=-\frac{1}{2}\left(\log 2\pi-\frac{1}{2}\log 3\right)-\frac{1}{2}\chi_1(l)\frac{L'}{L}(0,\chi_1).$$

Spira [9] gives the values $L(0,\chi_1) = \frac{1}{3}$ and $L'(0,\chi_1) \approx .31606627554754$. It follows from (8.4) and (8.5) that

$$|f(x + y_1 + \cdots + y_m)| < \frac{1}{2}\log\left(\frac{4x^2}{2x-1}\right) + 1.11839. \tag{8.6}$$

The theorem then follows from Lemma 3.4, (8.2), (8.3), (8.6) and the calculations of Brent and Davies mentioned in Section 5.

The first two sums involving zeros on the right of (8.1) may be calculated from the known zeros. From the tables of Davies [6], the author calculated that

$$\sum_{\substack{\rho \in z(\chi_1) \\ |\gamma| \leq H_1}} \frac{1}{|\rho|} < 8.7642, \tag{8.7}$$

and Schoenfeld [7] gave the estimate

$$\sum_{\substack{\rho \in z(\chi_0) \\ |\gamma| \leq H_0}} \frac{1}{|\rho|} < 7.93485. \tag{8.8}$$

From Schoenfeld we also obtain that $N_0(H_0) = 14384$.

Note that the bound given on the right of (8.1) is decreasing in x, so that for the purposes of (5.2) we may henceforth assume that $\log x = L$.

LEMMA 8.2. *If* $L \leq (m+1)R\log^2(A/17)$, *then*

$$x^{-1/2} \sum_{\substack{\rho \in z(\chi_0) \\ H_0 < |\gamma| \leq A}} \frac{1}{|\gamma|^{m+1}} + \sum_{\substack{\rho \in z(\chi_0) \\ A < |\gamma|}} \frac{x^{\beta-1}}{|\gamma|^{m+1}} < \varepsilon_1 + \varepsilon_2 + \varepsilon_3,$$

where

$$\varepsilon_1 = e^{-L/2}\left[G(H_0) - \tfrac{1}{2}G(A)\right],$$

$$G(t) = \frac{1}{t^m}\left[\frac{-N_0(t)}{t} + \frac{(m+1)(1+m\log(t/2\pi e))}{\pi m^2} + \frac{1}{t}\left(B_1\log t + \frac{B_1}{m+1} + B_2\right)\right],$$

$$\varepsilon_2 = \frac{1}{2}\exp\left(\frac{-L}{R\log(A/17)}\right)\left[F_0(A) + R_0(A) - N_0(A)\right]/A^{m+1},$$

$$\varepsilon_3 = \frac{L}{m\pi R 17^m}K_2\left(2\sqrt{\frac{mL}{R}}, V_1\right) + \frac{\log(17/2\pi)}{\pi 17^m}\sqrt{\frac{L}{mR}}\,K_1\left(2\sqrt{\frac{mL}{R}}, V_1\right)$$
$$+ \frac{B_1}{17^{m+1}}\sqrt{\frac{L}{(m+1)R}}\,K_1\left(2\sqrt{\frac{(m+1)L}{R}}, V_2\right),$$

$$V_1 = \sqrt{\frac{mR}{L}}\log\frac{A}{17}, \qquad V_2 = \sqrt{\frac{(m+1)R}{L}}\log\frac{A}{17}.$$

Proof. Theorem 1 of Rosser and Schoenfeld [5] states that if $\rho = \beta + i\gamma$ is a zero of $\zeta(s)$, then

$$\beta < 1 - \frac{1}{R\log(|\gamma|/17)}, \qquad |\gamma| > 21.$$

From this and the fact that $1 - \bar{\rho}$ is a zero whenever ρ is, we obtain

$$\sum_{\substack{\rho\in z(\chi_0)\\ A<|\gamma|}} \frac{x^{\beta-1}}{|\gamma|^{m+1}} < \frac{1}{2}x^{-1/2} \sum_{\substack{\rho\in z(\chi_0)\\ A<|\gamma|}} \frac{1}{|\gamma|^{m+1}} + \frac{1}{2} \sum_{\substack{\rho\in z(\chi_0)\\ A<|\gamma|}} \varphi_m(|\gamma|), \tag{8.9}$$

where

$$\varphi_m(t) = \frac{x^{-1/[R\log(t/17)]}}{t^{m+1}}.$$

If $H_0 \leqslant u < v$, then

$$\sum_{\substack{\rho\in z(\chi_0)\\ u<|\gamma|\leqslant v}} \frac{1}{|\gamma|^{m+1}} = \int_u^v \frac{dN_0(t)}{t^{m+1}}$$

$$= \frac{N_0(v)}{v^{m+1}} - \frac{N_0(u)}{u^{m+1}} + (m+1)\int_u^v \frac{N_0(t)}{t^{m+2}}\,dt.$$

If we use (7.1) to estimate the last integral, we obtain

$$\sum_{\substack{\rho\in z(\chi_0)\\ u<|\gamma|\leqslant v}} \frac{1}{|\gamma|^{m+1}} < G(u) - G(v). \tag{8.10}$$

We now write

$$\sum_{\substack{\rho\in z(\chi_0)\\ A<|\gamma|}} \varphi_m(|\gamma|) = \int_A^\infty \varphi_m(t)\,dN_0(t)$$

$$= -\varphi_m(A)N_0(A) - \int_A^\infty \varphi_m'(t)N_0(t)\,dt.$$

The condition $L \leqslant (m+1)R\log^2(A/17)$ implies that $\varphi_m'(t) < 0$ for $t > A$; hence (7.1) yields

$$\sum_{\substack{\rho\in z(\chi_0)\\ A<|\gamma|}} \varphi_m(|\gamma|) < -\varphi_m(A)N_0(A) - \int_A^\infty \varphi_m'(t)[F_0(t) + R_0(t)]\,dt$$

$$= \varphi_m(A)[F_0(A) + R_0(A) - N_0(A)] + \int_A^\infty \varphi_m(t)[F_0'(t) + R_0'(t)]\,dt$$

$$= 2(\varepsilon_2 + \varepsilon_3).$$

The lemma then follows from (8.9) and (8.10).

LEMMA 8.3. *If* $L \leqslant (m+1)R\log^2(H_1/4)$, *then*

$$\sum_{\substack{\rho\in z(\chi_1)\\ H_1<|\gamma|}} \frac{x^{\beta-1}}{|\gamma|^{m+1}} < \varepsilon_4 + \varepsilon_5 + \varepsilon_6,$$

where

$$\varepsilon_4 = \frac{1}{2}e^{-L/2}\left[\frac{-N_1(H_1)}{H_1^{m+1}} + \frac{(m+1)(1+m\log(3H_1/2\pi e))}{\pi m^2 H_1^m}\right.$$
$$\left. + \frac{1}{H_1^{m+1}}\left(C_1\log H_1 + \frac{C_1}{m+1} + C_2\right)\right],$$

$$\varepsilon_5 = \frac{1}{2}\exp\left(\frac{-L}{R\log(H_1/4)}\right)[F_1(H_1) + R_1(H_1) - N_1(H_1)]/H_1^{m+1},$$

$$\varepsilon_6 = \frac{L}{\pi m R 4^m}K_2\left(2\sqrt{\frac{mL}{R}}, V_3\right) + \frac{\log(6/\pi)}{\pi 4^m}\sqrt{\frac{L}{mR}}\,K_1\left(2\sqrt{\frac{mL}{R}}, V_3\right)$$
$$+ \frac{C_1}{4^{m+1}}\sqrt{\frac{L}{(m+1)R}}\,K_1\left(2\sqrt{\frac{(m+1)L}{R}}, V_4\right),$$

$$V_3 = \sqrt{\frac{mR}{L}}\log\frac{H_1}{4}, \qquad V_4 = \sqrt{\frac{(m+1)R}{L}}\log\frac{H_1}{4}.$$

The proof of Lemma 8.3 is virtually identical to that of Lemma 8.2, except that we use Theorem 6.1 for the zero-free region and (7.2) in place of (7.1).

As an immediate consequence of Theorem 8.1, Lemmas 8.2 and 8.3, and (8.7) and (8.8) we obtain the following.

COROLLARY 8.4. *If* $x \geqslant e^L$, $L \leqslant (m+1)R\log^2(H_1/4)$, $0 < \delta < (x-2)/mx$, *and* $l = 1$ *or* 2, *then*

$$\frac{1}{x}\left|\psi(x;3,l) - \frac{x}{2}\right| < \left(1 + \frac{m\delta}{2}\right)(8.349525)e^{-L/2} + \frac{m\delta}{4}$$
$$+ \frac{1}{2}A_m(\delta)\sum_{i=1}^{6}\varepsilon_i + \frac{1}{x}\left[1.39305 + \frac{1}{2}\log\left(\frac{4x^2}{2x-1}\right)\right].$$

Using Corollary 8.4 we tabulate in Table 4 values of ε and L for which (5.2) holds. The functions $K_\nu(z,y)$ that occur in ε_3 and ε_6 are easily estimated by the methods of Section 4. The values of m and δ used for each L were chosen experimentally and are also listed in the table.

Theorems 5.1, 5.2, and 5.3 follow from the estimates of Table 4 and direct computation of $\theta(x;\ 3,1)$ and $\theta(x;\ 3,2)$. The author calculated $\theta(p;\ 3,1)$ and $\theta(p;\ 3,2)$ for every prime p, $p \leqslant 10^8$. These calculations were performed in double precision on CDC Cyber computers at the University of Illinois and Michigan State University, using a standard sieve procedure to generate the primes.

If $x \geqslant 10^7$, then Table 4 yields

$$\theta(x;\ 3,2) < .5058681\,x.$$

Theorem 5.1 then follows by direct calculation for $x \leqslant 10^7$, since $\theta(x;\ 3,2)/x$ is decreasing for x between primes. The proof of Theorem 5.2 is similar, except that we use direct calculation for $x \leqslant 10^8$.

TABLE 4

L	ϵ	m	δ
13.815	1.1636(-2)	6	1.8391(-3)
16.118	5.8681(-3)	6	1.8325(-3)
18.420	4.0354(-3)	6	1.8239(-3)
20.723	3.4445(-3)	6	1.8147(-3)
23.025	3.2466(-3)	6	1.8055(-3)
27.631	3.1385(-3)	6	1.7870(-3)
35	3.0789(-3)	6	1.7577(-3)
50	2.9769(-3)	6	1.6996(-3)
75	2.8083(-3)	5	1.8704(-3)
100	2.6317(-3)	5	1.7529(-3)
150	2.3114(-3)	5	1.5397(-3)
200	1.9870(-3)	4	1.5883(-3)
250	1.7027(-3)	4	1.3612(-3)
300	1.4137(-3)	3	1.4127(-3)
350	1.1706(-3)	3	1.1706(-3)
400	9.6194(-4)	2	1.2818(-3)
450	7.5760(-4)	2	1.0096(-3)
500	5.9764(-4)	2	7.9653(-4)
600	3.7386(-4)	2	4.9834(-4)
700	2.3566(-4)	2	3.1415(-4)
800	1.4980(-4)	2	1.9971(-4)
900	9.6109(-5)	2	1.2814(-4)
1000	6.2280(-5)	2	8.3036(-5)

If $x \geqslant 10^8$, then from Table 4 and (4.1) we obtain

$$\theta(x;3,1) > .4959646\,x - 1.001093\,x^{1/2} - 3x^{1/3} > .49585\,x.$$

The estimates for $\theta(x;3,1)$ in Theorem 5.3 then follow from direct calculation for $x_0 \leqslant x \leqslant 10^8$.

If p is a prime, note that $p^{2m} \not\equiv 2 \pmod 3$ and $p^{2m+1} \equiv p \pmod 3$. It follows that

$$\psi(x;3,2) - \theta(x;3,2) = \sum_{m=1}^{\infty} \theta(x^{1/(2m+1)};3,2).$$

Let $x \geqslant 10^8$. The number of nonzero terms in the sum with $m > 1$ does not exceed $\log x/2\log 2 - 3/2$; hence from Theorem 5.1 we obtain

$$\psi(x;3,2) - \theta(x;3,2) < .50933118\left[x^{1/3} + x^{1/5}\left(\frac{\log x}{2\log 2} - \frac{3}{2}\right)\right] < 1.1\,x^{1/3}.$$

It follows from Table 4 that

$$\theta(x;3,2) > .4959646\,x - 1.1x^{1/3} > .49595\,x.$$

Theorem 5.3 then follows by direct calculation for $x_0 \leqslant x \leqslant 10^8$.

It is interesting to note that Theorems 5.1 and 5.3 are essentially best possible as they are stated, but Theorem 5.2 is probably not best possible. The calculations performed by the author show that for $x \leqslant 10^8$ the maximum of $\theta(x;\, 3,1)/x$ occurs at $x = 52553329$, and that $\theta(x;\, 3,1) < .499935x$ for $x \leqslant 10^8$. It would require significantly more calculation to determine the point at which $\theta(x;\, 3,1)/x$ assumes its maximum value.

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