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Numerical Solution of Two Transcendental Equations*

By Luciano Misici

Abstract. This paper deals with the study of the transcendental equations: $\sin(s+\nu)/(s+\nu) = \pm\sin(s-\nu)/(s-\nu)$, where $\nu = (s^2-\gamma^2)^{1/2}$. These equations are obtained in the study of some boundary value problems for a modified biharmonic equation using the Papkovich-Fadle series. Some numerical solutions obtained with an iterative procedure are given.

Introduction. It is well known that the governing equation of a bidimensional Stokes flow parallel to the x, y plane, in terms of the stream function $\Psi(x, y)$, is the biharmonic equation

$$\nabla^4\Psi = \frac{\partial^4\Psi}{\partial x^4} + \frac{\partial^4\Psi}{\partial x^2\partial y^2} + \frac{\partial^4\Psi}{\partial y^4} = 0. \tag{1}$$

The solution of Eq. (1) in a rectangle with homogeneous conditions on Ψ and on its normal derivative on two parallel sides and for sufficiently regular boundary conditions on the remaining two parallel sides is expressed using the Papkovich-Fadle (PF) series. The even and odd eigenfunctions are associated to the roots, in the complex plane, of the equations

$$\sin 2s = -2s, \tag{2a}$$

$$\sin 2s = 2s. \tag{2b}$$

A complete study of Eqs. (2a) and (2b) may be found in [1]. Another interesting case is represented by the study of the equation

$$\nabla^4\Psi - \gamma^2\nabla^2\Psi = 0, \tag{3}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ and γ is real.

Equation (3) occurs, for example, in the study of the fluid motion in porous media [2] when Brinkman's approximation [3], [4] is used. The same equation has been used by de Socio, Gaffuri and the author [5] in the study of a tridimensional Stokes flow. If the solution of Eq. (3) is expressed in terms of PF series, the equations corresponding to (2a), (2b) are

$$\sin(s+\nu)/(s+\nu) = -\sin(s-\nu)/(s-\nu), \tag{4a}$$

$$\sin(s+\nu)/(s+\nu) = \sin(s-\nu)/(s-\nu), \tag{4b}$$

where $\nu = (s^2-\gamma^2)^{1/2}$. This note deals with a numerical study of Eqs. (4a), (4b) and gives a method to find all the real and complex solutions for any real value of γ.

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1. Analysis. All the roots of Eqs. (2a) and (2b) are complex except for $s = 0$. We just notice that, for Eqs. (2a), (2b) and (4a), (4b), if s is a complex root, then $-s$ and $\bar{s}$ are also roots. This study deals only with the roots in the first quadrant of the complex plane. Let $s_n = x_n + iy_n$ be the complex roots of Eqs. (2a), (2b) and (4a), (4b) ordered for $\mathrm{Re}(s_n)$ increasing taking into account the appropriate multiplicity. An asymptotic [6] evaluation of s_n is

$$s_n = \left(n \mp \frac{1}{4}\right)\pi + \frac{i}{2}\ln[(4n \mp 1)\pi], \qquad n = 1, 2, \ldots, \tag{1.1}$$

for the two cases (2a) and (2b), respectively. The expression (1.1) gives an excellent initial guess for finding the roots of Eqs. (2a), (2b) through an iterative procedure. Since when $\gamma \to 0$ Eqs. (4a), (4b) approach Eqs. (2a), (2b), for γ sufficiently small the values (1.1) are good initial guesses for the roots of (4a) and (4b). Some problems arise when γ is large. In this case both (4a) and (4b) have real roots different from $s = 0$ and $s = \pm\gamma$. In the following we study the real positive roots different from $s = 0$ or $s = \pm\gamma$.

Case 4a. In this case it is easy to see that the real positive roots of (4a) correspond to the zeros of the real function $f(x)$ of the real variable x given by:

$$f_\gamma(x) = \begin{cases} \bar{f}_\gamma(x) \equiv x \sin x \cosh\sqrt{\gamma^2 - x^2} \\ \qquad + \sqrt{\gamma^2 - x^2}\cos x \sinh\sqrt{\gamma^2 - x^2}, & 0 \leqslant x \leqslant \gamma, \\ f^*_\gamma(x) \equiv x \sin x \cos\sqrt{x^2 - \gamma^2} \\ \qquad - \sqrt{x^2 - \gamma^2}\cos x \sin\sqrt{x^2 - \gamma^2}, & x > \gamma. \end{cases}$$

From the study of $f(x)$ one can see that there exists a value γ_1^* below which the function does not have any zeros, whereas, for $\gamma = \gamma_1^*$, $f(x)$ presents a double zero. Let $\gamma \in [0, \infty)$. It will be shown that there is an increasing sequence $\{\gamma_n^*\}_{n=1}^{\infty}$ such that the total number of the real positive zeros of f is given by:

$$\begin{array}{llll} & 2(n-1) & \text{simple roots} & \multirow{2}{*}{} \\ (2.1) & 1 & \text{double root} & \Big\} \text{ for } \gamma = \gamma_n^*, \\ & 2n & \text{simple roots} & \text{for } \gamma_n^* < \gamma < \gamma_{n+1}^*. \end{array}$$

The relevant value γ_n^* may be obtained by solving the nonlinear system

$$\begin{cases} F(x, \gamma) = 0, \\ \dfrac{\partial F(x, \gamma)}{\partial x} = 0, \end{cases} \tag{1.3}$$

where $F(x, \gamma) \equiv \bar{f}_\gamma(x)$.

Case 4b. As in the case (4a), the real positive roots correspond to the zeros of the function:

$$g_\gamma(x) = \begin{cases} \bar{g}_\gamma(x) \equiv \sqrt{\gamma^2 - x^2}\sin x \cosh\sqrt{\gamma^2 - x^2} \\ \qquad - x \cos x \sinh\sqrt{\gamma^2 - x^2}, & 0 \leqslant x \leqslant \gamma, \\ g^*_\gamma(x) \equiv \sqrt{x^2 - \gamma^2}\sin x \cos\sqrt{x^2 - \gamma^2} \\ \qquad - x \cos x \sin\sqrt{\gamma^2 - x^2}, & x > \gamma. \end{cases}$$

Also, in this case, the numerical solution shows that there is an increasing sequence $\{\hat{\gamma}_n\}_{n=1}^{\infty}$ of values of γ such that the root $x = \gamma$, which in general is simple, has multiplicity two. As before the total number of the positive real roots for (4b) increases with increasing γ following (1.2). The values $\hat{\gamma}_n$ may be simply obtained by computing the positive real roots of the function

$$G(\gamma) = \gamma \operatorname{cotg} \gamma - 1. \tag{1.4}$$

2. Numerical Results. The real positive roots of $f(x)$ can be found as the intersections between the functions

$$\begin{cases} h_1(x) = -x \operatorname{tg} x \\ h_2(x) = \sqrt{\gamma^2 - x^2} \operatorname{tgh} \sqrt{\gamma^2 - x^2} \end{cases} \quad \text{for } 0 \leqslant x \leqslant \gamma,$$

$$\begin{cases} h_3(x) = x \operatorname{tg} x \\ h_4(x) = \sqrt{x^2 - \gamma^2} \operatorname{tg} \sqrt{x^2 - \gamma^2} \end{cases} \quad \text{for } x > \gamma.$$

From Figure 1, since $h_1(x)$ is independent of γ, we have that the critical value γ_n^* of γ occurs when $h_1(x)$ is tangent to $h_2(x)$. Moreover, for $\gamma = \bar{\gamma}$ the following intersections are obtained:

$$\begin{array}{lll} n-1 & \text{for} & (2n-1)\frac{\pi}{2} < \bar{\gamma} < \gamma_n^*, \\ n+1 & \text{for} & \gamma_n^* \leqslant \bar{\gamma} \leqslant n\pi, \\ n & \text{for} & n\pi < \bar{\gamma} \leqslant (2n+1)\frac{\pi}{2}. \end{array}$$

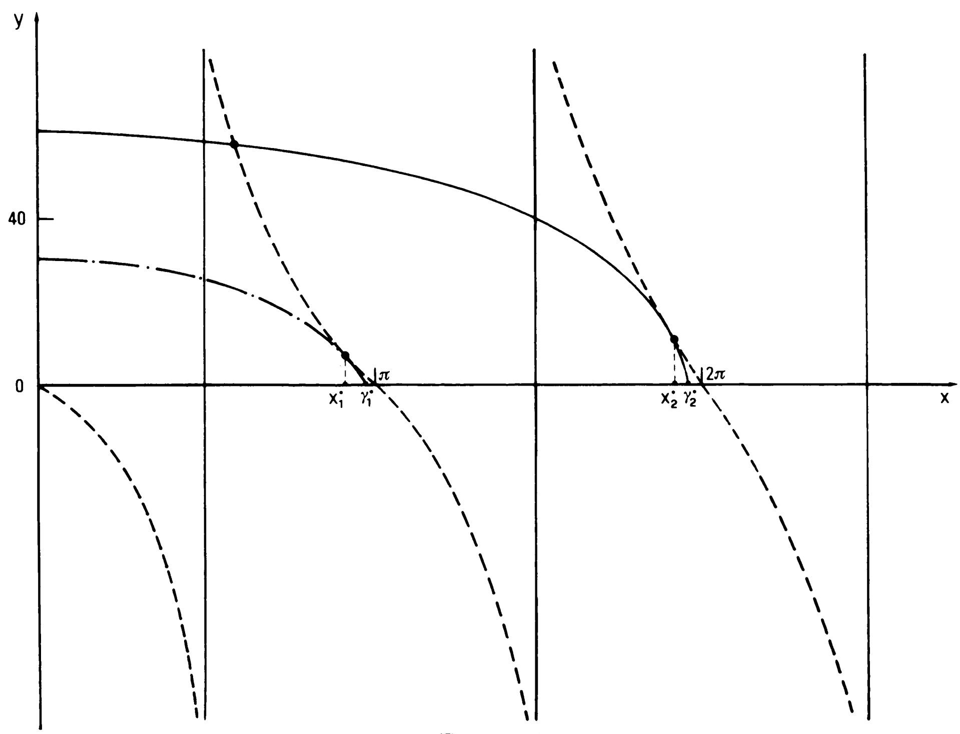

FIGURE 1

Qualitative behavior of: $h_1(x)$, (-----); $h_2(x)$ *for* $\gamma = \gamma_1^*$(-.-.-.-) *and* $\gamma = \gamma_2^*$(—).

TABLE 1

n	γ_n^*	x_n^*	$\hat{\gamma}_n$
1	3.044699213	2.798386046	4.493409458
2	6.237702792	6.121250467	7.725251837
3	9.394778563	9.317866462	10.904121659
4	12.543953611	12.486454395	14.066193913
5	15.690059952	15.644128370	17.220755272
6	18.834650141	18.796404366	20.371302959
7	21.978379228	21.945612880	23.519452499
8	25.121572041	25.092910412	26.666054259
9	28.264408144	28.238936575	29.811598791
10	31.406994934	31.384074018	32.956389039
11	34.549400602	34.528565755	36.100622244
12	37.691670537	37·672573565	39·244432361
13	40.833836124	40.816209326	42.387913568
14	43.975919763	43.959552889	45.531134014
15	47.117937867	47.102662770	48.674144232
16	50.259902741	50.245582838	51.816982487
17	53.401823788	53.388346622	54.959678288
18	56.543708319	56.530980194	58.102254754
19	59.685562107	59.673504130	61.244730260
20	62.827389766	62.815934889	64.387119591

Analogously it has been observed that a branch of $h_4(x)$ intersects a branch of $h_3(x)$ only if the zero of $h_4(x)$ is greater than the one of $h_3(x)$ and if the asymptote at the right of the first is smaller than the one at the right of the second. The converse is true if the left asymptote is considered. The number of intersections in this case is:

$$n - 1 \quad \text{for} \quad (2n-1)\frac{\pi}{2} < \bar{\gamma} \leqslant n\pi,$$

$$n \quad \text{for} \quad n\pi < \bar{\gamma} \leqslant (2n+1)\frac{\pi}{2}.$$

TABLE 2

Some roots of $\sin(s+\nu)/(s+\nu) = -\sin(s-\nu)/(s-\nu)$ *for different* γ

	$\gamma=2$		5		8		11	
n	x_n	y_n	x_n	y_n	x_n	y_n	x_n	y_n
1	2.437996	0.877279	1.977478	0	1.797406	0	1.728595	0
2	5.523739	1.492034	5.313822	0	5.464312	0	5.205321	0
3	8.647466	1.748061	6.271071	0.762808	8.165401	0	8.777944	0
4	11.781727	1.913401	9.184271	1.146239	11.447601	0	11.114004	0
5	14.919774	2.036307	12.194492	1.175362	9.232613	0.673713	13.095703	0
6	18.059488	2.134383	15.253600	1.933403	12.800335	0.848454	20.642927	0
7	21.200035	2.216087	18.339158	2.062323	15.814638	1.471467	12.343883	0.359468
8	24.341038	2.286155	21.440411	2.162697	18.825813	1.760598	15.552181	1.052172
9	27.482305	2.347515	24.551671	2.244949	21.866194	1.946863	18.455546	1.003263
10	30.623733	2.402107	27.669673	2.314711	24.928870	2.081945	22.335755	0.988899
11	33.765262	2.451285	30.792421	2.375349	28.007687	2.186887	25.421141	1.513242
12	36.906855	2.496031	33.918630	2.429026	31.098337	2.272254	28.467642	1.780566

TABLE 3

Some roots of $\sin(s+\nu)/(s+\nu) = \sin(s-\nu)/(s-\nu)$ *for different* γ

$\gamma =$	3		6		9		12	
n	x_n	y_n	x_n	y_n	x_n	y_n	x_n	y_n
1	4.227352	1.094263	3.834885	0	3.546650	0	3.431608	0
2	7.243661	1.571873	6.960890	0	7.212842	0	6.895312	0
3	10.329549	1.803743	7.963042	0.702769	9.546008	0	10.488174	0
4	13.440632	1.957677	10.904942	1.438448	14.888189	0	12.393439	0
5	16.563271	2.073438	13.903037	1.742506	10.860064	0.774565	14.943571	0
6	19.692069	2.166479	16.947589	1.930109	13.806829	0.565625	24.775731	0
7	22.824505	2.244399	20.020069	2.063755	17.508530	1.318499	13.893196	0.493383
8	25.959255	2.311505	23.110219	2.167006	20.525712	1.671658	17.224155	1.081474
9	29.095559	2.370475	26.212150	2.251019	23.560940	1.886380	20.098454	1.184221
10	32.232956	2.423096	29.322293	2.321855	26.616392	2.038154	23.147814	0.850624
11	35.371146	2.470618	32.438368	2.383133	29.687840	2.154083	27.077321	1.227797
12	38.509931	2.513954	35.558861	2.437169	32.771541	2.247156	30.150263	1.616494

One can note that in this case the relation (1.2) is verified and the values γ_n^* are obtained by (1.3). Similar considerations are valid for the case (4b). With such a study it is possible to isolate all the roots of (4a) and (4b), and then to obtain all their values. The solutions for (1.3) and (1.4) have been determined using Newton's method. Table 1 lists some values of γ_n^* and $\hat{\gamma}_n$, and for the case (4a) the relevant values of the double roots x_n^*. Tables 2 and 3 give the values of the first twelve roots of (4a) and (4b), for some values of γ, respectively. The numerical solutions have been obtained by a 370/168 IBM computer, using the iterative method developed by Ward [7] and modified by Bach [8]. The computed roots have been used in [2], [5] and, in the hypothesis that the boundary conditions are sufficiently smooth, give a good convergence to the data for the PF series solution of Eq. (3).

Department of Mathematics and Physics
University of Camerino
62032 Camerino, Italy

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