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Further Inequalities for the Gamma Function*

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Abstract. For $\lambda > 0$ and $k \geqslant 0$ we present a method which permits us to obtain inequalities of the type $(k+\alpha)^{\lambda-1} < \Gamma(k+\lambda)/\Gamma(k+1) < (k+\beta)^{\lambda-1}$, with the usual notation for the gamma function, where α and β are independent of k. Some examples are also given which improve well-known inequalities. Finally, we are also able to show in some cases that the values α and β in the inequalities that we obtain cannot be improved.

1. Introduction. In this paper we are concerned with some inequalities for the function $\Gamma(k+\lambda)/\Gamma(k+1)$, where $k \geqslant 0$ and $\lambda > 0$ is independent of k.

Many authors have studied inequalities for this function. For example, Gautschi [2, (7)] has proved that

$$\frac{1}{(k+1)^{1-\lambda}} < \frac{\Gamma(k+\lambda)}{\Gamma(k+1)} < \frac{1}{k^{1-\lambda}}, \qquad 0 < \lambda < 1, k = 1, 2, \ldots, \tag{1.1}$$

and, in the particular case $\lambda = 1/2$, Watson [5] has given the lower bound

$$\frac{\Gamma(k+1/2)}{\Gamma(k+1)} > \frac{1}{(k+4\pi^{-1}-1)^{1/2}}, \qquad k > 1, \tag{1.2}$$

where k is real.

Recently Lorch [4] has given some useful improvements of the bounds in (1.1) and has used his results to obtain a very interesting inequality for ultraspherical polynomials.

Here we show that Lorch's method can be sharpened somewhat, so as to obtain new inequalities and improvements upon known inequalities.

Moreover our results refer to the general case of real and positive k and not only integer k as in Lorch's results.

2. The Function $A_k(\lambda; \alpha)$. From the relation [1, p. 257, (6.1.46)]

$$\lim_{k\to\infty} k^{b-a}\frac{\Gamma(k+a)}{\Gamma(k+b)} = 1$$

we see that

$$\lim_{k\to\infty} f_k = 1,$$

where

$$f_k \equiv \frac{\Gamma(k+\lambda)}{\Gamma(k+1)}(k+\alpha)^{1-\lambda}, \qquad k \geqslant 0, \quad \lambda, \alpha > 0.$$

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Thus, if for some fixed α we can show that f_k ultimately increases (or decreases) to 1 for $k \to \infty$, we can conclude that $f_k < 1$ (or $f_k > 1$) for $k \geqslant k_0$, and we obtain the inequalities

$$\frac{\Gamma(k+\lambda)}{\Gamma(k+1)} < (k+\alpha)^{\lambda-1} \quad \text{or} \quad \frac{\Gamma(k+\lambda)}{\Gamma(k+1)} > (k+\alpha)^{\lambda-1}, \qquad k \geqslant k_0,$$

for the cases $f_k < 1$ and $f_k > 1$, respectively. This suggests studying the function $g_k \equiv f_{k+1}/f_k$ which, using the functional relation $\Gamma(z+1) = z\Gamma(z)$, can be written in the form

$$g_k \equiv \frac{f_{k+1}}{f_k} = \frac{k+\lambda}{k+1}\left(\frac{k+\alpha+1}{k+\alpha}\right)^{1-\lambda}$$

to see for what values of α, λ and k one has $g_k > 1$ or $g_k < 1$. Lorch in [4] has already used this method, but only for some particular values of α. Since $\lim_{k\to\infty} g_k = 1$, we are interested in the monotonicity of g_k with respect to k. Letting $k > 0$ be a continuous variable, we have

$$(k+\alpha)^{2-\lambda}(k+1)^2(k+\alpha+1)^{\lambda} g_k' = A_k(\lambda; \alpha),$$

where

$$A_k(\lambda;\alpha) = (1-\lambda)(-\lambda k + 2\alpha k - \lambda + \alpha^2 + \alpha). \tag{2.1}$$

Thus we need to study the sign of the function $A_k(\lambda; \alpha)$.

When $\alpha = \lambda/2$ we have

$$A_k(\lambda; \lambda/2) \equiv \frac{\lambda}{2}(1-\lambda)\left(\frac{\lambda}{2} - 1\right)$$

and

$$A_k(\lambda;\lambda/2) > 0, \qquad 1 < \lambda < 2,\, k \geqslant 0,$$
$$A_k(\lambda;\lambda/2) < 0, \qquad 0 < \lambda < 1 \text{ or } \lambda > 2,\, k \geqslant 0.$$

We thus obtain

$$\frac{\Gamma(k+\lambda)}{\Gamma(k+1)} < (k+\lambda/2)^{\lambda-1}, \qquad 0 < \lambda < 1 \text{ or } \lambda > 2,\, k \geqslant 0, \tag{2.2}$$

and

$$\left(k+\frac{\lambda}{2}\right)^{\lambda-1} < \frac{\Gamma(k+\lambda)}{\Gamma(k+1)}, \qquad 1 < \lambda < 2,\, k \geqslant 0, \tag{2.3}$$

which for integer k are the inequalities obtained by Lorch [4]. For the case $0 < \lambda < 1$ and real k the inequality (2.2) has also been proved independently by Kershaw [3]. We observe that the term $\lambda/2$ in the bounds (2.2) and (2.3) cannot be improved upon. For example, if $\lambda > 2$ and $\alpha < \lambda/2$, (2.1) shows that there exists a k_0 such that for $k \geqslant k_0$ the function $A_k(\lambda; \alpha)$ becomes, and remains, positive.

This proves that, for (2.2) to be valid for any nonnegative real k, the term $\lambda/2$ in (2.2) is best possible. Similarly, the term $\lambda/2$ in (2.3) and in (2.2), for $0 < \lambda < 1$, cannot be replaced by $\alpha > \lambda/2$.

3. New Inequalities. The study of the sign of the function $A_k(\lambda;\alpha)$ permits one to obtain new inequalities for $\Gamma(k+\lambda)/\Gamma(k+1)$. For example, for $\alpha=\frac{2}{3}\lambda$, (2.1) shows that

$$A_k\left(\lambda;\frac{2}{3}\lambda\right)\equiv\frac{\lambda}{3}(1-\lambda)\left(k+\frac{4}{3}\lambda-1\right)$$

is positive for $0<\lambda<1$ and $k\geqslant 1$, thus yielding the inequality

$$\left(k+\frac{2}{3}\lambda\right)^{\lambda-1}<\frac{\Gamma(k+\lambda)}{\Gamma(k+1)},\qquad 0<\lambda<1,\ k\geqslant 1. \tag{3.1}$$

This lower bound is more precise than the one in Gautschi's inequality (1.1).

It is clear that (3.1) can be further improved if we confine ourselves to values $k\geqslant k_0$ for some k_0. For example, if $k\geqslant 5$, it is sufficient to choose α such that $A_5(\lambda;\alpha)\geqslant 0$, that is, $\alpha\geqslant(-11+\sqrt{121+24\lambda})/2$. Since for these values of α the condition $2\alpha>\lambda$ is satisfied, we get $A_k(\lambda;\alpha)>0$ for $k>5$, and the inequality

$$\left(k+\frac{-11+\sqrt{121+24\lambda}}{2}\right)^{\lambda-1}<\frac{\Gamma(k+\lambda)}{\Gamma(k+1)},\qquad 0<\lambda<1,\ k\geqslant 5,$$

holds.

This lower bound is more precise than the one in (1.2), which is valid only for $\lambda=1/2$. In fact for $\lambda=1/2$ the term $4\pi^{-1}-1=0.273\cdots$ in (1.2) is replaced by $0.266\cdots$ in the new inequality. Generally, we have that for *every* $\varepsilon>0$ there exists a $k_0(\varepsilon)$ such that the inequality

$$\left(k+\frac{\lambda+\varepsilon}{2}\right)^{\lambda-1}<\frac{\Gamma(k+\lambda)}{\Gamma(k+1)},\qquad 0<\lambda<1,\ k\geqslant k_0,$$

holds.

Other useful inequalities can be obtained for the case $1<\lambda<2$. If we put $\alpha=\lambda/2+1/8$ we find $A_k(\lambda;\lambda/2+1/8)<0$ for $k\geqslant 0$, and this proves the inequality

$$\frac{\Gamma(k+\lambda)}{\Gamma(k+1)}<\left(k+\frac{\lambda}{2}+\frac{1}{8}\right)^{\lambda-1},\qquad 1<\lambda<2,\ k\geqslant 0.$$

Similarly, if $\alpha=\frac{\lambda}{2}+\frac{1}{10}$, one has $A_k(\lambda;\lambda/2+1/10)<0$ for $k\geqslant 1$, and we obtain

$$\frac{\Gamma(k+\lambda)}{\Gamma(k+1)}<\left(k+\frac{\lambda}{2}+\frac{1}{10}\right)^{\lambda-1},\qquad 1<\lambda<2,\ k\geqslant 1.$$

Our calculations show that the last upper bounds hold in the general case $\lambda>1$, but our interest is for $1<\lambda<2$, because (2.2) is more precise for $\lambda>2$.

It is clear that it is possible to obtain many other inequalities of a similar type.

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