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On Quartic Fields of Signature One With Small Discriminant. II

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Abstract. Tables of quartic fields, having two real and two complex conjugates, and with discriminants between -7776 and 0, are given.

1. In a paper published twenty-seven years ago [1] a table of quartic fields, having two real and two complex conjugates, was given. This, before the advent of computers, had been calculated by hand and it seemed desirable to check it by computer. The opportunity has been taken to extend the table, and some features, that do not occur for very small values of the discriminant, have appeared.

One error has been discovered in [1]: the polynomial $\theta^4 - 11\theta^3 + 43\theta^2 - 66\theta + 28$, defining the field with discriminant -1879, has index 2 and not 1 as stated. The field with discriminant -2787, for which the defining polynomial was given as $\theta^4 - 5\theta^3 + 15\theta^2 - 21\theta + 9$, with index 3, can also be defined by the polynomial $\theta^4 - 11\theta^3 + 38\theta^2 - 40\theta + 3$, with index 1.

For the sake of completeness, and with the kind permission of the Oxford Quarterly Journal of Mathematics, the original table (with the errors noted above corrected) is reprinted.

2. The method used, and established in [1], is that the field with discriminant Δ, and no subfield, can be defined by a polynomial $\theta^4 - a\theta^3 + b\theta^2 - c\theta + d$ with zeros α, β, $\gamma \pm i\delta$ such that $S = (\alpha - \beta)^2 + (\alpha - \gamma)^2 + (\beta - \gamma)^2 + 4\delta^2 \leqslant (-6\Delta)^{1/3}$. For the present paper all polynomials with $S \leqslant 36$ have been generated by computer, thus giving all fields with $-7776 \leqslant \Delta < 0$.

Since it is convenient to have defining polynomials with index 1, and these are not necessarily given by the smallest values of S, some defining polynomials with larger values of S are included. There may in fact be no defining polynomial with index 1, whether or not all indices have a common factor greater than 1. (See [3, pp. 328–337], for the relevant theory.) In Table 1 below indices other than 1 are given: an asterisk indicates that the given index is a common factor of the indices of all polynomials that define the field and have $S < 400$; it has not been proved that the given index is a common factor of all indices. Where there is no asterisk the indices examined have common factor 1 but no index 1 has been found.

An integral basis can usually be found by factorizing the polynomial modulo the index and discarding a repeated factor: for example, for $\Delta = -3407$, we have

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$\theta^4 - 9\theta^3 + 24\theta^2 - 34\theta + 4 \equiv \theta^2(\theta^2 + \theta + 1) \pmod 2$ and a basis is $1, \theta, \theta^2$, $\theta(\theta^2 + \theta + 1)/2$. The only exceptions are referred to by special notes.

Cases where there are distinct fields with the same discriminant and no subfield can arise in two ways, either because the fields are supported (see [2]) by distinct cubic fields (as obtains for $\Delta = -4027, -7344$) or because they are supported by numbers from the same cubic field whose ratio is not a square (as obtains for $\Delta = -3776, -6571, -6848, -6883, -6928$).

Quartic fields with subfields are of the form $K(\sqrt{\mu})$: those with discriminants in the relevant interval are given in Table 2.

TABLE 1

	a	b	c	d	Index
−283	3	7	5	1	
−331	5	10	6	1	
−491	7	15	9	1	
−563	6	12	7	1	
−643	7	18	18	5	
−688	4	10	6	1	
−731	5	11	11	3	
−751	6	13	11	1	
−848	8	23	26	9	
−976	6	15	16	5	
−1099	7	19	21	7	
−1107	7	18	18	3	
−1192	5	11	10	2	
−1255	7	15	8	1	
−1328	4	9	6	1	
−1371	4	8	7	1	
−1399	7	16	11	2	
−1423	7	19	22	7	
−1424	8	25	34	15	
−1456	8	22	22	5	
−1472	6	14	14	3	
−1588	3	9	8	2	
−1732	9	27	28	2	
−1791	7	17	14	1	
−1823	9	30	41	16	
−1856	10	35	46	11	
−1879	11	43	66	28	2*
−1927	9	28	31	4	
−1931	8	24	29	11	
−1963	7	20	26	11	
−1968	6	14	10	2	
−1984	6	14	12	2	
−2051	9	29	37	15	
−2068	9	28	33	11	
−2092	9	27	29	8	
−2096	8	22	22	6	
−2116	5	10	9	1	
−2151	8	23	25	4	
−2183	6	15	15	4	
−2191	5	12	13	4	
−2219	5	12	14	5	
−2243	9	28	32	5	
−2284	9	26	24	2	
−2319	9	28	33	12	
−2327	8	23	27	8	
−2412	9	29	39	16	
−2443	8	24	29	9	
−2480	8	24	30	10	
−2488	9	31	46	22	
−2563	6	16	19	7	
−2608	8	22	22	2	
−2619	9	27	29	9	
−2687	11	42	59	14	
−2696	7	18	17	3	
−2736	6	14	12	1	
−2763	5	11	7	1	
−2764	7	17	11	2	
−2767	7	18	17	4	
−2787	11	38	40	3	
−2816	12	50	80	31	
−2824	7	17	14	2	
−2843	5	11	9	1	
−2859	9	29	37	11	
−2911	8	25	35	16	
−2943	7	18	19	4	
−3052	7	19	23	8	
−3119	3	9	6	1	
−3163	10	34	41	7	
−3175	8	23	25	8	
−3188	7	19	20	6	
−3216	10	35	48	21	
−3223	9	29	36	11	
−3267	5	12	12	3	
−3271	4	9	9	2	
−3284	7	17	16	2	
−3303	8	23	25	7	
−3376	6	13	8	1	
−3407	9	29	34	4	2*
−3411	3	8	6	1	
−3424	8	21	18	2	
−3431	9	29	38	12	2*
−3436	9	28	34	10	
−3504	6	16	18	6	
−3544	11	41	58	26	
−3559	8	19	11	1	
−3571	5	13	7	1	

TABLE 1 (*continued*)

	a	*b*	*c*	*d*	Index		*a*	*b*	*c*	*d*	Index
−3632	10	36	52	23		−5260	9	31	45	20	
−3723	9	31	45	21		−5323	10	36	53	22	
−3747	7	19	21	5		−5343	10	35	49	19	
−3751	4	13	7	1		−5348	5	13	16	6	
−3776	2	9	6	1		−5371	10	34	43	15	
−3776	10	32	34	9		−5424	6	14	14	2	
−3816	11	44	73	37		−5431	8	19	9	1	
−3888	12	48	70	27		−5432	9	30	43	19	
−3891	5	14	12	3		−5448	13	59	106	58	
−3899	6	14	13	1		−5476	9	29	36	14	
−3919	8	25	33	13		−5548	9	30	40	14	
−3951	5	11	10	1		−5552	12	50	82	42	
−3967	7	19	18	4	2*	−5568	10	34	42	11	
−3984	6	19	24	9	3	−5591	11	38	41	9	2*
−4027	7	16	8	1		−5595	5	13	15	5	
−4027	8	22	23	4		−5636	7	23	36	18	3
−4063	11	42	61	22		−5644	11	43	65	22	
−4103	7	19	14	3		−5675	10	34	43	17	
−4108	7	18	18	2		−5732	5	15	20	8	2
−4152	8	23	26	4	2	−5748	11	37	36	8	2
−4192	8	25	34	14		−5755	9	30	40	17	
−4204	3	11	9	2		−5792	12	51	88	46	
−4287	7	20	27	12		−5816	9	28	31	5	
−4319	4	11	9	2		−5867	6	14	11	1	
−4384	12	47	60	2		−5932	11	43	69	34	
−4423	9	27	28	5		−5963	9	27	25	7	
−4432	10	43	44	13		−5987	6	16	17	5	
−4491	8	22	21	3		−6043	7	21	27	11	
−4492	5	14	10	2		−6064	10	37	56	25	
−4503	6	17	11	2		−6071	5	12	11	2	
−4564	9	26	21	5		−6075	9	30	40	15	
−4568	7	20	25	9		−6079	12	51	85	38	
−4595	10	36	51	19		−6091	8	26	37	17	
−4615	7	19	20	5		−6124	9	24	16	2	
−4648	5	12	7	1		−6199	7	19	22	4	2*
−4652	8	24	26	8	2	−6283	12	52	91	46	
−4663	7	20	23	8		−6331	7	21	29	13	
−4671	6	15	15	3		−6343	17	89	142	68	2*
−4675	13	60	110	55		−6371	9	28	32	11	
−4703	12	51	87	46		−6387	9	26	24	3	
−4744	7	21	24	8	2	−6399	8	21	15	3	
−4748	11	42	62	26		−6411	12	52	93	51	
−4752	8	21	18	3		−6444	7	20	26	10	
−4780	6	18	22	8	2	−6480	16	87	178	97	3
−4799	9	26	23	1	2*	−6484	11	41	52	2	
−4832	8	23	24	2		−6507	10	30	23	5	
−4864	8	24	28	10		−6571	4	10	7	1	
−4907	11	43	65	23		−6571	8	22	21	4	
−4944	8	23	24	3		−6571	9	28	32	9	
−4979	11	44	72	37		−6603	8	22	21	5	
−4999	10	35	47	17		−6604	11	43	67	32	
−5036	11	44	72	34		−6611	9	30	40	13	
−5184	6	15	14	3		−6656	8	24	28	9	
−5224	11	39	46	10		−6664	11	42	61	19	
−5231	9	26	23	6		−6687	9	29	36	13	
−5243	9	24	16	1		−6691	3	7	7	1	

TABLE 1 (*continued*)

	a	b	c	d	Index		a	b	c	d	Index
−6700	9	25	19	4		−7199	7	18	9	1	2*
−6739	11	24	20	3		−7259	4	10	11	3	
−6763	11	39	47	15		−7331	7	18	14	3	
−6791	11	44	71	32		−7335	10	35	47	13	
−6800	8	23	26	5		−7344	10	36	52	19	
−6848	2	8	6	1		−7344	8	21	14	1	3
−6848	6	17	22	9		−7351	10	39	62	28	4*††
−6863	11	41	54	7		−7407	8	25	33	12	
−6880	12	49	76	28	2†	−7412	5	12	13	3	
−6883	6	16	13	3		−7463	15	71	110	52	2*
−6883	7	16	10	1		−7472	8	26	38	18	
−6883	10	34	43	16		−7492	7	20	26	8	2*
−6896	12	47	64	21		−7528	5	13	10	2	
−6912	12	48	68	18		−7532	7	20	12	2	
−6924	9	29	35	8		−7571	7	19	19	5	
−6928	8	25	24	7		−7652	5	16	13	3	
−6928	8	19	10	1		−7668	9	27	28	6	
−6939	8	24	31	11		−7692	25	211	627	218	
−6967	9	30	43	17	2*	−7699	7	18	16	3	
−6976	10	37	56	26		−7703	8	23	27	7	
−6987	41	560	2598	723		−7715	10	32	33	5	
−7087	6	17	21	8		−7732	11	44	70	32	2*
−7088	8	20	14	2		−7744	10	33	36	2	
−7155	7	18	20	5		−7771	10	36	51	20	

† A basis for the field is $1, \theta, \theta^2, \theta^2(\theta - 1)/2$ but not $1, \theta, \theta^2, \theta(\theta^2 - 1)/2$.

†† Polynomials defining this field were examined for $S < 1000$. No index 2 occurs, but 2 is the highest common factor of all indices examined. A basis for the field is $1, \theta, \theta(\theta - 1)/2, \theta^2(\theta - 1)/2$.

TABLE 2

Δ	μ	Δ	μ
−275	$-(1 + 3\sqrt{5})/2$	−1984	$11 + 4\sqrt{2}$
−400	$(1 + \sqrt{5})/2$	−2000	$1\sqrt{5}$
−448	$-1 + 2\sqrt{2}$	−2048	$1\sqrt{2}$
−475	$-1 + 2\sqrt{5}$	−2312	$(3 + \sqrt{17})/2$
−507	$(-1 + \sqrt{13})/2$	−2375	$(-5 + 9\sqrt{5})/2$
−775	$(-1 + 5\sqrt{5})/2$	−2475	$(3 + 9\sqrt{5})/2$
−1024	$11 + \sqrt{2}$	−2704	$(3 + \sqrt{13})/2$
−1156	$14 + \sqrt{17}$	−3008	$-5 + 6\sqrt{2}$
−1323	$(3 + \sqrt{21})/2$	−3275	$(-9 + 11\sqrt{5})/2$
−1375	$15 + 4\sqrt{5}$	−3312	$15 + 4\sqrt{3}$
−1472	$-3 + 4\sqrt{2}$	−3312	$-5 + 4\sqrt{3}$
−1475	$(3 + 7\sqrt{5})/2$	−3475	$1(7 + 11\sqrt{5})/2$
−1600	$11 + \sqrt{5}$	−3600	$(3 + 3\sqrt{5})/2$
−1728	$13 + 2\sqrt{3}$	−3775	$(-1 + 11\sqrt{5})/2$
−1775	$-3 + 4\sqrt{5}$	−3875	$-5 + 6\sqrt{5}$
−1792	$11 + 2\sqrt{2}$	−3887	$(-5 + 3\sqrt{13})/2$
−1975	$(-17 + 11\sqrt{5})/2$	−4032	$13 + 6\sqrt{2}$

TABLE 2 (*continued*)

Δ	μ
-4107	$(-5+\sqrt{37})/2$
-4275	$13+6\sqrt{5}$
-4400	$(1+3\sqrt{5})/2$
-4475	$-1+6\sqrt{5}$
-4544	$-1+6\sqrt{2}$
-4608	$11+\sqrt{3}$
-4608	$-1+\sqrt{3}$
-4775	$(-9+13\sqrt{5})/2$
-4975	$(7+13\sqrt{5})/2$
-5056	$-7+8\sqrt{2}$
-5275	$(-1+13\sqrt{5})/2$
-5488	$121+8\sqrt{7}$
-5491	$17+2\sqrt{17}$
-5616	$13+4\sqrt{3}$
-5616	$-3+4\sqrt{3}$
-5887	$(-1+\sqrt{29})/2$
-5888	$13+4\sqrt{2}$
-5975	$(-13+15\sqrt{5})/2$
-6275	$(11+15\sqrt{5})/2$
-6336	$11+2\sqrt{3}$
-6336	$-1+2\sqrt{3}$
-6591	$(-13+5\sqrt{13})/2$
-6592	$15+8\sqrt{2}$
-6724	$132+5\sqrt{41}$
-6768	$11+4\sqrt{3}$
-6768	$-1+4\sqrt{3}$
-6775	$-7+8\sqrt{5}$
-6912	$1\sqrt{3}$
-6975	$(3+15\sqrt{5})/2$
-7267	$13+2\sqrt{13}$
-7375	$15+8\sqrt{5}$
-7600	$11+2\sqrt{5}$
-7616	$-3+8\sqrt{2}$
-7616	$-9+10\sqrt{2}$
-7775	$-3+8\sqrt{5}$

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1. H. J. GODWIN, "On quartic fields of signature one with small discriminant," *Quart. J. Math. Oxford Ser.* (2), v. 8, 1957, pp. 214–222.

2. H. J. GODWIN, "On relations between cubic and quartic fields," *Quart. J. Math. Oxford Ser.* (2), v. 13, 1962, pp. 206–212.

3. H. HASSE, *Zahlentheorie*, Akademie-Verlag, Berlin, 1949.