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Primitive α-Abundant Numbers

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Abstract. A number N is primitive α-abundant if $\sigma(M)/M < \alpha \leqslant \sigma(N)/N$ for all proper divisors M of N. In this paper, we tabulate, for $1 < \alpha \leqslant 5.4$, all such N for which $\sigma(N)/N$ is greatest. We show that, if N is primitive α-abundant and $\alpha > 1.6$, then $\sigma(N)/N < \alpha + \min\{\frac{2}{5}, 3\alpha/2e^{5\alpha/9}\}$.

Let N be a natural number ($N > 1$), let α be a real number ($\alpha > 1$) and let σ be the positive-divisor sum function. Put

$$H(N) = \frac{\sigma(N)}{N} \quad \text{and} \quad h(N) = \max\{H(M) \colon M | N, M < N\}.$$

We say that N is α-abundant if $\alpha \leqslant H(N)$ and that N is primitive α-abundant if also $h(N) < \alpha$. The set π_α of primitive α-abundant numbers has been investigated by a number of writers. For example, Erdös [2] showed that the number of elements of π_α which are less than x is $o(x/\log x)$ and Shapiro [3] (see also [4]) that there are only finitely many $N \in \pi_\alpha$ with a fixed number of distinct prime factors, provided that, when α is rational, N is relatively prime to the numerator of α.

When α is an integer, it is a classical problem to find members of π_α which are *least* α-abundant: such $N \in \pi_\alpha$ satisfy $H(N) = \alpha$, and we are referring to the search for perfect and multiply perfect numbers. In this note, we shall find all members of π_α which are *most* α-abundant, for $1 < \alpha \leqslant 5.4$. That is, we shall find those $N \in \pi_\alpha$ ($1 < \alpha \leqslant 5.4$) for which $H(N)$ is greatest.

Put

$$G(\alpha) = \max\{H(N) \colon N \in \pi_\alpha\}, \qquad \Gamma_\alpha = \{N \colon N \in \pi_\alpha, H(N) = G(\alpha)\},$$

so that Γ_α consists of those primitive α-abundant numbers N for which $\sigma(N)/N$ is greatest. Of more interest than G is the function $G_0(\alpha) = G(\alpha) - \alpha$. It was shown in [1] that Γ_α is nonempty and finite and that

$$G_0(\alpha) < \min\{\tfrac{1}{2}, 3\alpha e^{-\rho\alpha}/2\}, \tag{1}$$

where $\rho = e^{-\gamma}$ and γ is Euler's constant. (For $\alpha > 4.8$, $3\alpha e^{-\rho\alpha}/2 < \frac{1}{2}$.) It was also shown in [1] that

$$G_0(2) = \frac{62}{385} < .16104 \quad \text{and} \quad \Gamma_2 = \{3465\}.$$

Our present results, giving $G_0(\alpha)$ and Γ_α for $1 < \alpha \leqslant 5.4$, are given in Table 1. The following two theorems are the basis for the computations used to produce the

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table. Theorem 1 was proved in [1]; the proof is short and is repeated here for completeness.

THEOREM 1. *If $N \in \pi_\alpha$, then $H(N) < \alpha + \alpha/p^a$, where p^a is any component of N.*

Proof. Suppose $H(N) \geqslant \alpha + \alpha/p^a$, for some component p^a of N. Then, using the fact that σ is multiplicative,

$$\sigma\left(\frac{N}{p}\right) = \sigma\left(\frac{N}{p^a}\right)\sigma(p^{a-1}) = \frac{\sigma(N)}{\sigma(p^a)}\sigma(p^{a-1})$$
$$\geqslant \left(\alpha N + \frac{\alpha N}{p^a}\right)\frac{\sigma(p^{a-1})}{\sigma(p^a)} = \frac{\alpha N}{p}\,\frac{p^a+1}{p^{a-1}}\,\frac{\sigma(p^{a-1})}{\sigma(p^a)} \geqslant \frac{\alpha N}{p}.$$

Thus $h(N) \geqslant \alpha$, contradicting $N \in \pi_\alpha$.

From Theorem 1, if $N \in \pi_\alpha$ and we require $H(N) \geqslant \alpha + \varepsilon$ $(\varepsilon > 0)$, then $p^a < \alpha/\varepsilon$. With a little experimentation, ε can be chosen to give a small set of such numbers N, and then $G(\alpha)$ and Γ_α can be identified.

THEOREM 2. *Let $\alpha_+ > 1$ be any real number and define α_- by $\alpha_- = \max\{h(N)\colon N \in \Gamma_{\alpha_+}\}$. Then for any α, $\alpha_- < \alpha \leqslant \alpha_+$,*

$$G(\alpha) = G(\alpha_+) \quad \textit{and} \quad \Gamma_\alpha = \Gamma_{\alpha_+}.$$

Proof. Suppose $\alpha_- < \alpha \leqslant \alpha_+$, and take $N \in \Gamma_\alpha$, $N_+ \in \Gamma_{\alpha_+}$, so $H(N) = G(\alpha)$, $H(N_+) = G(\alpha_+)$. We have

$$h(N_+) \leqslant \alpha_- < \alpha \leqslant \alpha_+ \leqslant H(N_+),$$

so $N_+ \in \pi_\alpha$. Thus $H(N_+) \leqslant G(\alpha)$, or

$$G(\alpha_+) \leqslant G(\alpha). \tag{2}$$

We then have

$$h(N) < \alpha \leqslant \alpha_+ \leqslant G(\alpha_+) \leqslant G(\alpha) = H(N),$$

so $N \in \pi_{\alpha_+}$. Thus $H(N) \leqslant G(\alpha_+)$, or $G(\alpha) \leqslant G(\alpha_+)$. With (2), this shows that $G(\alpha) = G(\alpha_+)$, and it is then easy to see also that $\Gamma_\alpha = \Gamma_{\alpha_+}$.

In Table 1, the decreasing sequence (α_i), $0 \leqslant i \leqslant 145$, is defined by

$$\alpha_0 = 5.4, \quad \alpha_i = \max\left\{h(N)\colon N \in \Gamma_{\alpha_{i-1}}\right\} \qquad (1 \leqslant i \leqslant 145).$$

By Theorem 2, if $\alpha_{i+1} < \alpha \leqslant \alpha_i$ $(0 \leqslant i \leqslant 144)$, then

$$G_0(\alpha) = G_0(\alpha_i) + \alpha_i - \alpha \quad \text{and} \quad \Gamma_\alpha = \Gamma_{\alpha_i}.$$

Besides giving α_i, $G_0(\alpha_i)$ and Γ_{α_i} in Table 1, we have also given values for

$$g_i = G_0(\alpha_i) + \alpha_i - \alpha_{i+1} = \sup\{G_0(\alpha)\colon \alpha_{i+1} < \alpha \leqslant \alpha_i\}.$$

All values are exact, or, if given to five decimal places, are rounded. If required, the exact value of α_i $(1 \leqslant i \leqslant 145)$ may be found as follows: if $N \in \Gamma_{\alpha_{i-1}}$ has an underlined component p^a, then $\alpha_i = H(N/p)$.

The values $g_{141} = g_{144} = .5$ show that $\frac{1}{2}$ is best possible in (1). If $g_i < .5$, then $g_i \leqslant .4$, with equality when $i = 121$, 129 and 135. Since $3\alpha e^{-\rho\alpha}/2 < .4$ for $\alpha > 5.4$, we have

THEOREM 3. *If N is primitive α-abundant, for $\alpha > 1.6$, then*

$$\frac{\sigma(N)}{N} < \alpha + \min\left\{\frac{2}{5}, \frac{3\alpha e^{-\rho\alpha}}{2}\right\}.$$

We pose two questions, arising from Table 1.

(i) Can Γ_α have arbitrarily many elements? We do not know any α for which Γ_α has other than one or two elements.

(ii) Does there exist $\alpha > 2$ for which Γ_α has an odd member?

We suggest not, and propose the following more general conjecture. For any natural number n, there exists a rational number β_n such that if $\alpha > \beta_n$ and $N \in \Gamma_\alpha$, then N is divisible by $p_1 p_2 \cdots p_n$, where p_i is the ith prime. If this is so, then, from Table 1, $\beta_1 \geqslant 2$, $\beta_2 \geqslant 16/7 = H(2 \cdot 3 \cdot 7)$ (see $i = 132$) and $\beta_3 \geqslant 312/85 = H(2^2 3^2 5 \cdot 7 \cdot 17)$ (see $i = 101$). We conjecture further that equality holds in these three cases.

TABLE 1

i	α_i	$G_0(\alpha_i)$	$G_0(\alpha_i) + \alpha_i - \alpha_{i+1}$	Γ_{α_i}
0	5.4	.14550	.17330	$\{2^4 3^2 5^2 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 29\cdot \underline{31}\}$
1	5.37221	.13945	.14504	$\{2^3 3^2 5^2 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 29\cdot 31\cdot \underline{37}$,
				$2^4 3^2 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 29\cdot 31\cdot \underline{37}\}$
2	5.36662	.13892	.14488	$\{2^4 3^2 5^2 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 31\cdot \underline{37}\}$
3	5.36065	.13663	.17179	$\{2^4 3^2 5^2 7\cdot 11\cdot 13\cdot 17\cdot 23\cdot 29\cdot \underline{31}\}$
4	5.32549	.13964	.17079	$\{2^4 3^2 5^2 7\cdot 11\cdot 13\cdot 19\cdot 23\cdot 29\cdot \underline{31}\}$
5	5.29435	.13743	.14294	$\{2^3 3^2 5^2 7\cdot 11\cdot 13\cdot 19\cdot 23\cdot 29\cdot 31\cdot \underline{37}$,
				$2^4 3^2 5\cdot 7\cdot 11\cdot 13\cdot 19\cdot 23\cdot 29\cdot 31\cdot \underline{37}\}$
6	5.28884	.13691	.14278	$\{2^4 3^2 5^2 7\cdot 11\cdot 13\cdot 19\cdot 23\cdot 31\cdot \underline{37}\}$
7	5.28296	.14199	.17500	$\{2^3 3^2 \underline{5^2} 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29$,
				$\underline{2^4} 3^2 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\}$
8	5.24995	.16897	.17480	$\{\underline{2^4} 3^2 \underline{5^2} 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\}$
9	5.24412	.13613	.14159	$\{2^3 3^2 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 31\cdot \underline{37}\}$
10	5.23866	.14031	.14155	$\{2^4 3^2 5^2 7\cdot 11\cdot 13\cdot 19\cdot 29\cdot 31\cdot \underline{37}\}$
11	5.23742	.13586	.16792	$\{2^4 3^2 5^2 7\cdot 11\cdot 17\cdot 19\cdot 23\cdot 29\cdot \underline{31}\}$
12	5.20537	.16684	.17330	$\{\underline{2^4} 3^2 \underline{5^2} 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 29\}$
13	5.19891	.16174	.16752	$\{2^4 3^2 5^2 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot \underline{31}\}$
14	5.19313	.14836	.16692	$\{2^4 3\cdot 5^2 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\cdot \underline{31}\}$
15	5.17457	.15092	.17179	$\{\underline{2^4} 3^2 \underline{5^2} 7\cdot 11\cdot 13\cdot 17\cdot 23\cdot 29\}$

TABLE 1 (*continued*)

i	α_i	$G_0(\alpha_i)$	$G_0(\alpha_i) + \alpha_i - \alpha_{i+1}$	Γ_{α_i}
16	5.15370	.16034	.16606	$\{2^4 3^2 5^2 7\cdot 11\cdot 13\cdot 17\cdot 23\cdot \underline{31}\}$
17	5.14797	.15642	.16576	$\{2^4 3^2 5^2 7\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\cdot \underline{31}\}$
18	5.13863	.15572	.17079	$\{\underline{2}^4 3^2 \underline{5}^2 7\cdot 11\cdot 13\cdot 19\cdot 23\cdot 29\}$
19	5.12356	.15940	.16509	$\{2^4 3^2 5^2 7\cdot 11\cdot 13\cdot 19\cdot 23\cdot \underline{31}\}$
20	5.11787	.15036	.16463	$\{2^4 3^2 5^2 7\cdot 11\cdot 13\cdot 17\cdot 29\cdot \underline{31}\}$
21	5.10360	.14635	.17500	$\{2^3 3^2 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot \underline{29}\}$
22	5.07495	.16247	.16367	$\{2^4 3^2 5^2 7\cdot 11\cdot 13\cdot 19\cdot 29\cdot \underline{31}\}$
23	5.07375	.15287	.16333	$\{2^2 3^2 5^2 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\cdot \underline{31}\}$
24	5.06329	.14208	.16792	$\{\underline{2}^4 3^2 \underline{5}^2 7\cdot 11\cdot 17\cdot 19\cdot 23\cdot 29\}$
25	5.03745	.16146	.16771	$\{2^3 3^2 \underline{5}^2 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 29,$ $\underline{2}^4 3^2 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 29\}$
26	5.03120	.16193	.16752	$\{\underline{2}^4 3^2 \underline{5}^2 7\cdot 11\cdot 13\cdot 17\cdot 19\}$
27	5.02561	.14895	.16692	$\{\underline{2}^4 3\cdot \underline{5}^2 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\}$
28	5.00764	.15579	.16136	$\{2^4 3\cdot 5^2 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot \underline{31}\}$
29	5.00208	.15162	.16625	$\{2^3 3^2 \underline{5}^2 7\cdot 11\cdot 13\cdot 17\cdot 23\cdot 29,$ $\underline{2}^4 3^2 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 23\cdot 29\}$
30	4.98745	.16052	.16606	$\{\underline{2}^4 3^2 \underline{5}^2 7\cdot 11\cdot 13\cdot 17\cdot 23\}$
31	4.98191	.15672	.16576	$\{\underline{2}^4 3^2 \underline{5}^2 7\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\}$
32	4.97287	.15471	.16024	$\{2^4 3^2 5^2 7\cdot 13\cdot 17\cdot 19\cdot 23\cdot \underline{31}\}$
33	4.96734	.15622	.16528	$\{2^3 3^2 \underline{5}^2 7\cdot 11\cdot 13\cdot 19\cdot 23\cdot 29,$ $\underline{2}^4 3^2 5\cdot 7\cdot 11\cdot 13\cdot 19\cdot 23\cdot 29\}$
34	4.95829	.15958	.16509	$\{\underline{2}^4 3^2 \underline{5}^2 7\cdot 11\cdot 13\cdot 19\cdot 23\}$
35	4.95278	.15184	.15952	$\{2^4 3^2 5^2 7\cdot 11\cdot 17\cdot 23\cdot 29\cdot \underline{31}\}$
36	4.94510	.15850	.16463	$\{\underline{2}^4 3^2 \underline{5}^2 7\cdot 11\cdot 13\cdot 17\cdot 29\}$
37	4.93896	.15366	.15914	$\{2^4 3^2 5^2 7\cdot 11\cdot 13\cdot 17\cdot \underline{31}\}$
38	4.93348	.14990	.15886	$\{2^4 3^2 5^2 7\cdot 13\cdot 17\cdot 19\cdot 29\cdot \underline{31}\}$
39	4.92452	.15043	.21146	$\{2^3 3^2 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot \underline{23}\}$
40	4.86350	.15532	.15684	$\{2^4 3^2 5^2 7\cdot 11\cdot 13\cdot 23\cdot \underline{31}\}$
41	4.86198	.14773	.15655	$\{2^4 3^2 5^2 7\cdot 13\cdot 19\cdot 23\cdot 29\cdot \underline{31}\}$
42	4.85315	.15569	.15653	$\{2^2 3^2 5^2 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 29\cdot \underline{31}\}$
43	4.85232	.15533	.16154	$\{2^3 3\cdot \underline{5}^2 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29,$ $\underline{2}^4 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\}$
44	4.84611	.15597	.16136	$\{\underline{2}^4 3\cdot \underline{5}^2 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\}$

TABLE 1 (*continued*)

i	α_i	$G_0(\alpha_i)$	$G_0(\alpha_i) + \alpha_i - \alpha_{i+1}$	Γ_{α_i}
45	4.84072	.14775	.16092	$\{\underline{2}^4 3^2 \underline{5}^2 7\cdot 11\cdot 17\cdot 19\cdot 29\}$
46	4.82756	.15989	.16625	$\{2^3 3^2 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 23\cdot \underline{29}\}$
47	4.82120	.15434	.15549	$\{2^4 3^2 5^2 7\cdot 11\cdot 13\cdot 29\cdot \underline{31}\}$
48	4.82006	.15281	.16042	$\{2^3 3^2 \underline{5}^2 7\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29,$ $\underline{2}^4 3^2 5\cdot 7\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\}$
49	4.81245	.15489	.16024	$\{\underline{2}^4 3^2 \underline{5}^2 7\cdot 13\cdot 17\cdot 19\cdot 23\}$
50	4.80711	.15285	.15500	$\{2^4 3\cdot 5^2 7\cdot 11\cdot 17\cdot 19\cdot 23\cdot 29\cdot \underline{31}\}$
51	4.80495	.15401	.15997	$\{\underline{2}^4 3\cdot \underline{5}^2 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 29\}$
52	4.79899	.15929	.16528	$\{2^3 3^2 5\cdot 7\cdot 11\cdot 13\cdot 19\cdot 23\cdot \underline{29}\}$
53	4.79301	.15209	.15952	$\{\underline{2}^4 3^2 \underline{5}^2 7\cdot 11\cdot 17\cdot 23\cdot 29\}$
54	4.78558	.15338	.15932	$\{2^3 3^2 \underline{5}^2 7\cdot 11\cdot 13\cdot 17\cdot 29,$ $\underline{2}^4 3^2 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 29\}$
55	4.77964	.15383	.15914	$\{\underline{2}^4 3^2 \underline{5}^2 7\cdot 11\cdot 13\cdot 17\}$
56	4.77433	.15118	.15392	$\{2^4 3^2 5^2 7\cdot 17\cdot 19\cdot 23\cdot 29\cdot \underline{31}\}$
57	4.77159	.15294	.15886	$\{\underline{2}^4 3^2 \underline{5}^2 7\cdot 13\cdot 17\cdot 19\cdot 29\}$
58	4.76567	.15051	.15859	$\{\underline{2}^4 3^2 \underline{5}^2 7\cdot 11\cdot 19\cdot 23\cdot 29\}$
59	4.75759	.15824	.15858	$\{\underline{2}^4 3\cdot \underline{5}^2 7\cdot 11\cdot 13\cdot 17\cdot 23\cdot 29\}$
60	4.75726	.15282	.15839	$\{2^3 3^2 \underline{5}^2 7\cdot 11\cdot 13\cdot 19\cdot 29,$ $\underline{2}^4 3^2 5\cdot 7\cdot 11\cdot 13\cdot 19\cdot 29\}$
61	4.75169	.15337	.15823	$\{\underline{2}^4 3^2 \underline{5}^2 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\}$
62	4.74683	.15780	.15821	$\{\underline{2}^4 3^2 \underline{5}^2 7\cdot 11\cdot 13\cdot 19\}$
63	4.74641	.15354	.16333	$\{2^2 3^2 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot \underline{29}\}$
64	4.73662	.15047	.15765	$\{\underline{2}^4 3\cdot \underline{5}^2 7\cdot 11\cdot 13\cdot 19\cdot 23\cdot 29\}$
65	4.72944	.15226	.15747	$\{\underline{2}^4 3^2 \underline{5}^2 7\cdot 13\cdot 17\cdot 23\cdot 29\}$
66	4.72423	.15235	.15239	$\{2^4 3\cdot 5^2 7\cdot 11\cdot 13\cdot 19\cdot 23\cdot \underline{31}\}$
67	4.72419	.15077	.16250	$\{2^3 3^2 5\cdot 7\cdot 11\cdot 17\cdot 19\cdot 23\cdot \underline{29}\}$
68	4.71246	.15493	.15701	$\{2^3 3^2 \underline{5}^2 7\cdot 11\cdot 13\cdot 23\cdot 29,$ $\underline{2}^4 3^2 5\cdot 7\cdot 11\cdot 13\cdot 23\cdot 29\}$
69	4.71037	.15312	.24317	$\{2^3 3^2 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot \underline{19}\}$
70	4.62032	.17269	.19971	$\{2^3 3^2 5\cdot 7\cdot 11\cdot 13\cdot 19\cdot \underline{23}\}$
71	4.59330	.15005	.15301	$\{\underline{2}^4 3\cdot \underline{5}^2 7\cdot 13\cdot 17\cdot 19\cdot 23\cdot 29\}$
72	4.59034	.15121	.15295	$\{\underline{2}^4 3^2 \underline{5}^2 11\cdot 13\cdot 17\cdot 19\cdot 23\}$
73	4.58860	.15045	.15287	$\{\underline{2}^4 3^2 \underline{5}^2 7\cdot 11\cdot 17\cdot 29\}$

TABLE 1 (*continued*)

i	α_i	$G_0(\alpha_i)$	$G_0(\alpha_i) + \alpha_i - \alpha_{i+1}$	Γ_{α_i}
74	4.58618	.15044	.19736	$\{2^2 3^2 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot \underline{23}\}$
75	4.53926	.17319	.19635	$\{2^3 3^2 5\cdot 7\cdot 11\cdot 17\cdot 19\cdot \underline{23}\}$
76	4.51610	.16847	.19519	$\{2^3 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot \underline{23}\}$
77	4.48938	.16266	.19384	$\{2^3 3^2 5\cdot 7\cdot 13\cdot 17\cdot 19\cdot \underline{23}\}$
78	4.45820	.16212	.25668	$\{2^3 3^2 5\cdot 7\cdot 11\cdot 13\cdot \underline{17}\}$
79	4.36364	.17563	.22696	$\{2^2 3^2 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot \underline{19}\}$
80	4.31230	.20380	.22581	$\{2^3 3^2 5\cdot 7\cdot 11\cdot 17\cdot \underline{19}\}$
81	4.29030	.19908	.22447	$\{2^3 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot \underline{19}\}$
82	4.26491	.19329	.22291	$\{2^3 3^2 5\cdot 7\cdot 13\cdot 17\cdot \underline{19}\}$
83	4.23529	.16300	.18326	$\{2^2 3^2 5\cdot 7\cdot 11\cdot 17\cdot 19\cdot \underline{23}\}$
84	4.21503	.17856	.18307	$\{2^3 3^2 5\cdot 7\cdot 13\cdot 19\cdot \underline{23}\}$
85	4.21053	.16174	.18218	$\{2^2 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot \underline{23}\}$
86	4.19009	.17355	.29091	$\{\underline{2}^3 3^2 5\cdot 7\cdot 11\cdot 13\}$
87	4.07273	.21757	.23835	$\{2^3 3^2 5\cdot 7\cdot 11\cdot \underline{17}\}$
88	4.05195	.21296	.23694	$\{2^3 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot \underline{17}\}$
89	4.02797	.20732	.23529	$\{2^3 3^2 5\cdot 7\cdot 13\cdot \underline{17}\}$
90	4	.19009	.20950	$\{2^2 3\cdot 5\cdot 7\cdot 11\cdot 13\cdot 17\cdot \underline{19}\}$
91	3.98058	.18813	.20844	$\{2^3 3\cdot 5\cdot 7\cdot 11\cdot 17\cdot \underline{19}\}$
92	3.96028	.20072	.20805	$\{2^2 3^2 5\cdot 7\cdot 13\cdot 17\cdot \underline{19}\}$
93	3.95294	.18682	.20699	$\{2^3 3^2 5\cdot 7\cdot 17\cdot \underline{19}\}$
94	3.93277	.18249	.20576	$\{2^3 3\cdot 5\cdot 7\cdot 13\cdot 17\cdot \underline{19}\}$
95	3.90950	.16323	.29091	$\{2^2 3^2 5\cdot 7\cdot 11\cdot \underline{13}\}$
96	3.78182	.24615	.26853	$\{\underline{2}^3 3\cdot 5\cdot 7\cdot 11\cdot 13\}$
97	3.75944	.24056	.26667	$\{\underline{2}^3 3^2 5\cdot 7\cdot 13\}$
98	3.73333	.19944	.21849	$\{2^3 3^2 5\cdot 7\cdot \underline{17}\}$
99	3.71429	.19522	.21719	$\{2^3 3\cdot 5\cdot 7\cdot 13\cdot \underline{17}\}$
100	3.69231	.17147	.19319	$\{2^2 3^2 5\cdot 7\cdot 17\cdot \underline{19}\}$
101	3.67059	.17968	.21390	$\{2^3 3^2 7\cdot 11\cdot 13\cdot \underline{17}\}$
102	3.63636	.18495	.19107	$\{2^3 3\cdot 5\cdot 7\cdot 17\cdot \underline{19}\}$
103	3.63025	.18793	.25455	$\{\underline{2}^3 3^2 5\cdot 11\cdot 13\}$
104	3.56364	.21818	.29091	$\{2^2 \underline{3}^2 5\cdot 7\cdot 11\}$
105	3.49091	.24242	.26667	$\{2^2 3^2 5\cdot 7\cdot \underline{13}\}$
106	3.46667	.22564	.24615	$\{\underline{2}^3 3\cdot 5\cdot 7\cdot 13\}$

TABLE 1 (*continued*)

i	α_i	$G_0(\alpha_i)$	$G_0(\alpha_i) + \alpha_i - \alpha_{i+1}$	Γ_{α_i}
107	3.44615	.19021	.24242	$\{\underline{2^3}3^2 7\cdot 11\cdot 13\}$
108	3.39394	.18131	.19862	$\{2^3 3^2 7\cdot 11\cdot \underline{17}\}$
109	3.37622	.18701	.25455	$\{2^2 3^2 5\cdot 11\cdot \underline{13}\}$
110	3.30909	.21538	.23497	$\{\underline{2^3}3\cdot 5\cdot 11\cdot 13\}$
111	3.28951	.21049	.23333	$\{\underline{2^3}3^2 5\cdot 13\}$
112	3.26667	.22424	.24935	$\{2\cdot 3^2 5\cdot 7\cdot 11\cdot \underline{13},$ $2^2 3\cdot 5\cdot 7\cdot 11\}$
113	3.24156	.24935	.29091	$\{2^2 3\cdot 5\cdot 7\cdot \underline{11}\}$
114	3.2	.19394	.24242	$\{2^2 3^2 7\cdot 11\cdot \underline{13}\}$
115	3.15152	.20513	.22378	$\{\underline{2^3}3\cdot 7\cdot 11\cdot 13\}$
116	3.13287	.20047	.22222	$\{\underline{2^3}3^2 7\cdot 13\}$
117	3.11111	.19798	.25455	$\{2^2\underline{3^2}5\cdot 11\}$
118	3.05455	.21212	.23333	$\{2^2 3^2 5\cdot \underline{13}\}$
119	3.03333	.20823	.24935	$\{2\cdot \underline{3^2}5\cdot 7\cdot 11\}$
120	2.99221	.20779	.22857	$\{2\cdot 3^2 5\cdot 7\cdot \underline{13},$ $2^2 3\cdot 5\cdot 7\}$
121	2.97143	.22857	.4	$\{2^2 3\cdot 5\cdot \underline{7}\}$
122	2.8	.19221	.24935	$\{2\cdot 3\cdot 5\cdot 7\cdot \underline{11}\}$
123	2.74286	.19421	.19580	$\{\underline{2^3}3\cdot 11\cdot 13\}$
124	2.74126	.17541	.19444	$\{\underline{2^3}3^2 13\}$
125	2.72222	.18687	.20779	$\{2\cdot 3^2 7\cdot 11\cdot \underline{13},$ $2^2 3\cdot 7\cdot 11\}$
126	2.70130	.20779	.24242	$\{2^2 3\cdot 7\cdot \underline{11}\}$
127	2.66667	.16970	.21818	$\{2\cdot \underline{3^2}5\cdot 11\}$
128	2.61818	.18182	.2	$\{2\cdot 3^2 5\cdot \underline{13},$ $2^2 3\cdot 5\}$
129	2.6	.2	.4	$\{\underline{2^2}3\cdot 5\}$
130	2.4	.26667	.33333	$\{2^2 3\cdot \underline{7}\}$
131	2.33333	.16017	.20779	$\{2\cdot 3\cdot 7\cdot \underline{11}\}$
132	2.28571	.13736	.16154	$\{\underline{2^3}5\cdot 13\}$
133	2.26154	.15524	.17263	$\{2\cdot 5\cdot 7\cdot 11\cdot \underline{13}\}$
134	2.24416	.15584	.3	$\{2\cdot 3\cdot 5,\ 2^2 5\cdot \underline{7}\}$
135	2.1	.3	.4	$\{2\cdot 3\cdot \underline{5}\}$

TABLE 1 (*continued*)

i	α_i	$G_0(\alpha_i)$	$G_0(\alpha_i) + \alpha_i - \alpha_{i+1}$	Γ_{α_i}
136	2	.16104	.16623	$\{\underline{3^2}5\cdot 7\cdot 11\}$
137	1.99481	.13853	.15238	$\{3^2 5\cdot 7\cdot \underline{13}\}$
138	1.98095	.13373	.15105	$\{2\cdot 5\cdot 11\cdot \underline{13}\}$
139	1.96364	.13636	.3	$\{\underline{2^2}5\}$
140	1.8	.2	.25	$\{2\cdot 3,\ 2^2\underline{7}\}$
141	1.75	.25	.5	$\{2\cdot \underline{3}\}$
142	1.5	.11119	.11508	$\{5\cdot 7\cdot 11\cdot \underline{13}\}$
143	1.49610	.10390	.26667	$\{3\cdot \underline{5}\}$
144	1.33333	.16667	.5	$\{\underline{2}\}$
145	1	—	—	—

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